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Voting with evaluations: When should we sum? What should we sum? *

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Abstract

Most studies of the voting literature take place in the arrovian framework, in which voters rank the available alternatives, and where Arrow's impossibility theorem prevails. I consider a different informational basis for social decisions, by allowing individuals to evaluate alternatives rather than to rank them. Voters express their opinion by assigning to each alternative an evaluation from a given set. I focus on additive rules, which follow the utilitarian paradigm. If the evaluations are numbers, the elected alternative is the one with the highest sum of evaluations. I generalize this notion to any set of evaluations, taking into account the possibility of qualitative ones. I provide an axiomatization for each of the two main additive rules: "Range Voting" when the set of evaluations is $[0, 1]$ and "Evaluative Voting" when the set of evaluations is finite.

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1 Introduction

I consider a common situation in which a group of individuals, an electorate or a committee, has to choose among several alternatives or candidates. The main feature of the model is that individuals evaluate each candidate by assigning her an evaluation, or a grade, which has to be chosen in a pre-defined set. A well-known example is the system of Approval Voting (Brams and Fishburn, 1978), in which two evaluations are available. A voter can approve or disapprove of each candidate and the winning candidate is the one with the largest number of approvals. This system is known to possess several advantages over the plurality rule (Myerson and Weber, 1993; Laslier, 2012), which is used in many countries for political elections. The goal of this article is to investigate extensions of the Approval Voting system, by allowing voters to use more than two evaluations. This means that voters can, to some extent, indicate the intensity of their preferences.

This is, I consider an information structure to take *social decisions* which allows to make interpersonal comparisons of evaluations. This contrasts with a large part of the voting literature, in which *social decisions* rest on ordinal information. In that case, any voting rule suffers from Arrow's theorem (Arrow, 1951), which asserts a general impossibility result for the aggregation of individual preferences. The broadening[1] of the information structure considered here follows the work on the relevant informational basis of *social judgments* (d'Aspremont and Gevers, 1977). As Sen (1970) argued, the introduction of individual utilities in that case allows to escape from the arrovian impossibility result. For similar reasons, Arrow's conditions are compatible in the framework of evaluation voting (Balinski and Laraki, 2011).

What rule should be used to select the winning candidate in such an election? If the evaluations are numerical grades, it seems natural to choose the candidate with the highest total

[1] The information structure is indeed richer than in the arrovian setup if the set of evaluations is large. For instance, for any vector of evaluations from $[0, 1]$, one can deduce an ordering of the alternatives, but the converse is not true. If the set evaluations is small, the information structure is just different than in the arrovian setup.

grade. If the evaluations are qualitative, for instance “A”, “B”, “C”, the previous rule may be adapted by assigning a numerical value to each evaluation. But other procedures are conceivable as well, such as comparing the candidates according to their median evaluations (Balinski and Laraki, 2011) or according to their minimal evaluations (Aleskerov et al., 2010). As many rules are possible in the context of evaluation voting, one needs to distinguish between available rules, that is why I conduct an axiomatic analysis. The method consists in characterizing a rule or a class of rules with a set of axioms, while holding the balloting procedure[2] fixed (Goodin and List, 2006).

I focus on rules which involve some summation of the evaluations. When the evaluations are numerical grades, the additive rule selects as a winning candidate the one with the highest sum of evaluations, a feature reminiscent of the utilitarian principle. This rule is called Range Voting (Smith, 2000) when the evaluations lie in $[0, 1]$, and Evaluative Voting (Hillinger, 2005) when the evaluations are the natural numbers between 1 and some number K. More generally, an additively separable rule assigns a value to each evaluation, and two candidates are compared according to the sum of their values. Thus, the value describes “what” we should sum. This allows to extend the Evaluative Voting rule to the case of qualitative evaluations.

When the evaluations are chosen in $[0, 1]$, the key property to characterize additively separable rules is an axiom of *Separability*. If a voter is indifferent between two candidates, in the sense that she assigns the same evaluation to both, the level of this evaluation should not matter for the social choice between these two candidates. This property has already been used in the theory of measurement to obtain additive structures (Krantz et al., 1971), relying on various sophisticated techniques (Debreu, 1960; Gorman, 1968; Wakker, 1989; Gonzales, 1996). By treating voters symmetrically, I provide a new proof for this result, that I illustrate with an analogy with an elementary problem of mass measurement. Then, I identify two invariance properties that single out the Range Voting rule among additively separable rules. In this respect, the approach is similar to the derivation of the utilitarian rule by Maskin (1978). This result also complements a recent characterization of the Range Voting rule that has been

[2] In my framework, the balloting procedure is described by the set of evaluations available to the voters.

obtained by Pivato (2013a) in a variable-population framework.

When the set of evaluations is finite, I provide a first axiomatization of Evaluative Voting, based on an axiom of *Compensation*. An increase in the evaluation of a voter for a candidate can be compensated by a decrease in the evaluation of another voter for the same candidate. This result differs from the axiomatization of Evaluative Voting proposed by Gaertner and Xu (2012) as it applies to any finite set of evaluations. It also generalizes a characterization of the Evaluative Voting rule that has been proposed for three evaluations (Alcantud and Laruelle, 2013). A second axiomatization is proposed with an axiom of *Independence of a Common Increase of the Evaluations*, making the link with the axiomatization of Range Voting.

In Section 2, I introduce the framework of evaluation voting systems. A simple rule ranks any two candidates by comparing the evaluations that they receive. Many well-known rules fall in this category. In Section 3, I consider the case in which evaluations are chosen in the interval $[0, 1]$. I obtain a characterization of additively separable rules and an axiomatization of the Range Voting rule. In Section 4, the evaluations lie in a finite set and I provide two simple axiomatizations of the Evaluative Voting rule. Then, in Section 5, I introduce a broader notion of aggregation procedures and characterize the class of simple rules. This step enables to state the previous axiomatizations of voting rules in this more general framework. Section 6 relates the article to the literature on scoring rules and provides a discussion on the strategic incentives faced by voters using additive rules.

2 A simple framework for evaluation voting systems

In this section, I introduce a simple framework for the analysis of aggregation rules in the evaluation voting context. The main concept is the notion of simple rule, which is used in Section 3 and Section 4, and then extended in Section 5.

2.1 Notations and definitions

The election involves a population of voters, denoted by $I = \{1, \dots, \#I\}$, that I call the electorate. This set is finite ($\#I < \infty$) and a typical voter is $i \in I$. The electorate proceeds to a vote in order to choose among a set X of candidates, or social alternatives[3]. This set is finite and a typical candidate is $x \in X$.

A voting system is composed of two elements: the balloting procedure defines the set of ballots available to the voters and the aggregation procedure describes how the ballots are aggregated into a voting outcome. I introduce formally the balloting and aggregation procedures for evaluation voting systems.

Balloting procedure Voters assign to each candidate an evaluation taken in a set Γ. This set is a language of evaluations, through which voters are allowed to express their opinion on the candidates. The set Γ is endowed with a linear order $>_\Gamma$ and with the associated order topology. If $a >_\Gamma b$, a is a better evaluation than b. Voters are free in the sense that they can assign evaluations to candidates without any restriction, for instance giving the same evaluation to several candidates. The set of ballots available to the voters is thus Γ^X. An evaluation profile is a collection of such ballots. Throughout the article, I assume that voters cast their ballot sincerely, this assumption is then discussed in Subsection 6.2.

I consider two cases: either $\Gamma = [0, 1]$ or Γ is a finite set[4]. In the first case, voters are allowed to express very finely their opinion on the candidates, but Γ is bounded. In the second case, Γ may be composed of numbers or not, in particular the evaluations may be qualitative.

Aggregation procedure As every candidate receives an evaluation vector (in Γ^I) in this election, one can compare these evaluation vectors: this idea is captured by the following

[3]For simplicity, I keep using the terms *electorate* and *candidates* throughout the paper. However, it should be clear that the scope of the analysis is large: the electorate can be any type of committee, in which members have the same power, and the elements in the set X can be any kind of social alternatives.

[4]The set $[0, 1]$ is endowed with the usual order on real numbers. If Γ is finite, the order topology is just the discrete topology. In a similar context of "judging with evaluations", Bhattacharya (2015) uses the same two sets of evaluations.

definition of a simple rule [5] .

Definition 1. *Simple rule.*
A simple (aggregation) rule is a complete and transitive binary relation $\succeq$ on the set of evaluation vectors Γ^I. I denote by $\sim$ its symmetric part and by $\succ$ its asymmetric part.

A simple rule is thus a social ranking of all possible evaluation vectors. I assume that the winning candidate at a given evaluation profile is the one with the highest ranked evaluation vector.

In this article, I focus on rules exhibiting some form of additivity, as captured in the following definition.

Definition 2. *A simple rule $\succeq$ is additively separable if there exists a continuous and increasing function $val : \Gamma \to \mathbb{R}$ such that*

$$\forall u, v \in \Gamma^I, \qquad u \succeq v \quad \Leftrightarrow \quad \sum_{i \in I} val(u_i) \geq \sum_{i \in I} val(v_i).$$

The simplest example is the additive rule, which is well-defined when the evaluations are quantitative ($\Gamma \subset \mathbb{R}$): $u \succeq v \Leftrightarrow \sum_i u_i \geq \sum_i v_i$. The notion of additive separability extends this rule, notably to the case in which the evaluations are qualitative. The function val can be interpreted as a value: $val(a)$ stands for the value of the evaluation a, and two evaluation vectors are compared according to their total values. Note that this definition is specific in two respects: the function val is assumed to be the same for every voter and it is assumed to be continuous[6].

Evaluation voting system An evaluation voting system is a couple $(\Gamma, \succeq)$, where the set of evaluations Γ describes the balloting procedure and the simple rule $\succeq$ defines the aggregation procedure.

[5] This notion corresponds to the concept of social welfare ordering in the economic theories of justice (d'Aspremont and Gevers, 2002).
[6] This last assumption is innocuous when Γ is finite.

2.2 Examples

I illustrate the previous definitions by providing several examples of evaluation voting systems.

Approval Voting (AV) This system, introduced by Brams and Fishburn (1978), is the simplest and most studied one (Laslier and Sanver, 2010). There are two evaluations: $\Gamma = \{0, 1\}$. The evaluations 1 and 0 are interpreted as an approval and a disapproval respectively. The simple rule is additive:

$$u \succeq^{AV} v \quad \Leftrightarrow \quad \sum_{i \in I} u_i \geq \sum_{i \in I} v_i. \tag{AV}$$

Thus, the candidate with the highest number of approvals wins the election. This system is an extension of the Plurality Rule. The difference is that voters have to assign evaluation 1 to exactly one candidate under Plurality Rule, whereas this constraint is removed with Approval Voting.

As voters cannot express any preference intensity under Approval Voting, the elected candidate may be far from being the utilitarian optimal one. To improve upon Approval Voting, the two following examples provide extensions in which more than two evaluations are available.

Range Voting (RV) In this system, proposed by Smith (2000), the set of evaluations is $\Gamma = [0, 1]$. The Range Voting rule is defined by

$$u \succeq^{RV} v \quad \Leftrightarrow \quad \sum_{i \in I} u_i \geq \sum_{i \in I} v_i. \tag{RV}$$

This additive rule formally follows the paradigm of utilitarianism from the theories of justice (Blackorby et al., 2002), according to which the best social alternative is the one which maximizes the total sum of utilities in the society. The difference between the Range Voting rule and the utilitarian social welfare ordering is the domain on which they are defined. On the one hand, evaluations lie in the interval $[0, 1]$ for the Range Voting rule. On the other hand, utilities can take any value in $\mathbb{R}$ for utilitarianism. This formal difference is of practical importance:

since the evaluations are bounded, by 0 below and 1 above, a single voter cannot dictate her choice, for instance by assigning a very high evaluation to her preferred candidate[7].

Evaluative Voting (EV) The system of Evaluative Voting, advocated by Hillinger (2005), can be first defined for $\Gamma = \{1, \dots, K\}$, with K being a natural number. The simple rule is

$$u \succeq^{EV} v \quad \Leftrightarrow \quad \sum_{i \in I} u_i \geq \sum_{i \in I} v_i. \tag{EV}$$

As for Range Voting, this additive rule follows the utilitarian doctrine. More generally, Evaluative Voting (EV) can be defined for any finite set of evaluations Γ, we note $K = \#\Gamma$. Let us define the evaluative value as the function

$$val^{EV} : \left| \begin{array}{lcl} \Gamma & \to & \{1, \dots, K\} \\ a & \mapsto & val^{EV}(a) = \#\{b \in \Gamma | a \geq_\Gamma b\}. \end{array} \right.$$

This function mimics the Borda score defined for ordinal rankings : it associates the value 1 to the worst evaluation in Γ, the value 2 to the second worst evaluation in Γ, ..., and the value K to the best evaluation in Γ. Then, the Evaluative Voting rule is the additively separable rule defined by

$$u \succeq^{EV} v \quad \Leftrightarrow \quad \sum_{i \in I} val^{EV}(u_i) \geq \sum_{i \in I} val^{EV}(v_i). \tag{EV}$$

The Evaluative Voting system can be viewed a generalization of the Borda voting system (Young, 1974) when $\#\Gamma = \#X$. Indeed, the only difference between the two in that case rests on their balloting procedures. In the Borda voting system, a voter assigns a rank (between 1 and $\#X = K$) to each candidate, this is equivalent to assigning each evaluation to exactly one candidate. This constraint is removed under Evaluative Voting, because a given evaluation can by assigned to several candidates. In both cases, the aggregation procedure is the same: each evaluation (resp. rank) is assigned a value (resp. a score), such that the difference in value (resp. score) between two consecutive evaluations (resp. ranks) is constant. Then, candidates

[7]This property has already been emphasized by Pivato (2013a), who refers to it as the *absence of minority overrides*.

are compared according to their total values (resp. total scores).

