Continuous-Time Methods and Distributional Macroeconomics

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Lectures: Monday 2pm-4pm

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Course Description and Learning Objectives

Modern macroeconomic theories emphasize the need to understand the macroeconomy through the lens of distributions of microeconomic variables, moving beyond representative agent frameworks and simple aggregates. A growing body of research has underscored the critical role of heterogeneous agent (HA) models in capturing key economic phenomena, such as long-run growth patterns, income and wealth inequality, consumption dynamics, occupational choices, and the nuanced transmission of monetary and fiscal policies. Given the complexity of models in which the relevant state variable is a joint distribution, continuous-time methods have emerged as a powerful toolkit offering both analytical clarity and numerical efficiency in modeling dynamic economic systems with rich heterogeneity.

This 12-hour PhD course introduces students to foundational concepts as well as recent developments in continuous-time methods, with a specific focus on their application to HA models in macroeconomics. Delivered over six 2-hour sessions, the course is designed to provide participants with a theoretical grounding in stochastic dynamic programming, alongside hands-on exposure to numerical techniques tailored for solving multi-dimensional problems.

The objectives of this course are twofold. First, students will develop an understanding of the mathematical underpinnings of continuous-time methods, such as notions of stochastic calculus, Itô's Lemma, viscosity solutions, and the Hamilton-Jacobi-Bellman and Kolmogorov Forward equations. Second, they will acquire valuable practical skills in numerical methods to solve and estimate HA models efficiently, such as robust Finite Difference Methods and the Sequence-Space Jacobian approach. By the end of the course students will be equipped to integrate continuous-time methods into their own research and contribute to the evolving frontier of quantitative macroeconomic theory.

Class Attendance and Participation

Attendance is essential and expected. Student participation and collaboration are highly encouraged and will greatly benefit their learning experience. Students should bring their personal computer with a working internet connection, as well as the Anaconda distribution of Python and/or Matlab. I will provide computer codes for all examples presented in the course.

You can always reach out to me by email about any questions you may have. I will be holding unofficial office hours after the end of each lecture, in addition to in-person meetings by appointment.

Assessment

The final grade for this course will be the average of two (relatively long) homework assignments involving extensions of the models we solve in class. You are strongly encouraged to consult with each other and collaborate by forming study groups. However, each student must turn in their own work.

Tentative Lecture Outline

▶ **Lecture 1 – Mathematical Foundations**
Review of basic notions in probability and measure theory – Brownian motion and stochastic processes – Introduction to Stochastic Calculus; Itˆo' s Lemma; Fundamental theorems of stochastic integration and differentiation.

▶ **Lecture 2 – Dynamic Programming in Continuous Time**
Review of optimal control methods, Hamiltonians, Maximum Principle, and deterministic dynamic programming – Stochastic dynamic programming, the Hamilton-Jacobi-Bellman and Kolmogorov Forward equations – Viscosity solutions.

▶ **Lecture 3 – Heterogeneous Agent Models with Idiosyncratic Shocks (Part 1)**
Numerical methods, the Finite Difference Method (FDM) – FDM with Poisson processes, FDM with general Ito processes – Warmup by solving the stochastic neoclassical growth model – Solving the Huggett model with Poisson shocks.

▶ **Lecture 4 – Heterogeneous Agent Models with Idiosyncratic Shocks (Part 2)**
Solving the Aiyagari model with general drift-diffusion processes – Transitional dynamics in the Aiyagari model – Extensions, models with non-convexities and occupational choice.

▶ **Lecture 5 – Heterogeneous Agent Models with Aggregate Shocks (Part 1)**
The problem with aggregate shocks and the Master equation – Perturbation methods and MIT shocks – Prelude to the sequence-space approach.

▶ **Lecture 6 – Heterogeneous Agent Models with Aggregate Shocks (Part 2)**
Sequence-Space Jacobian method – Some pleasant sequence-space arithmetic in continuous time – Solving a HANK model.

Useful References

The course has been designed to be mostly self-contained, and a large collection of lecture slides will be available. Apart from a required reading list of papers, I have decided to include a non-exhaustive list of references based on both personal favorites and widely revered textbooks. I do believe you will find them illuminating and particularly helpful, not only during the course of your Ph.D. but also throughout your academic and professional career.

DYNAMIC PROGRAMMING AND STOCHASTIC CALCULUS

Dixit, A. and R. Pyndick (1994), ***Investment Under Uncertainty***, Princeton University Press.

Fleming, W. H. and H. M. Soner (2006), ***Controlled Markov Processes and Viscosity Solutions***, Springer.

Karatzas, I. and S. E. Shreve (1991), ***Brownian Motion and Stochastic Calculus***, Springer.

Oksendal, B. (2003), ***Stochastic Differential Equations: An Introduction with Applications***, Springer.

Oksendal, B. and A. Sulem (2007), ***Applied Stochastic Control of Jump Diffusions***, Universitext.

Stokey, N. (2008), ***The Economics of Inaction: Stochastic Control Models with Fixed Costs***, Princeton University Press.

NUMERICAL METHODS AND DYNAMIC PROGRAMMING

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2022), ***Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach***, Review of Economic Studies, 89(1): 45-86.

Auclert, A., B. Bard´oczy, M. Rognlie, and L. Straub (2021), ***Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models***, Econometrica, 89(5), 2375-2408.

Barles, G. and P. E. Souganidis (1991), ***Convergence of Approximation Schemes for Fully Nonlinear Second Order Equations***, Asymptotic Analysis, 4(3), 271-283.

Bilal, A. and G. Shlok (2025), ***Some Pleasant Sequence-Space Arithmetic In Continuous Time***, National Bureau of Economic Research Working Paper Series, No. 33525.

Boppart, T., P. Krusell, and K. Mitman (2018), ***Exploiting MIT Shocks in Heterogeneous-Agent Economies: The Impulse Response as a Numerical Derivative***, Journal of Economic Dynamics and Control, 89, 68-92.

Kushner, H. J. and P. Dupuis (2013), ***Numerical Methods for Stochastic Control Problems in Continuous Time***, Springer.

HETEROGENEOUS AGENT MODELS

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2022), ***Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach***, Review of Economic Studies, 89(1): 45-86.

Aiyagari, S. R. (1994), ***Uninsured Idiosyncratic Risk and Aggregate Saving***, The Quarterly Journal of Economics, 109(3), 659-684.

Bilal, A. (2023), ***Solving Heterogeneous Agent Models with the Master Equation***, National Bureau of Economic Research Working Paper Series, No. 31103.

Buera, F. J., and Y. Shin (2013), ***Financial Frictions and the Persistence of History: A Quantitative Exploration***, Journal of Political Economy, 121(2), 221-272.

Huggett, M. (1993), ***The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies***, Journal of Economic Dynamics and Control, 17(5-6), 953-969.

Kaplan, G., B. Moll, and G. Violante (2018), ***Monetary Policy According to HANK***, American Economic Review, 108(3), 697-743.

Krusell, P., and A. A. Smith (1998), ***Income and Wealth Heterogeneity in the Macroeconomy***, Journal of Political Economy, 106(5), 867-896.

Nuno, G. and B. Moll (2018), ***Social Optima in Economies with Heterogeneous Agents***, Review of Economic Dynamics, 28, 150-180.