Threshold aggregation (TA) This system follows the leximin principle from the theories of justice (d'Aspremont and Gevers, 2002). It has been studied by Aleskerov et al. (2010) for a finite set of evaluations, but it can be defined for any set Γ.

For any vector $u \in \Gamma^I$, let $\tilde{u}$ be the non-decreasing vector of Γ^I which can be obtained by a permutation of vector u. Then, the Threshold Aggregation rule is defined by

$$u \succeq^{TA} v \quad \Leftrightarrow \quad \tilde{u} \geq_{Lex} \tilde{v}, \tag{TA}$$

where $\geq_{Lex}$ denotes the lexicographic ordering between two vectors. The interpretation is that the aggregation procedure gives priority to the worst-off voter in each evaluation vector. As it has been argued (Aleskerov et al., 2010), this procedure provides a fairly good description of editors' acceptance decisions for submissions in peer-reviewed journals, as based on peers evaluations. In that case, the set of evaluations is {Reject, Major Revision, Minor Revision, Accept}. We observe that the Threshold Aggregation rule is additively separable when Γ is finite[8], but it is not for $\Gamma = [0, 1]$.

Majority Judgment (MJ) This system, introduced by Balinski and Laraki (2011), uses the set of evaluations Γ = {Excellent, Very Good, Good, Fair, Poor, Very Poor, To Reject} with the order Excellent$>_\Gamma$Very Good$>_\Gamma$Good$>_\Gamma$Fair$>_\Gamma$Poor$>_\Gamma$Very Poor$>_\Gamma$To Reject. The aggregation procedure focuses on the median of the evaluation vector. If the lower median of vector u is attained by some individual $i \in I$, the majority-value of u is the vector $MV(u)$

[8]For $\Gamma = \{1, \ldots, K\}$, a value is given by the function

$$val^{TA} : \left| \begin{array}{lll} \{1, \ldots, K\} & \rightarrow & \mathbb{R} \\ k & \mapsto & val^{TA}(k) = \sum_{p=1}^{k} \frac{1}{(\#I)^p}. \end{array} \right.$$

The intuition is that the difference in value between two consecutive evaluations should decrease sharply as evaluations increase. Then, a single low evaluation in a vector renders the total value of the vector low too.

defined inductively by $MV(u) = \big(u_i, MV(u_{-i})\big)$. Then the rule is

$$u \succeq^{MJ} v \quad \Leftrightarrow \quad MV(u) \geq_{Lex} MV(v). \tag{MJ}$$

This rule is used in various fields, ranging from skate competitions to wine tasting (Balinski and Laraki, 2011).

3 Characterizing the Range Voting rule

In this section, I focus on the case in which $\Gamma = [0, 1]$. However, most of the axioms remain well-defined for any set of evaluations.

3.1 Characterization of additively separable rules

I introduce four natural axioms which are satisfied by any additively separable rule. In the next definition, Σ_I is the set of all permutations on the set I. For a permutation $\sigma \in \Sigma_I$ and an evaluation vector $u \in \Gamma^I$, the vector u_σ is defined by $\forall i \in I,\ (u_\sigma)_i = u_{\sigma(i)}$.

Axiom 1. *Anonymity (A)*
For any vectors $u, v \in \Gamma^I$ and permutation $\sigma \in \Sigma_I$, $\quad u \succeq v \quad \Rightarrow \quad u_\sigma \succeq v_\sigma.$

This axiom states that the ranking between two evaluation vectors is not affected by a permutation of the voters. It constitutes a natural requirement of fairness among voters.

Axiom 2. *Strong Pareto (SP)*
For any vectors $u, v \in \Gamma^I$,
$$\begin{cases} \forall i \in I, & u_i \geq_\Gamma v_i \\ \exists j \in I, & u_j >_\Gamma v_j \end{cases} \quad \Rightarrow \quad u \succ v.$$

This axiom conveys a classical notion of efficiency of the rule. An other interpretation is that it implies the positive responsiveness (May, 1952) of the voting system: when a voter increases her evaluation in favor of a candidate, this change is beneficial for this candidate.

Axiom 3. *Continuity (Cont)*

For any vector $u \in \Gamma^I$*, the sets* $\{v \in \Gamma^I | v \succeq u\}$ *and* $\{v \in \Gamma^I | u \succeq v\}$ *are closed with respect to the product topology on* Γ^I*.*

This property expresses the rather natural view that small changes in the inputs of the rule, the evaluations, should not lead to dramatic changes in the output of the rule, the ranking between candidates' evaluation vectors.

One important feature of additively separable rules is the treatment of indifferent voters. A voter is said to be indifferent between two candidates if she assigns the same evaluation to both. The next axiom introduces a notion of subsidiarity for the rule: to compare two candidates, we do not need to take into account the evaluations assigned by indifferent voters.

Axiom 4. *Separability (Sep)*

For any vectors $u, v, u', v' \in \Gamma^I$ *and subset of voters* $J \subseteq I$*,*

$$\begin{cases} \forall j \in J, & u_j = v_j \quad \text{and} \quad u'_j = v'_j \\ \forall i \in I \backslash J, & u_i = u'_i \quad \text{and} \quad v_i = v'_i \end{cases}, \qquad u \succeq v \quad \Leftrightarrow \quad u' \succeq v'.$$

In this axiom, voters in J are indifferent between u and v, and also between u' and v'. A violation of this axiom is a rather odd situation in which the evaluations given by indifferent voters are decisive for the social ranking, as we observe below with the system of Majority Judgment.

Counter-example of (Sep) with the system of Majority Judgment Consider the following evaluation vectors u, v, u', v', with three voters $1, 2, 3$:

	u	v
1	Good	Very Good
2	Poor	Very Poor
3	Excellent	Excellent

	u'	v'
1	Good	Very Good
2	Poor	Very Poor
3	To Reject	To Reject

In the first situation, we have $u \prec^{MJ} v$, as v has a better median evaluation than u (Very Good $>_\Gamma$ Good). However, we have $u' \succ^{MJ} v'$ in the second situation as Poor (median of u')

dominates Very Poor (median of v'). With this rule, the evaluation of an indifferent voter (here voter 3 is indifferent between u and v and between u' and v') is decisive for the social ranking between two candidates (note that voters 1 and 2 give the same evaluations in vectors u, u' and in vectors v, v'). Indeed, when voter 3 assigns Excellent to both candidates, v is preferred, whereas u' is preferred when voter 3 assigns To Reject to both of them.

The following result provides a characterization of additively separable rules.

Theorem 1. *Let $\Gamma = [0,1]$ and assume $\#I \geq 3$. A simple rule $\succeq$ satisfies the axioms (A), (SP), (Cont) and (Sep) if and only if it is additively separable. Moreover, the four axioms are mutually independent.*

The axiom (Sep), also known as "elimination of indifferent voters", has already been used as a decisive tool in the theory of additive conjoint measurement, which aims at providing conditions under which a binary relation on a cartesian product admits an additively separable representation. Economists have been interested in this theory in its early developments. Debreu (1960) and Gorman (1968) studied the existence of an additively separable utility representation of individual preferences over bundles of different goods. This theory also provided a characterization of the class of generalized utilitarianism rules in the theories of justice (Blackorby et al., 2002) and is used extensively in the area of multi-criteria decision making (Bouyssou and Pirlot, 2005).

The proofs of the existence of an additively separable representation led to various mathematical developments (Gonzales, 1996): an algebraic approach (Krantz et al., 1971), several topological approaches (Debreu, 1960; Wakker, 1989) and an approach with functional equations (Aczél, 1966; Gorman, 1968).

Although the result of Theorem 1 could have been deduced from these works, I choose to provide a new proof[9] in the present paper, adapted to the current model, and in particular to the set of evaluations $\Gamma = [0,1]$. The proof is constructive (the function *val* is explicitly defined) and follows the topological approach of Wakker (1989). I exploit the symmetry of the model given by axiom (A) to obtain a uni-dimensional proof: there is only one function *val* to

[9] The proof only works for $\#I \geq 4$.

construct here, whereas one needs to find a function val^i for each voter i when axiom (A) is dropped, as previous works did. I provide in the next section a sketch of the proof, which is illustrated with an analogy with the measurement of mass.

3.2 An analogy with an elementary problem of mass measurement

An analogy. Let us consider the following simple rule defined on pairs of evaluations[10]:

$$\forall a < b < c < d \in [0,1], \qquad (a,d) \succeq (b,c) \quad \Leftrightarrow \quad val(a) + val(d) \geq val(b) + val(c)$$

This is an additively separable rule, the value is the function $val : [0,1] \to [0,1]$.

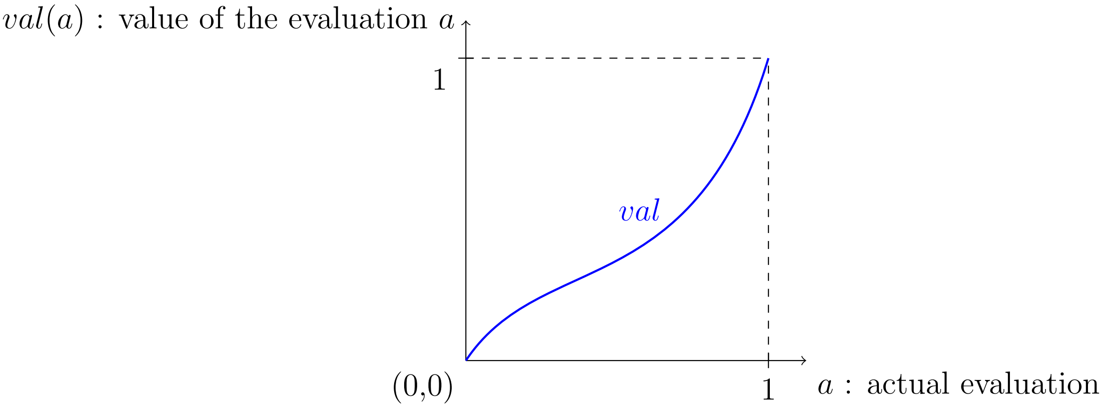

Figure 1: Function val for the translation of evaluations.

Let us assume that we can compare pairs of evaluations according to this rule, but that the function val is unknown. Can we recover the value function from the rule $\succeq$? I prove in the sequel that this is possible, by providing an analogy between the previous measurement structure and the measurement of mass.

Let us consider a stick of heterogeneous mass density, one extremity of the stick has coordinate 0, whereas the other has coordinate 1. The total mass of the stick is normalized to 1. The density of mass in the stick has a cumulative distribution F, assumed to be continuous and

[10] Although it is assumed in the proof of Theorem 1 that $\#I \geq 4$, we can restrict our attention for this part of the proof to comparisons between pairs of evaluations. This is made possible by axiom (Sep): if there are four voters, among which two are indifferent, the rule allows to compare pairs of evaluations.

increasing. This function translates the coordinate on the stick a to the mass of the portion $(0, a)$ of the stick (Figure 2).

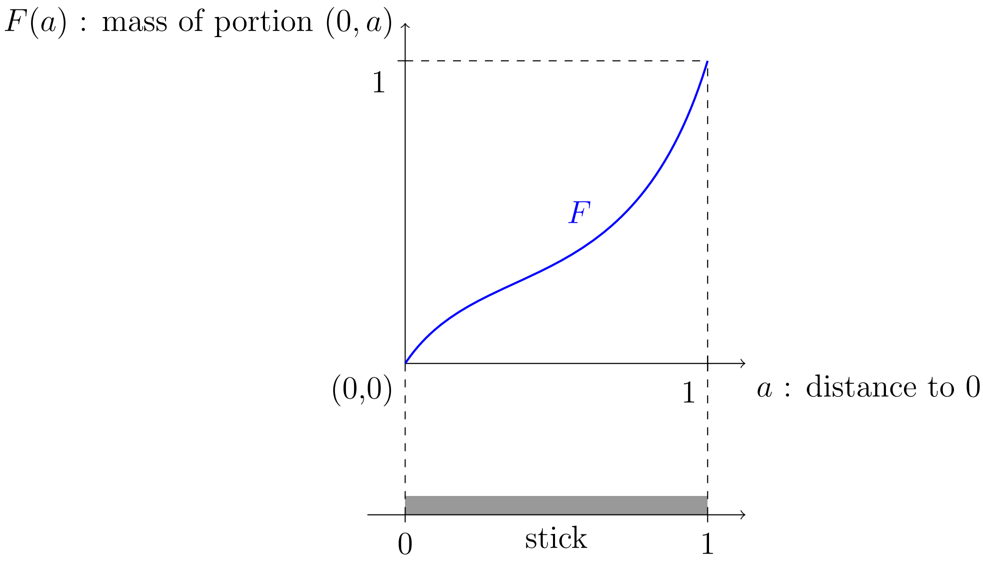

Figure 2: Function F for the measurement of mass.

For any coordinates $a < b$, let $Mass(a, b)$ denote the mass of the portion of the stick between a and b. Since $Mass(a, b) = F(b) - F(a)$, we have

$$\begin{aligned} \forall a < b < c < d \in [0, 1], \qquad Mass(c, d) \geq Mass(a, b) \quad &\Leftrightarrow \quad F(d) - F(c) \geq F(b) - F(a) \\ &\Leftrightarrow \quad F(a) + F(d) \geq F(b) + F(c) \end{aligned}$$

The conclusion is that we have a formal analogy between the measurement of mass and the social measurement of evaluations. Indeed, if $F = val$, we have the following equivalence:

$$\forall a < b < c < d \in [0, 1], \qquad (a, d) \succeq (b, c) \quad \Leftrightarrow \quad Mass(c, d) \geq Mass(a, b).$$

The interpretation is that the pair of evaluations (a, d) dominates the pair of evaluations (b, c) if the advantage given by d over c is greater than the one given by b over a. In the sequel, I assume that F and val are not known, but that we can compare the mass of two portions of the stick and that we can rank pairs of evaluations. The point is that describing a way to discover F in the first problem enables to elicit val in the second problem.

Identification of the density of mass inside the stick The process to discover F is very simple as long as we are endowed with a weighing scale (triangle in gray on Figure 3). First, we already know that the portions $(0,0)$ and $(0,1)$ of the stick have masses 0 and 1 respectively. It follows that $F(0) = 0$ and $F(1) = 1$, implying in turn that $F^{-1}(0) = 0$ and $F^{-1}(1) = 1$. Then, we know that $F^{-1}(\frac{1}{2})$ is the unique number $z \in (0,1)$ such that the mass between 0 and z is the same as the mass between z and 1. By iteration, we can similarly obtain $F^{-1}(\frac{1}{4})$ and $F^{-1}(\frac{3}{4})$, as illustrated on Figure 3 (on each line, the portions in blue and in red have the same mass).

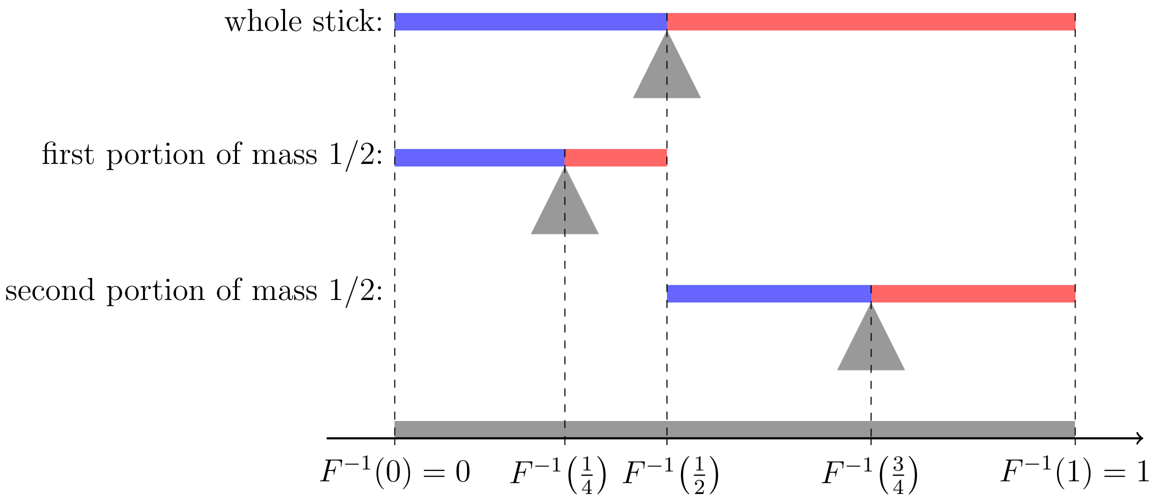

Figure 3: Identifying F^{-1} on the dyadic numbers $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$.

Finally, we obtain $F^{-1}(d)$ for any dyadic number[11] d between 0 and 1. Because F^{-1} is continuous, this defines uniquely F^{-1} on $[0,1]$, and we can finally recover F. The exactly same process allows to identify the function val^{-1} on $[0,1]$, and then to find the function val. In the proof, I first show that we can similarly construct a function val, under the assumption that the rule $\succeq$ satisfies (A), (SP), (Cont) and (Sep). Then, I prove that the rule $\succeq$ is indeed represented[12] by the function $VAL(u) = \sum_{i \in I} val(u_i)$.

[11] A dyadic number is a number d of the form $d = \frac{k}{2^p}$, with $k, p \in \mathbb{N}$.

[12] For instance, if we note $a = F^{-1}(\frac{1}{4})$, $b = F^{-1}(\frac{1}{2})$ and $c = F^{-1}(\frac{3}{4})$, we observe that $2F(b) = F(a) + F(c)$. We can prove that $Mass(a,b) = Mass(b,c)$, using the relations that helped us constructing (a,b,c) : $Mass(0,a) = Mass(a,b)$, $Mass(b,c) = Mass(c,1)$ and $Mass(0,b) = Mass(b,1)$. This argument is the "first reasoning" illustrated in the proof on Figure 8.

3.3 An axiomatization of Range Voting

I elicited in the previous section a set of axioms, necessary and sufficient for a rule to be additively separable when the set of evaluations is $[0, 1]$. A next question arises : under which additional conditions should the function val be the identity? In other words, when should we sum exactly the evaluations?

In the case of social welfare orderings, Maskin (1978) shows that the axiom of Cardinal Full Comparability[13] allows to characterize the utilitarian rule among the class of generalized utilitarian rules. Following this approach, I provide in this section two axioms, that are specific to the set of evaluations $[0, 1]$, allowing to single out the additive rule among the class of additively separable rules.

Axiom 5. *Contraction Invariance (CI)*

For any vectors $u, v \in [0, 1]^I$ *and scalar* $\alpha \in (0, 1)$, $\quad u \sim v \quad \Rightarrow \quad \alpha \cdot u \sim \alpha \cdot v.$

According to this axiom[14], if two candidates are equally ranked by the rule, they should remain equally ranked after a contraction of all their evaluations by the same scalar $\alpha \in (0, 1)$. In the following definition, the vector $\mathbb{1}$ is defined by $\forall i \in I,\ \mathbb{1}_i = 1$.

Axiom 6. *Symmetry Invariance (SI)*

For any vectors $u, v \in [0, 1]^I$, $\quad u \sim v \quad \Rightarrow \quad \mathbb{1} - u \sim \mathbb{1} - v.$

This axiom expresses the symmetry of the rule with respect to the set of evaluations $[0, 1]$. With this set, the minimal evaluation that a voter can assign is 0 and the maximal one is 1. Thus, if a candidate x receives an evaluation a from a voter, this evaluation can be interpreted either as vote of a in favor of x or as a vote of $(1-a)$ against x. According to this interpretation, the axiom (SI) requires the rule to treat symmetrically the votes in favor of a candidate and the votes against her[15]. This axiom is reminiscent of the property of self-duality in the literature on the bankruptcy problem (Thomson, 2003).

[13] $\forall u, v, u', v' \in \mathbb{R}^I$ such that $\exists \alpha > 0, \exists \beta \in \mathbb{R}$ with $u' = \alpha \cdot u + \beta \cdot \mathbb{1}$, $v' = \alpha \cdot v + \beta \cdot \mathbb{1}$, $\quad u \succeq v \ \Leftrightarrow\ u' \succeq v'$.

[14] In the literature on social welfare orderings, a similar property is captured by the axiom of Independence of the Common Utility Scale (Moulin, 1991).

[15] If u is equivalent to v in favor of a candidate, then (SI) implies that u is equivalent to v against a candidate, this means that $\mathbb{1} - v$ is equivalent to $\mathbb{1} - u$ in favor of a candidate.

Theorem 2. *Let $\Gamma = [0,1]$ and assume that $\#I \geq 3$. A simple rule $\succeq$ satisfies the axioms (A), (SP), (Cont), (Sep), (CI) and (SI) if and only if it is the Range Voting rule $\succeq^{RV}$. Moreover, the six axioms are mutually independent.*

This axiomatization of the Range Voting rule differs from the one provided in Pivato (2013a). In that article, Range Voting is characterized up to an equivalence relation between rules, and this equivalence class corresponds to the class of additively separable rules in my framework. Thus, Theorem 2 goes beyond, as it explicitly takes into account the set of evaluations $[0,1]$ and describes invariance properties satisfied by the Range Voting rule, which are encoded in the axioms (CI) and (SI).

4 From qualitative to quantitative evaluations: Evaluative Voting

In this section, Γ is a finite set of evaluations, of cardinality $\#\Gamma = K$. This setting encompasses in particular the case of qualitative evaluations, in which a natural question appears: should we treat these qualitative evaluations as quantitative ones and can we sum them? I discuss the notion of additive separability in this setting and provide two axiomatizations of Evaluative Voting.

4.1 Additively separable rules

The first result of this section extends Theorem 1 only to the case with no more than three evaluations. It is adapted from Sertel and Slinko (2007).

Proposition 1. *Assume that $\#\Gamma \leq 3$. A simple rule satisfies axioms (A), (SP) and (Sep) if and only if it is additively separable.*

We observe that the axioms (A), (SP) and (Sep) are not sufficient to obtain additive separability with more than three evaluations. In the following example, adapted from Sertel and Slinko (2007), 11 stands for the vector $(1,1)$.

Counter-example with four evaluations Consider the following rule, for $\Gamma = \{1, 2, 3, 4\}$, and $\#I = 2$: $11 \prec 12 \sim 21 \prec 13 \sim 31 \prec 22 \prec 23 \sim 32 \prec 14 \sim 41 \prec 24 \sim 42 \prec 33 \prec 34 \sim 43 \prec 44$. This rule satisfies (A), (SP) and (Sep), but is not additively separable, since we have

$$13 \prec 22$$
$$23 \prec 14$$
$$24 \prec 33.$$

If the rule was additively separable, we would be able to sum the values of the vectors on each column and obtain $val(1)+2val(2)+2val(3)+val(4) < val(1)+2val(2)+2val(3)+val(4)$, hence a contradiction.

This example is not specific to the choice of $\#I$ or $\#\Gamma$ and many such counter-examples can be found. The literature on conjoint measurement for finite structures (Krantz et al., 1971) identifies a family of axioms extending the property (Sep), called the "cancellation conditions". These axioms prevent this type of "cycle" to occur, and guarantee together the existence of an additive representation. On the one hand, one needs an infinite number of these axioms to obtain additive separability in all generality, on the other hand a finite number of these conditions is sufficient for fixed $\#I$ and $\#\Gamma$ (Fishburn, 1997). In the sequel, I do not refer to this literature and I provide a direct characterization of Evaluative Voting.

4.2 Characterizing the Evaluative Voting rule

If a is an evaluation in Γ, I define the successor of a, $succ(a)$, as the lowest evaluation above a and the predecessor of a, $pred(a)$, as the highest evaluation below a, when they exist. For instance, if $\Gamma = \{A, B, C\}$, with $A >_\Gamma B >_\Gamma C$, we have: $succ(B) = A$ and $pred(B) = C$.

Axiom 7. *Compensation (Comp)*

$$\text{For any vectors } u, u' \in \Gamma^I, \quad \begin{cases} \exists i \in I, & u'_i = succ(u_i) \\ \exists j \in I, & u'_j = pred(u_j) \\ \forall k \neq i, j, & u'_k = u_k \end{cases} \quad \Rightarrow \quad u' \sim u.$$

This axiom states that if a voter i increases her evaluation for a candidate, by assigning the evaluation just above, whereas another voter j decreases her evaluation for the same candidate by assigning the evaluation just below, both changes compensate and the relative position of the candidate's evaluation vector in the social ranking remains the same.

This axiom plays the same role as the axiom of Cancellation, introduced in Young (1974) to characterize the Borda rule. Indeed, it implies that the rule must be anonymous and that the difference in value between two consecutive evaluations must be constant, if the rule is additively separable. This axiom also appears as a natural generalization of the axiom of Compensation introduced in Alcantud and Laruelle (2013) to characterize the Evaluative Voting rule with three evaluations[16]. I obtain a first axiomatization of the Evaluative Voting rule.

Theorem 3. *Let Γ be a finite set. A simple rule $\succeq$ satisfies axioms (SP) and (Comp) if and only if it is the Evaluative Voting rule $\succeq^{EV}$.*

This result is also in line with the axiomatization of Evaluative Voting in Gaertner and Xu (2012), with the axiom (SP) and an axiom of Cancellation Independence, which plays the same role as the axiom (Comp) here. One advantage of Theorem 3 is that it makes no assumption on the structure of the set Γ, in particular the evaluations may be qualitative.

I now propose another axiom, which does not include any requirement of anonymity, and which permits to make the link with the analysis performed in the previous section.

Axiom 8. *Independence of a Common Increase of the Evaluations (ICIE)*
For any vectors $u, v, u', v' \in \Gamma^I$ such that $\begin{cases} \exists i \in I, \quad u'_i = succ(u_i), \quad v'_i = succ(v_i) \\ \forall j \neq i, \quad u'_j = u_j, \qquad v'_j = v_j \end{cases}$, ,

$$u \succeq v \quad \Leftrightarrow \quad u' \succeq v'.$$

This axiom requires that the ranking between two evaluation vectors u and v remains

[16]The axiom of Compensation in Alcantud and Laruelle (2013) is in fact slightly weaker as it applies only to vectors u such that $\#\{i \in I | u_i = \min \Gamma\} = \#\{i \in I | u_i = \max \Gamma\}$, or equivalently $\frac{1}{\# I} \sum_{i \in I} val^{EV}(u_i) = 2$. In that article, the requirement is then obtained on all vectors $u \in \Gamma^I$ with the axiom of Consistency. In my framework, it would have been possible to introduce the same weakening of axiom (Comp), and then to obtain the full axiom (Comp) with the axiom (Sep).

unaffected if a voter similarly increases her evaluations, by assigning the evaluations just above, in both vectors. The axiom (ICIE) appears as a strengthening of the axiom (Sep): indeed (Sep) requires the previous property to hold only for vectors (u, v) such that $u_i = v_i$. This is, (Sep) requires the rule to be independent of a common increase of the evaluations, only if this common increase is made by an indifferent voter.

Proposition 2. *Let Γ be a finite set. If a simple rule $\succeq$ satisfies axioms (A) and (ICIE), then it satisfies the axiom (Comp).*

As a corollary of Theorem 3 and Proposition 2, we finally obtain the following axiomatization of the Evaluative Voting rule.

Theorem 4. *Let Γ be a finite set. A simple rule $\succeq$ satisfies axioms (A), (SP) and (ICIE) if and only if it is the Evaluative Voting rule $\succeq^{EV}$. Moreover, the three axioms are mutually independent.*

This result can be viewed as a discrete analog of a characterization of the utilitarian social welfare ordering in Moulin (1991), in which the axiom of Zero Independence[17] plays a similar role as the axiom (ICIE) here.

5 A general framework for evaluation voting systems

I assumed in the previous sections that the aggregation procedure consists in binary comparisons of the evaluation vectors obtained by the candidates. In this section, I extend the notion of aggregation rule and discuss the previous assumption. This finally leads to characterization theorems in a more general context.

5.1 General rules

As I am concerned with the choice of a candidate by the electorate, I define a general rule as a function mapping each evaluation profile to a nonempty subset of winning candidates[18]. An

[17] $\forall u, v, w \in \mathbb{R}^I, \quad u \succeq v \quad \Leftrightarrow \quad u + w \succeq v + w.$

[18] My concern is to elect only one candidate, but this definition takes into account the possibility of a tie between some candidates. If a tie occurs, the complete description of the social choice rule should describe how

evaluation profile is a matrix $m = (m_i^x)_{i\in I}^{x\in X}$, with $m_i \in \Gamma^X$ being the ballot cast by voter i and $m^x \in \Gamma^I$ being the evaluation vector received by candidate x. I denote by $M = \Gamma^{(X\times I)}$ the set of all evaluation profiles.

Definition 3. *General rule.*

A general (aggregation) rule is a correspondence $\Phi : M \rightrightarrows X$, such that for any profile $m \in M$, $\Phi(m) \neq \emptyset$.

This notion of general rule is reminiscent of the concept of Social Welfare Functional, or SWFL (d'Aspremont and Gevers, 2002), the difference being that the outcome is a choice (a subset of candidates) in the voting context, rather than a ranking of the candidates. The following definition makes the link between the notions of simple and general rules.

Definition 4. *A general rule Φ is represented by a simple rule $\succeq$ if:*

$$\forall m \in M, \qquad \Phi(m) = \{x \in X \mid \forall y \in X, m^x \succeq m^y\}.$$

It follows that every simple rule induces a general rule, and we shall see that the converse is not true. If a general rule Φ is represented by a simple rule, the simple rule provides a much more compact description of the aggregation procedure than Φ.

5.2 Examples

Range Voting and Evaluative Voting. The general rule Φ^{RV} is the rule represented by $\succeq^{RV}$ for $\Gamma = [0, 1]$. This means that under Range Voting, the elected candidate is the one with the highest total evaluation. Similarly, for Γ finite, Φ^{EV} is the rule represented by $\succeq^{EV}$.

it is broken. Fair coin tossing is one possibility.

Relative Utilitarianism. For $\Gamma = [0,1]$, the general rule following formally[19] the principle of relative utilitarianism (Dhillon and Mertens, 1999) is defined[20] by

$$\forall m \in M, \quad \Phi^{RU}(m) = \arg\max_{x \in X} \sum_{i \in I} val^x(m_i), \qquad \text{with } val^x(m_i) = \frac{m_i^x - \min_y m_i^y}{\max_y m_i^y - \min_y m_i^y}. \quad \text{(RU)}$$

This rule associates a value to each evaluation m_i^x, which depends on the evaluations assigned by voter i to the other candidates, so that i's preferred candidate receives the value 1 and i's least preferred candidate receives the value 0.

5.3 From binary comparisons to choice procedures : a representation theorem

I introduce two axioms which are satisfied when a general rule is represented by a simple rule.

Axiom 9. *Pareto Indifference (PI*)*

$$\forall m \in M, \forall x, y \in X, \qquad \text{if} \quad m^x = m^y, \quad \text{then} \quad \Big(x \in \Phi(m) \Leftrightarrow y \in \Phi(m)\Big).$$

This axiom states that if two candidates have the same evaluation vector at a given profile, then either they are both in the set of winning candidates or no one is winning.

Axiom 10. *Binary Independence of Irrelevant Candidates (BIIC*)*

$$\forall m, n \in M, \forall x, y \in X, \qquad \text{if} \begin{cases} n^x = m^x \\ n^y = m^y \end{cases}, \text{ then} \left(\begin{cases} x \in \Phi(m) \\ y \in \Phi(n) \end{cases} \Rightarrow y \in \Phi(m) \right).$$

This is, if each of the two candidates x and y receives the same evaluation vector in profiles m and n, then if one candidate is chosen in one profile and the other is chosen in the other profile,

[19] The rule Φ^{RU} defined here is introduced as a benchmark, rather than a desirable rule. Whereas it is legitimate to re-scale individual utilities when there exists no common utility scale, it seems not necessarily appropriate to re-scale the evaluations here, as the voters have a common scale of evaluations Γ.

[20] I take by convention $\frac{0}{0} = 0$.

both candidates should be chosen in both profiles. The idea is that a change in the evaluations obtained by the other candidates should not change the revealed preference between m^x and m^y. Indeed, m^x is revealed to be preferred to m^y in profile m, as $x \in \Phi(m)$; and $m^y = n^y$ is revealed to be preferred to $m^x = n^x$ in profile n, as $y \in \Phi(n)$.

Both axioms (PI*) and (BIIC*) are particularly mild in the voting context, as they appear as natural requirements of consistency of the aggregation procedure. The following result shows that they are nevertheless sufficient to characterize the class of simple rules.

Theorem 5. *Let $\#X \geq 3$. A general rule Φ satisfies axioms (PI*) and (BIIC*) if and only if it is represented by a simple rule $\succeq$.*

This result[21] is similar to the welfarist lemma of the theories of justice (d'Aspremont and Gevers, 1977), for the representation of a Social Welfare Functional by a Social Welfare Ordering.

The two axioms are independent. A constant general rule, electing always the same candidate, satisfies (BIIC*) but not (PI*). In turn, the rule Φ^{RU} satisfies (PI*) but not (BIIC*), as the value of an evaluation given by a voter depends on the evaluations assigned by the same voter to the other candidates.

5.4 Axiomatizations of voting systems in the general framework

In this section, I provide an adaptation of the axioms previously introduced to the framework of general rules.

Axiom 11. *Anonymity (A*)*

For any profile $m \in M$ and permutation $\sigma \in \Sigma_I$, $\quad \Phi(m_\sigma) = \Phi(m)$.

Axiom 12. *Strong Pareto (SP*)*

For any profile $m \in M$ and candidate $x \in X$, $\left(\forall z \neq x, \begin{cases} \forall i \in I, & m_i^z \succeq m_i^x \\ \exists j \in I, & m_j^z \succ m_j^x \end{cases} \right) \Rightarrow x \notin \Phi(m).$

[21]It appears that the result is in fact slightly more general than Theorem 5 : we do not use in the proof the structure of the set of evaluations vectors Γ^I, the result would still hold if the attributes of the candidates $(m^x)_{x \in X}$ lie in any abstract set.

This axiom states that a candidate should not be elected if her evaluation vector is (strictly) Pareto-dominated by the evaluation vectors of all other candidates.

Axiom 13. *Continuity (Cont*)*

For any sequence $(m^{(p)})_{p\in\mathbb{N}} \in M^{\mathbb{N}}$, $\begin{cases} m^{(p)} \to_{p\to\infty} m \\ \forall p \in \mathbb{N},\ x \in \Phi(m^{(p)}) \end{cases} \quad \Rightarrow \quad x \in \Phi(m).$

This is, if the candidate x is chosen in every evaluation profile $m^{(p)}$, then x should be chosen in the limit profile m. This definition corresponds to the upper hemicontinuity[22] of the correspondence Φ. We observe that some of the previous axioms imply together the property of *Pareto Indifference*.

Proposition 3. *Let* $\Gamma = [0, 1]$*. If a general rule* Φ *satisfies axioms (BIIC*), (SP*) and (Cont*), then it satisfies (PI*).*

Axiom 14. *Separability (Sep*)*

For any profiles $m, n \in M$,

$$\text{if} \quad \exists J \subseteq I, \begin{cases} \forall i \in J, \forall x, y \in X, & m_i^x = m_i^y, \quad n_i^x = n_i^y \\ \forall i \in I \backslash J, & m_i = n_i \end{cases}, \ \text{then} \quad \Phi(m) = \Phi(n).$$

Under axiom (Sep*), a voter has no influence on the social decision if she is indifferent between all candidates. In the next two axioms, the set of evaluations is $[0, 1]$.

Axiom 15. *Contraction Invariance (CI*)*

For any profile $m \in M$*, candidate* $x \in X$ *and scalar* $\alpha \in (0, 1)$*,*

$$\Phi(m) = X \quad \Rightarrow \quad \Phi(\alpha \cdot m) = X.$$

Axiom 16. *Symmetry Invariance (SI*)*

For any profile $m \in M$*,* $\qquad \Phi(m) = X \quad \Rightarrow \quad \Phi(\mathbb{1} - m) = X.$

[22] For instance, if the general rule Φ is derived from a continuous preference relation on Γ^I, the correspondence is upper hemicontinuous. For a comprehensive survey of this notion, see Ok (2007).

The axioms (CI*) and (SI*) impose weak invariance requirements: if all the candidates are chosen in one profile, then they should also be all chosen in the modified profile. The following axiom applies only when Γ is finite.

Axiom 17. *Independence of a Common Increase of the Evaluations (ICIE*)*
For any profiles $m, n \in M$ and voter $i \in I$,

$$\text{if} \quad \begin{cases} \forall x \in X, & n_i^x = succ(m_i^x) \\ \forall j \neq i, & n_j = m_j \end{cases}, \qquad \text{then} \quad \Phi(m) = \Phi(n).$$

This means that if one voter increases her evaluation for each candidate, by assigning the evaluation just above, the social choice does not change.

It should be clear that when a general rule Φ is represented by a simple rule $\succeq$, the rule $\succeq$ satisfies one of the axioms introduced in this article if and only if the rule Φ satisfies its starred counterpart. The axiomatizations of the Range Voting and Evaluative Voting rules are thus extended to the general framework. These results are direct corollaries of Theorem 1, Theorem 2, Theorem 4, Theorem 5 and Proposition 3.

Corollary 1. *Let $\Gamma = [0, 1]$ and assume $\#I \geq 3$ and $\#X \geq 3$. A general rule Φ satisfies axioms (BIIC*), (A*), (SP*), (Cont*) and (Sep*) if and only if it is represented by an additively separable simple rule. Moreover, the five axioms are mutually independent.*

Corollary 2. *Let $\Gamma = [0, 1]$ and assume $\#I \geq 3$ and $\#X \geq 3$. A general rule Φ satisfies axioms (BIIC*), (A*), (SP*), (Cont*), (Sep*), (CI*) and (SI*) if and only if it is the Range Voting rule. Moreover, the seven axioms are mutually independent.*

Corollary 3. *Let Γ be a finite set and assume $\#X \geq 3$. A general rule Φ satisfies axioms (PI*), (BIIC*), (A*), (SP*) and (ICIE*) if and only if it is the Evaluative Voting rule. Moreover, the five axioms are mutually independent.*

6 Discussion

6.1 Relationship with the literature on scoring rules

I focused in this article on additively separable voting rules under evaluation balloting. This topic has already been studied in a variable-population framework, notably in two recent works (Pivato, 2013b; Balinski and Laraki, 2011). These two research pieces closely follow the literature on scoring rules, that has been first developed for arrovian ballots (Smith, 1973; Young, 1975), and then extended to more general ballots (Myerson, 1995). A scoring rule transforms any ballot into a vector of scores, a score for each candidate, and compare candidates according to their total scores. All the aforementioned articles emphasize the central role played by the axiom of Reinforcement to characterize scoring rules. This axiom states that if two sub-electorates agree on a ranking between two candidates, then the rule applied to the joint-electorate should produce the same ranking.

Balinski and Laraki (2011) consider the case of voting rules ranking any two candidates, in an environment where voters assign evaluations to the candidates. They axiomatize the class of lexicographic point-summing methods (general scoring rules) with the Reinforcement axiom, coined join-consistency. They offer variations of this result with the related axioms of proper cancellation and participant-consistency. Pivato (2013b) considers voting rules, defined as choice correspondences, taking as inputs any kind of ballots, in an abstract setting. General scoring rules are again characterized with the Reinforcement axiom.

On the one hand, the Reinforcement axiom immediately implies the axiom (Sep) of this article. On the other hand, it is possible to construct rules for a fixed electorate that satisfy (Sep), but that cannot be extended to variable-population rules satisfying Reinforcement, as put forth in the counter-example of Subsection 4.1. Indeed, the Reinforcement axiom implies all the "cancellation conditions" identified as necessary requirements for additivity in the conjoint measurement literature, whereas the axiom (Sep) is just one of these conditions. Therefore, the results obtained in Section 3 cannot be deduced from the works using the Reinforcement axiom, they rather provide a complementary view on the same objects, additively separable voting rules.

6.2 Additive rules and strategic incentives

The introduction of additive rules such as Range Voting or Evaluative Voting raises the important issue of the strategic incentives created by these voting systems. For instance, if there are only two candidates, the assumption of sincere balloting seems naive. In that case, any non-indifferent voter's optimal strategy is to assign extreme evaluations: the highest possible evaluation to her favorite candidate and the lowest possible evaluation to the other one. Hence, if voters are strategic, the outcome is the same as the outcome of Approval Voting[23].

Núñez and Laslier (2014) generalize this observation when there are more than two candidates. They show that, for a large electorate of strategic voters, the outcomes of the equilibria in the Evaluative Voting game coincide with the outcomes of the equilibria in the Approval Voting game. Do non-extreme grades matter then and is it useful to extend Approval Voting by introducing more than two evaluations? Núñez and Laslier (2014) observe that the result does not automatically hold for a fixed population of voters. More importantly, voters do not necessarily vote for instrumental motives in political elections, but in many cases to express their opinion, as it has been argued in the theory of expressive voting (Brennan and Buchanan, 1984). Along this interpretation, evaluation voting systems seem to be particularly desirable as they allow a great flexibility in the expression of voters' opinions. This argument is in line with the findings of an *in situ* experiment on Evaluative Voting conducted during the 2012 French presidential elections (Baujard et al., 2014), in which many voters used intermediate evaluations in their ballots.

7 Conclusion

I offered in this article a detailed axiomatic analysis of additively separable rules for evaluation voting systems.

[23]This is also the outcome of the majority rule if abstention is allowed.

When the set of evaluations is $[0, 1]$, the questions on "when" and "what" we should sum can be answered separately. I argued that a compelling consistency condition, the axiom of *Separability*, leads to adopt an additively separable rule (Theorem 1). To answer the question on "what" we should sum, I observed that Range Voting is the only rule among this class to satisfy two natural invariance properties: *Contraction Invariance* and *Symmetry Invariance* (Theorem 2).

When the set of evaluations is finite, I answered to the questions "what" and "when" simultaneously. The Evaluative Voting rule can be characterized either with an axiom of *Compensation* (Theorem 3), or with an axiom of *Independence of a Common Increase of the Evaluations* (Theorem 4), making the link with the analysis of Range Voting.

The implicit assumption behind these results is that candidates are compared according to their evaluation vectors, a property reminiscent of the notion of *welfarism* in the theories of justice. The axioms of Pareto Indifference and Binary Independence of Irrelevant Candidates can justify this assumption in a general framework for evaluation voting systems (Theorem 5). Previous characterization results are thus extended to this framework (Corollary 1, Corollary 2 and Corollary 3).

The scope of this analysis is rather general as it covers the different relevant cases of evaluation voting systems, concerning the set of evaluations and the notion of aggregation rule. Moreover, as this is a formal approach, it can be applied to any model where some alternatives are compared according to several criteria, provided that these criteria are written in a common scale. An important hypothesis of the analysis is that voters can freely choose the evaluations they assign to the candidates, an assumption coined *unrestricted domain* in the literature. This view is challenged for instance by the system of cumulative voting (Glasser, 1959), in which the grades given by a voter have to sum up to some constant value, for instance to 1 if the set of evaluations is $[0, 1]$. This leaves the area open for further research.

8 Appendix : proofs

8.1 Theorem 1

Proof. Throughout the proof, the set of evaluations is $[0,1]$ and $\#I \geq 4$. The fact that an additively separable rule satisfies axioms (A), (SP), (Cont) and (Sep) is rather straightforward: we obtain (SP) and (Cont) because val is increasing and continuous; (Sep) comes from the additive structure of the rule.

I begin by deriving a property $(*)$, that any additively separable rule satisfies (Step 0). Then, for a given rule $\succeq$ satisfying axioms (A), (SP), (Cont) and (Sep), and using this property $(*)$, I construct an increasing function f on the set of dyadic rationals D (Step 1). I show that f and $\succeq$ must necessarily satisfy $(*)$ on D (Step 2). Finally, f can be extended on $[0,1]$ and satisfies $(*)$ on $[0,1]$ (Step 3). This allows to prove that $\succeq$ is indeed an additively separable rule, with value $val = f^{-1}$ (Step 4).

As a remark, I slightly abuse notation in this proof. Here, any considered rule $\succeq$ satisfies the axiom (Sep), this means that the social ranking is not sensitive to the votes of indifferent voters. This property allows to naturally define a complete and binary relation on the sets $([0,1]^J)_{J\subseteq I}$. For $J \subseteq I$ and $u, v \in [0,1]^J$, I write $u \succeq v$ for $(u, 0_{I\setminus J}) \succeq (v, 0_{I\setminus J})$.

Step 0. Defining the property $(*)$

Let $\succeq$ be an additively separable rule. By definition, there exists a continuous and increasing function $val : [0,1] \to \mathbb{R}$, such that:

$$\forall u, v \in [0,1]^I, \qquad u \succeq v \quad \Leftrightarrow \quad \sum_{i\in I} val(u_i) \geq \sum_{i\in I} val(v_i).$$

In particular, for all $u, v \in [0,1]^2$, we have

$$val(u_1) + val(u_2) = val(v_1) + val(v_2) \quad \Rightarrow \quad u \sim v.$$

Moreover, we can assume without loss of generality that $val([0,1]) = [0,1]$, and we know that the function val is one-to-one. If we denote by f the inverse function of val, the previous line can be written

$$\forall x, y, z, t \in [0,1], \quad x + y = z + t \quad \Rightarrow \quad \big(f(x), f(y)\big) \sim \big(f(z), f(t)\big). \tag{$*$}$$

For the remainder of the proof, let $\succeq$ be a simple rule satisfying axioms (A), (SP), (Cont) and (Sep).

Step 1. Construction of f on the set of dyadic rationals D.

I denote by D the set of dyadic rationals between 0 and 1: $D = \{\frac{k}{2^p} \mid p \in \mathbb{N},\ 0 \le k \le 2^p\}$. This set is generated by the sets $(D_p)_{p\in\mathbb{N}}$ defined by

$$\forall p \in \mathbb{N}, \qquad D_p = \left\{\frac{k}{2^p} \mid 0 \le k \le 2^p\right\}.$$

We have $D = \cup_{p\in\mathbb{N}} D_p$ with $\forall p \in \mathbb{N},\ D_p \subset D_{p+1}$.

Step 1.1 Construction of f on the set $\mathbf{D_0}$

For $p = 0$, I define f on $D_0 = \{0, 1\}$ by $f(0) = 0$ and $f(1) = 1$. The function f is increasing on D_0.

Step 1.2 Construction of f on the sets $\mathbf{D_p, p \ge 1}$

I proceed by induction. Let us assume that the function f is defined and increasing on the set D_{p-1} for some $p \ge 1$.

- For any even integer k in $\{0, \ldots, 2^p\}$, the number $f\big(\frac{k}{2^p}\big)$ is already defined since $\frac{k}{2^p} \in D_{p-1}$. For sake of simplicity, I note this number:

$$x_p^k = f\big(\frac{k}{2^p}\big).$$

- For any odd integer k in $\{1, \ldots, 2^p - 1\}$, I define the number $f\big(\frac{k}{2^p}\big)$ as the unique number x_p^k in $[0,1]$ such that

$$\big(x_p^k, x_p^k\big) \sim \big(x_p^{k-1}, x_p^{k+1}\big).$$

The existence and uniqueness of x_p^k are due to a generalized version of the Intermediate

Value Theorem, which follows from standard arguments[24]. Moreover, we know that $x_p^k \in (x_p^{k-1}, x_p^{k+1})$, which means that

$$f\big(\frac{k-1}{2^p}\big) < f\big(\frac{k}{2^p}\big) < f\big(\frac{k+1}{2^p}\big).$$

I represent the definition of x_p^k by Figure 4, or alternatively by Figure 5.

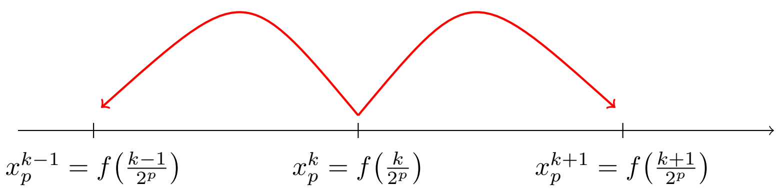

Figure 4: Definition of x_p^k: $\big(x_p^k, x_p^k\big) \sim \big(x_p^{k-1}, x_p^{k+1}\big)$.

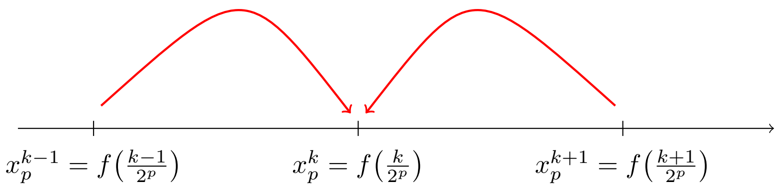

Figure 5: Definition of x_p^k: $\big(x_p^{k-1}, x_p^{k+1}\big) \sim \big(x_p^k, x_p^k\big)$.

Finally, the function f is defined by induction on each D_p for $p \geq 0$, and is increasing on these sets, thus f is defined and increasing on the set D. The definition of f on D is represented in the Figure 6 below.

Step 2. Property $(*)$ on the set D.

I show that for any $x, y, z, t \in D$,

$$x + y = z + t \quad \Rightarrow \quad \big(f(x), f(y)\big) \sim \big(f(z), f(t)\big). \qquad (*_D)$$

Step 2.1. Decomposition of the property $(*_\mathbf{D})$.

[24]the sets $\{z \in [0,1] | (z,z) \succeq (x_p^{k-1}, x_p^{k+1})\}$ and $\{z \in [0,1] | (x_p^{k-1}, x_p^{k+1}) \succeq (z,z)\}$ are closed, from axiom (Cont). Moreover, these sets are non-empty (we know that $x_p^{k-1} < x_p^{k+1}$) , from axiom (SP), and their union is $[0,1]$, from the completeness of the binary relation $\succeq$. Because $[0,1]$ is connected, the two sets must intersect, but this intersection cannot include more than one element according to axiom (SP).

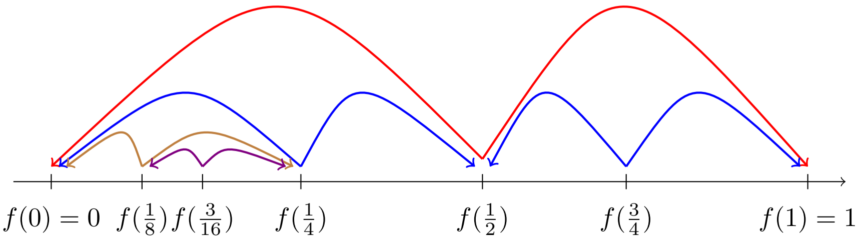

Figure 6: Definition of f on the set D.

It is sufficient to prove that for any $p \geq 0$, $k \in \{0, \dots, 2^p\}$ and $q \leq \min(k, 2^p - k)$, we have:

$$\Big(f\big(\frac{k}{2^p}\big), f\big(\frac{k}{2^p}\big)\Big) \sim \Big(f\big(\frac{k-q}{2^p}\big), f\big(\frac{k+q}{2^p}\big)\Big). \qquad (\mathcal{P}_p^{k,q})$$

This property can be written

$$\big(x_p^k, x_p^k\big) \sim \big(x_p^{k-q}, x_p^{k+q}\big) \qquad (\mathcal{P}_p^{k,q})$$

and is represented on Figure 7 below.

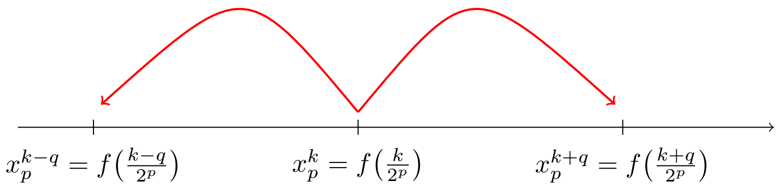

Figure 7: Property $(\mathcal{P}_p^{k,q})$.

By doing this, I provide a decomposition of the property $(*_D)$:

$$(*_D) = \bigcap_{p\geq 0} \bigcap_{0<k<2^p} \bigcap_{0<q\leq min(k,2^p-k)} (\mathcal{P}_p^{k,q})$$

where the intersection means that f satisfies $(*_D)$ if and only if it satisfies every $(\mathcal{P}_p^{k,q})$.

Step 2.2. f satisfies properties $(\mathcal{P}_\mathbf{p}^{\mathbf{k},\mathbf{1}})_{\mathbf{0<k<2^p}}$ on $\mathbf{D_p}$

I show by induction on p that the property $(\mathcal{P}_p^{k,1})$ is satisfied for all $k \in \{1, \dots, 2^p - 1\}$. If $p = 1$, the property is trivially satisfied by definition of $f\left(\frac{1}{2}\right)$.

Assume that for some $p \geq 2$, the property $(\mathcal{P}_{p'}^{k,1})$ is true for all $p' \leq p - 1$ and $k \in$

$\{1,\ldots,2^{p'}-1\}$. Let k be an integer in $\{1,\ldots,2^p-1\}$, we have to distinguish two cases: if k is odd, the property is trivially satisfied by definition of x_p^k. If k is even, we know that $k-1$ and $k+1$ are odd and we can write by definition:

$$\left(x_p^{k-1},x_p^{k-1}\right)\sim\left(x_p^{k-2},x_p^{k}\right) \tag{$\mathcal{P}_p^{k-1,1}$}$$

$$\left(x_p^{k+1},x_p^{k+1}\right)\sim\left(x_p^{k},x_p^{k+2}\right) \tag{$\mathcal{P}_p^{k+1,1}$}$$

Moreover, because k is even, it follows from the induction hypothesis that $(\mathcal{P}_{p-1}^{\frac{k}{2},1})=(\mathcal{P}_p^{k,2})$ holds:

$$\left(x_p^{k},x_p^{k}\right)\sim\left(x_p^{k-2},x_p^{k+2}\right). \tag{$\mathcal{P}_p^{k,2}$}$$

For sake of simplicity, let us omit the p in the following computations: x^k stands for x_p^k. We obtain successively:

$$\begin{aligned}
(x^k,x^k,x^k,x^k) &\sim (x^{k-2},x^{k+2},x^k,x^k) && (\mathcal{P}^{k,2}),\ (\text{EIV})\\
&\sim (x^{k-2},x^k,x^k,x^{k+2}) && (\text{A})\\
&\sim (x^{k-2},x^k,x^{k+1},x^{k+1}) && (\mathcal{P}^{k+1,1}),\ (\text{EIV})\\
&\sim (x^{k-1},x^{k-1},x^{k+1},x^{k+1}) && (\mathcal{P}^{k-1,1}),\ (\text{EIV})
\end{aligned}$$

The reasoning is illustrated on Figure 8 below, where the arrows above the axis are the assumptions, and the arrows below the axis are the conclusion.

Assume by contradiction, and without loss of generality, that $(x^{k-1},x^{k+1})\succ(x^k,x^k)$. This would imply, by axiom (Sep):

$$(x^{k-1},x^{k+1},x^{k-1},x^{k+1})\succ(x^{k-1},x^{k+1},x^k,x^k)$$

$$(x^{k-1},x^{k+1},x^k,x^k)\succ(x^k,x^k,x^k,x^k)$$

and then by transitivity of $\succ$, $(x^{k-1},x^{k+1},x^{k-1},x^{k+1})\succ(x^k,x^k,x^k,x^k)$, which is false from the

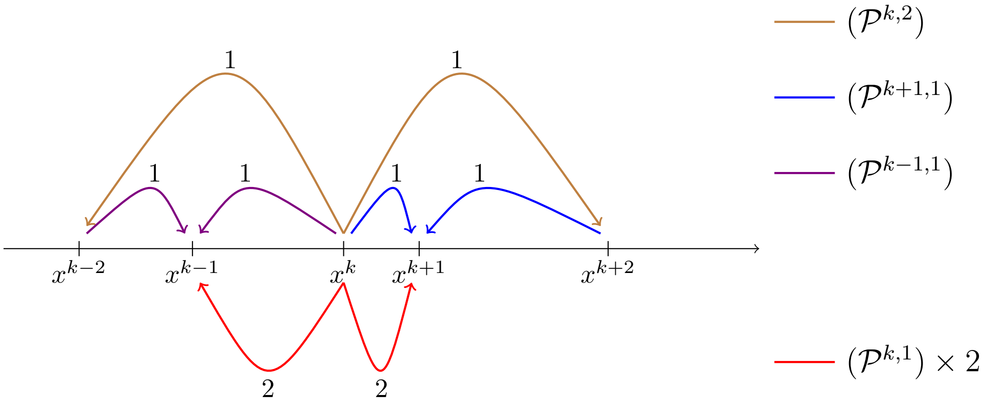

Figure 8: First reasoning: proof of $(\mathcal{P}_p^{k,1})$ by induction on p.

previous computations. Finally, we obtain

$$\left(x_p^k, x_p^k\right) \sim \left(x_p^{k-1}, x_p^{k+1}\right). \tag{$\mathcal{P}_p^{k,1}$}$$

Thus, the property $(\mathcal{P}_p^{k,1})$ is satisfied for all $p \geq 0$ and $k \in \{1, \dots, 2^p - 1\}$.

Step 2.3. f satisfies properties $(\mathcal{P}_{\mathbf{p}}^{\mathbf{k,q}})_{\mathbf{0<k<2^p,\ 0<q<\min(k,2^p-k)}}$ on $\mathbf{D_p}$

Let $p \geq 0$ and $k \in \{0, \dots, 2^p\}$. I show by induction on q that the property $(\mathcal{P}_p^{k,q})$ is satisfied for all $q \leq \min(k, 2^p - k)$. The property $(\mathcal{P}_p^{k,1})$ is already known to be satisfied (Step 2.2).

Assume that there exists q with $1 \leq q < \min(k, 2^p - k)$, such that $(\mathcal{P}_p^{k,q'})$ is satisfied for any q' such that $0 \leq q' \leq q$. It follows that

$$\left(x_p^k, x_p^k\right) \sim \left(x_p^{k-q}, x_p^{k+q}\right) \tag{$\mathcal{P}_p^{k,q}$}$$

$$\left(x_p^k, x_p^k\right) \sim \left(x_p^{k-q+1}, x_p^{k+q-1}\right). \tag{$\mathcal{P}_p^{k,q-1}$}$$

Moreover, we know that

$$\left(x_p^{k-q}, x_p^{k-q}\right) \sim \left(x_p^{k-q-1}, x_p^{k-q+1}\right) \tag{$\mathcal{P}_p^{k-q,1}$}$$

$$\left(x_p^{k+q}, x_p^{k+q}\right) \sim \left(x_p^{k+q-1}, x_p^{k+q+1}\right). \tag{$\mathcal{P}_p^{k+q,1}$}$$

Let us omit the p in the following computations: x^k stands for x_p^k.

$$\begin{aligned}
\left(x^k, x^k, x^k, x^k\right) &\sim \left(x^{k-q}\quad, x^{k+q}\quad, x^{k-q}\quad, x^{k+q}\quad\right) && (\mathcal{P}^{k,q}) \times 2\\
&\sim \left(x^{k-q}\quad, x^{k-q}\quad, x^{k+q}\quad, x^{k+q}\quad\right) && (\mathrm{A})\\
&\sim \left(x^{k-q-1}, x^{k-q+1}, x^{k+q-1}, x^{k+q+1}\right) && (\mathcal{P}^{k-q,1}),\ (\mathcal{P}^{k+q,1})\\
&\sim \left(x^{k-q+1}, x^{k+q-1}, x^{k-q-1}, x^{k+q+1}\right) && (\mathrm{A})\\
&\sim \left(x^{k}\quad\quad, x^{k}\quad\quad, x^{k-q-1}, x^{k+q+1}\right). && (\mathcal{P}^{k,q-1})
\end{aligned}$$

The reasoning is illustrated on Figure 9 below.

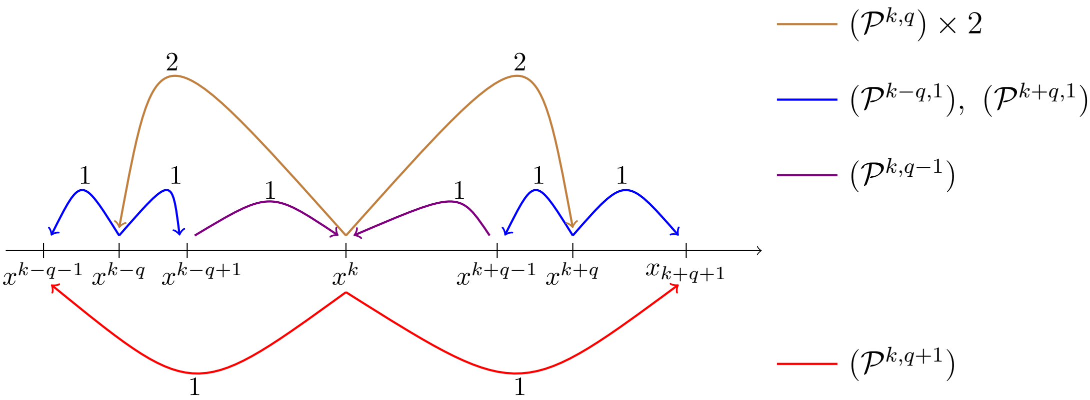

Figure 9: Second reasoning: proof of $(\mathcal{P}_p^{k,q})$ by induction on q.

Thus, we have:

$$\left(x_p^k, x_p^k\right) \sim \left(x_p^{k-q-1}, x_p^{k+q+1}\right). \qquad (\mathcal{P}_p^{k,q+1})$$

Finally, the condition $(\mathcal{P}_p^{k,q})$ holds for all $p \geq 0$, $k \in \{0, \ldots, 2^p\}$ and $q \leq \min(k, 2^p - k)$: this implies that the function f satisfies the property $(*_D)$.

Step 3. Definition and properties of the function f on $[0,1]$

Let us show that we can extend f into a continuous and increasing function $f : [0,1] \to [0,1]$, which satisfies condition $(*)$ on $[0,1]$.

Let x be a real number in $(0,1)$. Because f is increasing on D, the limits $\underline{l} = \lim_{d \to x,\ d<x,\ d \in D} f(d)$ and $\overline{l} = \lim_{d \to x,\ d>x,\ d \in D} f(d)$ are well-defined and $\underline{l} \leq \overline{l}$. Let us show that these two limits are equal.

I define the sequences of dyadic numbers $(d_p)_{p\in\mathbb{N}}$, $(d_p^-)_{p\in\mathbb{N}}$ and $(d_p^+)_{p\in\mathbb{N}}$ by:

$$\forall p \geq 0, \qquad d_p = \max\{d \in D_p | d < x\}, \quad d_p^- = d_p - \frac{1}{2^p}, \quad d_p^+ = d_p + \frac{1}{2^p}.$$

We know that $\lim_{p\to\infty} f(d_p) = \lim_{p\to\infty} f(d_p^-) = \underline{l}$, whereas $\lim_{p\to\infty} f(d_p^+) = \bar{l}$. Moreover, since f satisfies $(*_D)$, we have

$$\forall p \geq 0, \qquad \big(f(d_p), f(d_p)\big) \sim \big(f(d_p^-), f(d_p^+)\big).$$

By axiom (Cont), we obtain: $(\underline{l}, \underline{l}) \sim (\underline{l}, \bar{l})$. By axiom (SP), the two limits must be equal. Finally, I define $f(x)$ by

$$f(x) = \lim_{d\to x, d\in D} f(d).$$

The function f is defined on $[0,1]$, it is continuous and increasing. Moreover, the continuity of f implies that the condition $(*)$, which holds on D, holds also on $[0,1]$.

Step 4. The rule $\succeq$ is additively separable

We know that the function f is a continuous and increasing mapping from $[0,1]$ to $[0,1]$. Thus, f is one-to-one and it follows that the inverse function of f, that I denote by $val = f^{-1}$, is well-defined, increasing and continuous on $[0,1]$.

Let us show that the binary relation $\succeq$ is represented by the function VAL defined by

$$VAL : \left| \begin{array}{ccc} [0,1]^I & \to & \mathbb{R} \\ u & \mapsto & \displaystyle\sum_{i\in I} val(u_i) \end{array} \right.$$

I will prove:

$$\forall k \leq \#I, \forall u, v \in [0,1]^k, \qquad u \succeq v \quad \Leftrightarrow \quad \sum_{i=1}^{k} val(u_i) \geq \sum_{i=1}^{k} val(v_i)$$

To prove this, let us show by induction on k the following property

$$\forall u \in [0,1]^k, \qquad u \sim u^{VAL} \tag{k}$$

where u^{VAL} is defined[25] by

$$u^{VAL} = \Big(\underbrace{1,\dots,1}_{\lfloor VAL(u)\rfloor}, \underbrace{0,\dots,0}_{k-\lfloor VAL(u)\rfloor-1}, val^{-1}\big(VAL(u) - \lfloor VAL(u)\rfloor\big)\Big),$$

with the abuse of notation $VAL(u) = \sum_{i=1}^{k} val(u_i)$ for $u \in [0,1]^k$. For $k = 2$, the property (2) is derived from the fact that the function f satisfies property $(*)$ on $[0,1]$.

Assume that the property (k) is true for some $k \geq 2$. Let $u \in [0,1]^{k+1}$. I first apply the property (k) on the vector $u_{\{1,k\}} = (u_1,\dots,u_k) \in [0,1]^k$ and I get:

$$\begin{aligned}(u_1,\dots,u_k,u_{k+1}) &\sim \Big(\underbrace{1,\dots,1}_{\lfloor VAL(u_{\{1,k\}})\rfloor}, \underbrace{0,\dots,0}_{k-\lfloor VAL(u_{\{1,k\}})\rfloor-1}, val^{-1}\big(VAL(u_{\{1,k\}}) - \lfloor VAL(u_{\{1,k\}})\rfloor\big), u_{k+1}\Big) && (k)\\ &\sim \Big(\underbrace{1,\dots,1}_{\lfloor VAL(u_{\{1,k\}})\rfloor}, \underbrace{0,\dots,0}_{k-\lfloor VAL(u_{\{1,k\}})\rfloor-1}, v\Big)\end{aligned}$$

where $v = \Big(val^{-1}\big(VAL(u_{\{1,k\}}) - \lfloor VAL(u_{\{1,k\}})\rfloor\big), u_{k+1}\Big)$. Then, I apply the property (2) on the vector v. We know that $VAL(v) = VAL(u) - \lfloor VAL(u_{\{1,k\}})\rfloor \in [0,2)$ and we can write

$$v \sim \Big(\underbrace{1}_{\lfloor VAL(v)\rfloor}, \underbrace{0}_{1-\lfloor VAL(v)\rfloor}, val^{-1}\big(VAL(v) - \lfloor VAL(v)\rfloor\big)\Big) \tag{2}$$

[25] Remark that we obtain with this definition

$$\sum_{i=1}^{k} val(u_i^{VAL}) = \lfloor VAL(u)\rfloor \times 1 + (val \circ val^{-1})\big(VAL(u) - \lfloor VAL(u)\rfloor\big) = \sum_{i=1}^{k} val(u_i).$$

Thus

$$u \sim \Big(\underbrace{1,\dots,1}_{\lfloor VAL(u_{\{1,k\}})\rfloor}, \underbrace{1}_{\lfloor VAL(v)\rfloor}, \underbrace{0,\dots,0}_{k-\lfloor VAL(u_{\{1,k\}})\rfloor-1}, \underbrace{0}_{1-\lfloor VAL(v)\rfloor}, val^{-1}\big(VAL(v) - \lfloor VAL(v)\rfloor\big)\Big) \quad \text{(A)}$$

We remark from the expression of $VAL(v)$ that the two following equalities hold

$$\lfloor VAL(v)\rfloor = \lfloor VAL(u)\rfloor - \lfloor VAL(u_{\{1,k\}})\rfloor$$
$$VAL(v) - \lfloor VAL(v)\rfloor = VAL(u) - \lfloor VAL(u)\rfloor$$

It follows that

$$u \sim \Big(\underbrace{1,\dots,1,}_{\lfloor VAL(u)\rfloor} \underbrace{0,\dots,0}_{k-\lfloor VAL(u)\rfloor-1}, val^{-1}\big(VAL(u) - \lfloor VAL(u)\rfloor\big)\Big) \qquad (k+1)$$

Finally, the property (k) is always true. It follows from property (k) for $k = \#I$ and axiom (SP) that

$$\sum_{i\in I} val(u_i) = \sum_{i\in I} val(v_i) \quad \Rightarrow \quad u \sim v$$
$$\sum_{i\in I} val(u_i) > \sum_{i\in I} val(v_i) \quad \Rightarrow \quad u \succ v.$$

Finally, the rule $\succeq$ is represented by the function $VAL(u) = \sum_{i\in I} val(u_i)$, this means that $\succeq$ is an additively separable rule, with value val. The independence of the axioms follows from the independence of the larger set of axioms used in Theorem 2.

□

8.2 Theorem 2

Proof. Let $\succeq$ be a simple rule satisfying axioms (A), (SP), (Cont), (Sep), (CI) and (SI). We know from Theorem 1 that $\succeq$ is additively separable, meaning that there exists a continuous and increasing function $val : [0,1] \to \mathbb{R}$ such that $\succeq$ is represented by $VAL(u) = \sum_{i\in I} val(u_i)$. Let us assume without loss of generality that $val([0,1]) = [0,1]$. I show that $\forall x \in [0,1],\ val(x) = x$.

Step 1. $\mathbf{val(x) + val(1-x) = 1}$ **for all** $\mathbf{x \in [0,1]}$

Let $x \in [0,1]$ and define $z \in [0,1]$ as the unique number such that

$$val(z) = \frac{val(x) + val(1-x)}{2}$$

We obtain

$$(z,z) \sim (x, 1-x)$$

It follows from axioms (SI) and (A) that we have

$$(1-z, 1-z) \sim (1-x, x) \sim (x, 1-x) \sim (z,z)$$

An application of axiom (SP) gives $z = 1-z$ and thus $z = \frac{1}{2}$. Finally we obtain that for all $x \in [0,1]$,

$$val\Big(\frac{1}{2}\Big) = \frac{val(x) + val(1-x)}{2}$$

Taking $x = 0$ and knowing that $val(0) = 0$ and $val(1) = 1$, we find that $val\big(\frac{1}{2}\big) = \frac{1}{2}$. Thus:

$$\forall x \in [0,1], \qquad val(x) + val(1-x) = 1.$$

Step 2. $\mathbf{val(x) = x}$ **for all** $\mathbf{x \in D_p,\ p \geq 1.}$

We know that $val(x) = x$ for all $x \in D_1$. Let $p \geq 2$ and assume that val is the identity function on the set D_{p-1}. Let $d \in D_p$: d can be written as $d = \frac{k}{2^p}$ with $k \in \{0, \dots, 2^p\}$. If k is even, we know that $d \in D_{p-1}$ and thus $val(d) = d$. Let us first consider the case where k is odd and $k < 2^{p-1}$, so that $d < \frac{1}{2}$. We have:

$$\left\{\frac{k-1}{2^{p-1}}, \frac{k}{2^{p-1}}, \frac{k+1}{2^{p-1}}\right\} \subseteq D_{p-1}$$

It follows that

$$val\Big(\frac{k-1}{2^{p-1}}\Big) = \frac{k-1}{2^{p-1}}, \qquad val\Big(\frac{k}{2^{p-1}}\Big) = \frac{k}{2^{p-1}}, \qquad val\Big(\frac{k+1}{2^{p-1}}\Big) = \frac{k+1}{2^{p-1}}$$

Thus,

$$\Big(\frac{k}{2^{p-1}}, \frac{k}{2^{p-1}}\Big) \sim \Big(\frac{k-1}{2^{p-1}}, \frac{k+1}{2^{p-1}}\Big)$$

Applying the axiom (CI) with scalar $\alpha = \frac{1}{2}$, we get

$$\Big(\frac{k}{2^{p}}, \frac{k}{2^{p}}\Big) \sim \Big(\frac{k-1}{2^{p}}, \frac{k+1}{2^{p}}\Big)$$

Moreover, since k is odd, $k-1$ and $k+1$ are even, meaning that

$$\Big\{\frac{k-1}{2^{p}}, \frac{k+1}{2^{p}}\Big\} \subseteq D_{p-1}$$

So, we know that

$$val\Big(\frac{k-1}{2^{p}}\Big) = \frac{k-1}{2^{p}}, \qquad val\Big(\frac{k+1}{2^{p}}\Big) = \frac{k+1}{2^{p}}$$

As a conclusion, we obtain

$$\begin{aligned} 2 \times val\Big(\frac{k}{2^{p}}\Big) &= \frac{k-1}{2^{p}} + \frac{k+1}{2^{p}} \\ &= 2 \times \frac{k}{2^{p}} \end{aligned}$$

Finally $val(d) = d$. If $d > \frac{1}{2}$, we obtain $val(d) = d$ by application of Step 1.

Conclusion : $\mathbf{val(x) = x}$ **for all** $\mathbf{x \in [0, 1]}$**.** This follows trivially from the continuity of val. Thus, the rule $\succeq$ is the Range Voting rule. □

8.3 Theorem 2: independence of the axioms

To show the independence of the axioms, I introduce six different rules, each satisfying exactly five of the six axioms. The failures and satisfactions of each axiom are summarized in the following table:

Axioms/rules	RV	WU	CT	TA	WRU	SYM	SQ
A	×		×	×	×	×	×
SP	×	×		×	×	×	×
Cont	×	×	×		×	×	×
Sep	×	×	×	×		×	×
CI	×	×	×	×	×		×
SI	×	×	×	×	×	×	

where WU, CT, WRU, SYM, SQ are simple rules to be defined below. **WU** is a rule following the paradigm of weighted utilitarianism (d'Aspremont and Gevers, 2002): it is represented by the function $V(u) = \sum_{i\in I} \lambda_i u_i$ where the coefficients $(\lambda_i)_{i\in I}$ are positive and not all equal. By linearity of V, the rule satisfies all the axioms, except (A), since voters have different social weights (the weight of voter i is given by the coefficient λ_i). **CT** is a constant rule in the sense that $\forall u, v \in [0,1]^I, u \sim v$. This rule trivially satisfies all the axioms but (SP). The leximin rule, **TA**, introduced in Section 2, is not continuous for $\Gamma = [0,1]$, as we have $\forall p \in \mathbb{N}, (\frac{1}{2^p}, \frac{1}{2}) \succ^{TA} (0,1)$ but $(0, \frac{1}{2}) \prec^{TA} (0,1)$. However, it is known to satisfy (Sep) and it satisfies (SI) as $u \sim^{TA} v \;\Leftrightarrow\; u = v$. **WRU** is a peculiar rule following the principle of weighted rank utilitarianism (d'Aspremont and Gevers, 2002): it is represented by the function $V(u) = \sum_{i\in I} \lambda_i \tilde{u}_i$ (recall that $\tilde{u}$ is the non-decreasing evaluation vector which is obtained by a permutation of u) where the social weights $(\lambda_i)_{i\in I}$ satisfy the following conditions: (a) $\forall i \in I, \lambda_i > 0$; (b) $\forall i \in I, \lambda_i = \lambda_{\#I-i}$; (c) $\exists i, j \in I, \lambda_i \neq \lambda_j$. Here, (a) implies that the rule satisfies (SP) and (c) is necessary to have a rule distinct from RV. Condition (b) ensures that the rule satisfies (SI), as we have:

$$V(\mathbb{1} - u) = \sum_{i\in I} \lambda_i (\widetilde{\mathbb{1} - u})_i = \sum_{i\in I} \lambda_i (1 - \tilde{u}_{\#I-i}) = (\sum_{i\in I} \lambda_i) - V(u)$$

SYM is an additively separable rule, represented by $V(u) = \sum_{i\in I} val(u_i)$ where the function val is given by $val(x) = 2x^2$ if $x \leq \frac{1}{2}$ and $val(x) = 1 - 2(1-x)^2$ if $x > \frac{1}{2}$, as illustrated on

Figure 10. Because the graph of val is invariant by symmetry with respect to $(\frac{1}{2}, \frac{1}{2})$, we have for any $x \in [0,1]$, $val(x) + val(1-x) = 1$, thus (SI) is satisfied. However, as $2val(1/2) = 1 = val(0) + val(1)$ and $2val(1/4) = 1/4 < 1/2 = val(0) + val(1/2)$, the rule violates (CI). **SQ** is the additively separable rule represented by the function $V(u) = \sum_{i \in I} u_i^2$.

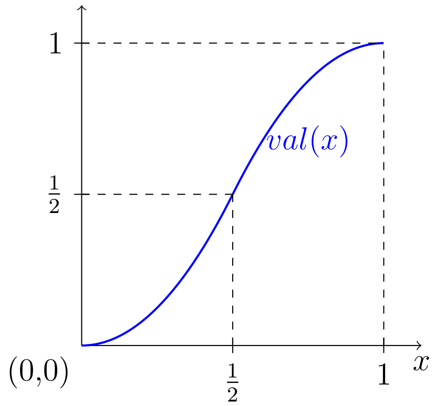

Figure 10: Function val for SYM.

8.4 Proposition 1

Proof. If $\#\Gamma = 2$, the statement is straightforward. Let us assume that $\#\Gamma = 3$ and that the simple rule $\succeq$ satisfies (A), (SP) and (Sep). As the rule $\succeq$ is anonymous, it induces naturally a binary relation on the set of all multisets of cardinality $\#I$ on the set Γ.

A multiset is an application $\mu : \Gamma \to \mathbb{N}$. The cardinality of a multiset is $|\mu| = \sum_{a \in \Gamma} \mu(a)$. The notion of multiset is just the generalization of the notion of set, in which an element can appear several times. For each vector $u \in \Gamma^I$, we can associate the multiset μ^u defined by

$$\forall a \in \Gamma, \qquad \mu^u(a) = \#\{i \in I | u_i = a\}.$$

If v is a permutation of u, it is clear than $\mu^u = \mu^v$, it follows that the relation on multisets is well-defined.

Now, axioms (Sep) and (SP) on the relation $\succeq$ correspond together to the notion of Consistency, as defined in Sertel and Slinko (2007), on the binary relation between multisets. Because $\#\Gamma = 3$, the additive separability follows from Theorem 1 in Sertel and Slinko (2007), and applies to the relation $\succeq$. $\square$

8.5 Theorem 3

Proof. It is straightforward to see that Evaluative Voting satisfies all the mentioned axioms. Let $\succeq$ be a rule satisfying axioms (SP) and (Comp). Let $u, v \in \Gamma^I$ be two evaluation vectors such that $\sum_{i \in I} val^{EV}(u_i) = \sum_{i \in I} val^{EV}(v_i)$. The vector v can be obtained from u by a finite sequence of transformations of the form $\{(u_i, u_j) \mapsto (succ(u_i), prec(u_j))\}$. It follows by iterated application of axiom (Comp) that $u \sim v$.

As a result, there exists a function $f : \{\#I, \dots, K \times (\#I)\} \to \mathbb{R}$ such that:

$$u \succeq v \quad \Leftrightarrow \quad f\Big(\sum_{i \in I} val^{EV}(u_i)\Big) \geq f\Big(\sum_{i \in I} val^{EV}(v_i)\Big).$$

By axiom (SP), the function f is increasing. Finally, the rule $\succeq$ is the Evaluative Voting rule. $\square$

8.6 Proposition 2

Proof. Let $\succeq$ be a simple rule satisfying axioms (A) and (ICIE). Let u and u' be two evaluation vectors such that $u'_1 = succ(u_1),\ u'_2 = prec(u_2),\ \forall k \geq 3, u'_k = u_k$. We obtain successively:

$$\begin{aligned}
\Big(u_1, u'_2, u_{-1,2}\Big) &\sim \Big(u'_2, u_1, u_{-1,2}\Big) && \text{(A)}\\
\Big(u_1, succ(u'_2), u_{-1,2}\Big) &\sim \Big(u'_2, succ(u_1), u_{-1,2}\Big) && \text{(ICIE)}\\
\Big(u_1, u_2, u_{-1,2}\Big) &\sim \Big(u'_2, u'_1, u_{-1,2}\Big) &&\\
\Big(u_1, u_2, u_{-1,2}\Big) &\sim \Big(u'_1, u'_2, u_{-1,2}\Big) && \text{(A)}\\
u &\sim u' &&
\end{aligned}$$

By applying this reasoning for arbitrary i, j, we obtain that the rule $\succeq$ satisfies axiom (Comp). $\square$

8.7 Theorem 4 : independence of the axioms

Axiom (A) is violated by the rule $u \succeq v \Leftrightarrow \sum_{i\in I} \lambda_i val^{EV}(u_i) \geq \sum_{i\in I} \lambda_i val^{EV}(v_i)$. Axiom (SP) is not satisfied by the constant rule. Finally, axiom (ICIE) is violated by the TA rule.

8.8 Theorem 5

Proof. Let Φ be a general rule satisfying axioms (PI*) and (BIIC*). I begin by showing a property called Strong Neutrality, which extends the property (BIIC*) to the case where the revealed preferences between two evaluation vectors are obtained with two different couples of candidates.

Step 1: The rule Φ satisfies the Strong Neutrality property (SN):

$$\forall m, n \in M, \forall x, y, z, t \in X, \quad \text{if} \begin{cases} n^z = m^x \\ n^t = m^y \end{cases}, \text{ then: } \left(\begin{cases} x \in \Phi(m) \\ t \in \Phi(n) \end{cases} \Rightarrow y \in \Phi(m) \right).$$

The intuition behind this axiom is the following: if $m^y = n^t$ is revealed to be preferred to $m^x = n^z$ in profile n, then whenever x is chosen in profile m, y should also be chosen in m. Let m and n be two evaluation profiles satisfying the previous conditions. I note $u = m^x = n^z$ and $v = m^y = n^t$. Let r be a candidate in $X \backslash \{y, t\}$, the existence of r is guaranteed because $\#X \geq 3$. I construct three new profiles m^1, m^2, m^3 as explained in the following table, in which each line represents an evaluation profile.

	x	y	z	t	r	$\ldots$
m	u	v	$\cdot$	$\cdot$	$\cdot$	$\cdot$
m^1	u	v	$\cdot\vert u$	$\cdot$	u	$\cdot$
m^2	$\times\vert u$	v	$\cdot\vert u$	v	u	$\cdot$
m^3	$\times\vert u$	$\times$	u	v	u	$\times$
n	$\times$	$\times$	u	v	$\times$	$\times$

It should be read in the table that m^1, m^2 and m^3 are defined by:

$$\begin{cases} (m^1)^x = (m^1)^r = u \\ (m^1)^y = v \\ (m^1)^z = \begin{cases} u & \text{if } r = z \\ m^z & \text{if } r \neq z \end{cases} \\ (m^1)^s = m^s,\ \forall s \notin \{x,y,z,r\} \end{cases}, \qquad \begin{cases} (m^2)^y = (m^2)^t = v \\ (m^2)^r = u \\ (m^2)^x = \begin{cases} u & \text{if } r = x \\ n^x & \text{if } r \neq x \end{cases} \\ (m^2)^z = \begin{cases} u & \text{if } r = z \\ m^z & \text{if } r \neq z \end{cases} \\ (m^2)^s = m^s,\ \forall s \notin \{x,y,z,t,r\} \end{cases},$$

$$\begin{cases} (m^3)^z = (m^3)^r = u \\ (m^3)^t = v \\ (m^3)^x = \begin{cases} u & \text{if } r = x \\ n^x & \text{if } r \neq x \end{cases} \\ (m^3)^s = n^s,\ \forall s \notin \{x,z,t,r\}. \end{cases}$$

I obtain successively the following properties

- $r \in \Phi(m^1)$: let $s \in \Phi(m^1)$. If $s \neq r$, by application of (BIIC*) to profiles m, m^1 and candidates x, s, we get $x \in \Phi(m^1)$, then (PI*) yields $r \in \Phi(m^1)$.

- $t \in \Phi(m^3)$: let $s \in \Phi(m^3)$. If $s \neq r$, by application of (BIIC*) to profiles n, m^3 and candidates t, s, we get $t \in \Phi(m^3)$. If $r \in \Phi(m^3)$, (PI*) yields $z \in \Phi(m^3)$. By application of (BIIC*) to profiles n, m^3 and alternatives z, t, we get $t \in \Phi(m^3)$.

- $\{y,t,r\} \in \Phi(m^2)$: it is sufficient to show that $t \in \Phi(m^2)$ or $r \in \Phi(m^2)$. Indeed, by (PI*) we have $(y \in \Phi(m^2) \Leftrightarrow t \in \Phi(m^2))$. Moreover, if $r \in \Phi(m^2)$, by application of (BIIC*) to m^2, m^3 and t, r, we get $t \in \Phi(m^2)$. Conversely, if $t \in \Phi(m^2)$, then $y \in \Phi(m^2)$, and by application of (BIIC*) to profiles m^1, m^2 and y, r, we obtain $r \in \Phi(m^2)$.

  Let $s \in \Phi(m^2)$ and consider all possible cases:

  - $s \in \{y,t,r\}$: already done.

- $s = x \neq r$: (BIIC*) applied to m^2, m^3 and x, t yields $t \in \Phi(m^2)$.
- $s \notin \{x, y, t, r\}$. (BIIC*) applied to m^1, m^2 and s, r gives $r \in \Phi(m^2)$.

- $y \in \Phi(m^1)$: application of (BIIC*) to m^1, m^2 and y, r.
- $y \in \Phi(m)$: application of (BIIC*) to m, m^1 and x, y.

Finally, we have $y \in \Phi(m)$: the rule Φ satisfies the Strong Neutrality property.

Let us define for the following step the revealed preference relation $\succeq^\Phi$ on Γ^I associated to the general rule Φ by

$$u \succeq^\Phi v \quad \text{if} \quad \exists m \in M, \exists x, y \in X, \quad \begin{cases} m^x = u \\ m^y = v \\ x \in \Phi(m). \end{cases}$$

Step 2. The relation $\succeq^\Phi$ is complete and transitive on Γ^I.

Take $u, v \in \Gamma^I$, and let $x \in X$. Consider the profile m defined by $m^x = u$ and $\forall y \neq x,\ m^y = v$. By definition of a general rule, there must be at least one element in $\Phi(m)$. Thus, by definition of the relation $\succeq^\Phi$, we get either $u \succeq^\Phi v$ or $v \succeq^\Phi u$: the relation $\succeq^\Phi$ is complete.

In order to show that $\succeq^\Phi$ is transitive, assume that $u \succeq^\Phi v$ and $v \succeq^\Phi w$. By definition, the relation $u \succeq^\Phi v$ (resp. $v \succeq^\Phi w$) is obtained for a profile $m \in M$ (resp. $n \in M$), and candidates x and y (resp. z and t) such that

$$\begin{cases} m^x = u \\ m^y = v \\ x \in \Phi(m). \end{cases} \quad \text{and} \quad \begin{cases} n^z = v \\ n^t = w \\ z \in \Phi(n). \end{cases}$$

Consider the profile l defined by

$$\begin{cases} l^x = u \\ l^y = v \\ l^s = w, \forall s \notin \{x, y\} \end{cases}$$

Let $s \in \Phi(l)$. We have three cases to consider:

- $s = x$, then by definition of $\succeq^\Phi$, we have $u \succeq^\Phi w$.
- $s = y$, then by application of (BIIC*) to profiles m, l and candidates x, y, we get $x \in \Phi(l)$, and we are back to the first case.
- $s \notin \{x, y\}$, then by application of Strong Neutrality to profiles n, l and candidates $(z, t), (y, s)$, we get $y \in \Phi(l)$ and we are back to the previous case.

Finally, the relation $\succeq^\Phi$ is complete and transitive.

Step 3. The general rule Φ is represented by the simple rule $\succeq^\Phi$.

If $x \in \Phi(m)$, we have by definition of $\succeq^\Phi$: $\forall y \in X$, $m^x \succeq^\Phi m^y$.

Conversely, let $m \in M$, $x \in X$ be such that $\forall y \in X,\ m^x \succeq^\Phi m^y$. Let $y \in \Phi(m)$. We have by assumption $m^x \succeq^\Phi m^y$. This relation is obtained for some profile $n \in M$ and candidates $z, t \in X$, such that $n^z = m^x$, $n^t = m^y$ and $z \in \Phi(n)$. By application of Strong Neutrality to profiles m and n for candidates $(x, y), (z, t)$, we obtain that $x \in \Phi(m)$.

□

8.9 Proposition 3

Proof. Let $\Gamma = [0, 1]$ and let Φ be a general rule satisfying axioms (BIIC*), (SP*) and (Cont*). Let $m \in M$ and $x, y \in X$ such that $m^x = m^y$ and $x \in \Phi(m)$. We note $u = m^x = m^y$. Let n be the evaluation profile defined by $\forall z \in X,\ n^z = u$. We first show that $y \in \Phi(n)$.

If $u \neq 0_{[0,1]^I}$, we define the sequence of evaluation vectors $(u^{(p)})_{p \in \mathbb{N}}$ by

$$\forall p \in \mathbb{N}, \forall i \in I, \qquad u_i^{(p)} = u_i\big(1 - \frac{1}{2^p}\big)$$

and the profiles $(n^{(p)})_{p \in \mathbb{N}}$ by $(n^{(p)})^y = u$, $\forall z \neq y,\ (n^{(p)})^z = u^{(p)}$. By axiom (SP*), we obtain for all p, $\Phi(n^{(p)}) = \{y\}$. By axiom (Cont), we obtain $y \in \Phi(n)$.

If $u = 0_{[0,1]^I}$, we define $(n^{(p)})_{p \in \mathbb{N}}$ by $(n^{(p)})_1^y = \frac{1}{2^p}$ and $\forall (i, z) \neq (1, y), (n^{(p)})_i^z = 0$. By axiom (SP*), we obtain for all p, $\Phi(n^{(p)}) = \{y\}$. By axiom (Cont), we obtain $y \in \Phi(n)$.

Finally, since $y \in \Phi(n)$, by application of (BIIC*) to profiles (m, n) and candidates (x, y), we obtain $y \in \Phi(m)$. $\square$

8.10 Corollary 2, Corollary 3 : independence of the axioms

Proof. For $\Gamma = [0, 1]$, let us define the rule Φ^{DIC}, dependent of irrelevant candidates, by

$$\forall m \in M, \qquad \Phi^{DIC}(m) = \arg\max_{x\in X} \sum_{i\in I} val^x(m_i)$$

where $val^x(m_i) = \frac{1}{2}m_i^x + \frac{1}{5}\Big(\min_{y\neq x} m_i^y + \max_{y\neq x} m_i^y\Big)$. With this rule the value of an evaluation is a weighted sum of this evaluation and the lowest and highest evaluations given by the same voter to the other candidates. The weights are chosen so that Φ^{DIC} satisfies (SP*): indeed $m_i^x = m_i^y \Leftrightarrow val^x(m_i) = val^y(m_i)$ and $m_i^x > m_i^y \Leftrightarrow val^x(m_i) > val^y(m_i)$. We observe that Φ^{DIC} also satisfies (A*), (Cont*), (Sep*), (CI*) and (SI*) since $val^x(\mathbb{1} - m_i) = \frac{9}{10} - val^x(m_i)$. Moreover, Φ^{DIC} fails to satisfy (BIIC*). Consider profile m such that $m_1 = (0, 1, 0, 1)$ and $m_2 = (1, 0, 0, 0)$ and profile n such that $n_1 = (0, 1, 0, 0)$ and $n_2 = (1, 0, 0, 0)$ (candidates are ranged in the following order: x, y, z, t). We have $n^x = m^x$ and $n^y = m^y$. Moreover, $\Phi(m) = \{y, t\}$ and $\Phi(n) = \{x, y\}$, therefore Φ violates (BIIC*). Finally, the independence of the axioms in Corollary 2 is deduced from the independence of the axioms of Theorem 2.

For Γ finite, let $>_X$ be a linear order on X and define:

$$\forall m \in M, \qquad \Phi^{NPI}(m) = \max\Big\{x \in X \Big| \forall y \in X,\ \sum_{i\in I} val^{EV}(m_i^x) \geq \sum_{i\in I} val^{EV}(m_i^y)\Big\}$$

The rule Φ^{NPI} satisfies axioms (BIIC*), (A*), (SP*) and (ICIE*) but not (PI*).

Let us define a discrete analog of the rule Φ^{RU} by

$$\forall m \in M, \quad \Phi^{RUd}(m) = \arg\max_{x\in X} \sum_{i\in I} val^x(m_i),$$

with $val^x(m_i) = \dfrac{val^{EV}(m_i^x) - val^{EV}(\min_y m_i^y)}{val^{EV}(\max_y m_i^y) - val^{EV}(\min_y m_i^y)}$. The rule Φ^{RUd} satisfies axioms (PI*), (A*), (SP*) and (ICIE*), but not (BIIC*). Finally, the independence of the axioms in Corollary 3 is deduced from the independence of the axioms of Theorem 4.

□

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