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Working Papers / Documents de travail

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WP 2016 - Nr 07

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First version: *July 2015*; Revised: *December 2015*

Abstract: *We analyze a version of the Benhabib and Farmer [3] two-sector model with sector-specific externalities in which we consider a class of utility functions inspired from the one considered in Jaimovich and Rebelo [14] which is flexible enough to encompass varying degrees of income effect. First, we show that local indeterminacy and sunspot fluctuations occur in 2-sector models under plausible configurations regarding all structural parameters – in particular regarding the intensity of income effects. Second, we prove that there even exist some configurations for which local indeterminacy arises* under any degree of income effect. *More precisely, for any given size of income effect, we show that there is a non-empty range of values for the Frisch elasticity of labor and the elasticity of intertemporal substitution in consumption such that indeterminacy occurs. This contrasts with the results obtained in one-sector models in both Nishimura* et al. *[19], in which it is shown that indeterminacy cannot occur under either GHH and KPR preferences, and in Jaimovich [13] in which local indeterminacy only arises for intermediary income effects.*

Keywords: *Indeterminacy, sunspots, income and substitution effects, sector-specific externalities, infinite-horizon two-sector model*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

*This work has been carried out thanks to the support of the A*MIDEX project (n° ANR-11-IDEX-0001-02) funded by the "Investissements d'Avenir" French Government program, managed by the French National Research Agency (ANR). We would like to thank an anonymous referee together with X. Raurich and T. Seegmuller for useful comments and suggestions.

1 Introduction

It is well-known since Benhabib and Farmer [2, 3] that local indeterminacy and sunspot fluctuations arise in two-sector models under much more empirically plausible configurations regarding structural parameters than in their one-sector equivalents. In particular, indeterminacy occurs for calibrations consistent with a low degree of increasing returns to scale and a standard (negatively sloped) equilibrium labor demand function. Besides, it has been shown that two-sector models submitted to correlated sunspot and technological shocks are able to account for many empirical regularities regarding the comovements of consumption and investment over the business cycle, and regarding the allocation of labor across these two sectors (Dufourt *et al.* [6]). Yet, these results were obtained under a specification of individual preferences derived from Greenwood *et al.* [9] (thereafter GHH), which implies that there is no income effect on labor supply.

From a theoretical point of view, one may thus wonder whether results obtained under GHH preferences can be extended to a framework in which the magnitude of the income effect on labor supply differs from zero. While this issue has been the subject of particular attention in one-sector models (see in particular Jaimovich [13]), no systematic study of the role of income effects in two-sector models has been provided so far.[1] The aim of this chapter is to undertake such an analysis.

We analyze a version of the Benhabib and Farmer [3] two-sector model with sector-specific externalities in which we consider a class of utility functions which is flexible enough to encompass varying degrees of income effect. Our specification of individual preferences is inspired from – but slightly differs from – the one considered in Jaimovich and Rebelo [14] (JR). This specification admits as particular (and polar) cases the GHH formulation without income effect and the canonical specification of King *et al.* [16] (KPR) used in many DSGE models. We analyze how the local stability properties of the model change when we vary the parameter governing the intensity of the income effect, and we determine the conditions under which local indeterminacy arises. We perform this analysis for different configurations regarding the other structural parameters influencing the wage elasticity of labor supply, the elasticity of intertemporal substitution (EIS) in consumption, and the degree of increasing returns to scale (IRS).

Our main results can be described as follows. First, we show that local indeterminacy and sunspot fluctuations occur in 2-sector models under plausible configurations regarding all structural parameters – in particular regarding the intensity of income effects. Second, we show that there even exist some configurations for which local indeterminacy arises *under any degree of income effect*. More precisely, for any given size of income effect, we show that there is a non-empty range of values for the Frisch elasticity of labor and the elasticity of intertemporal substitution in consumption such that indeterminacy occurs. This contrasts with the results obtained in one-sector models in both Nishimura *et al.* [19], in which it is shown that indeterminacy cannot occur under either GHH and KPR preferences as long as realistic parameter values are considered, in particular when the slope of the labor demand function is negative, and in Jaimovich [13] in

[1] Nishimura and Venditti [20] show that local indeterminacy can occur under both GHH and KPR preferences − the latter displaying positive income effect − but there is no clear picture of the impact of the income effect on the occurrence of sunspot fluctuations.

which local indeterminacy only arises for intermediary income effects.

The rest of this Chapter is organized as follows. We present the model and we characterize the intertemporal equilibrium in the next Section. In Section 3, we prove the existence of a unique steady state and we provide the expression of the characteristic polynomial. The complete set of conditions for local indeterminacy are derived in Section 4. Section 5 provides some numerical illustrations, while economic intuitions underlying our main theoretical results are given in Section 6. Some concluding remarks are stated in Section 7, whereas all the technical details are given in a final Appendix.

2 The model

We consider a standard two-sector infinite-horizon model with productive externalities and JR-type preferences (see Jaimovich [13] and Jaimovich and Rebelo [14]). Households are infinitely-lived, accumulate capital, and derive utility from consumption and leisure. Firms produce differentiated consumption and investment goods using capital and labor, and sell them to consumers. All markets are perfectly competitive.

2.1 The production structure

Firms in the consumption sector produce output $Y_c(t)$ according to a Cobb-Douglas production function:

$$Y_c(t) = K_c(t)^{\alpha} L_c(t)^{1-\alpha} \tag{1}$$

where $K_c(t)$ and $L_c(t)$ are capital and labor allocated to the consumption sector.

In the investment sector, output $Y_I(t)$ is also produced according to a Cobb-Douglas production function but which is affected by a productive externality

$$Y_I(t) = A(t) K_I(t)^{\alpha} L_I(t)^{1-\alpha} \tag{2}$$

where $K_I(t)$ and $L_I(t)$ are the numbers of capital and labor units used in the production of the investment good, and $A(t)$ is the externality parameter. Following Benhabib and Farmer [3], we assume that the externality is sector-specific and depends on the average levels $\bar{K}_I(t)$ and $\bar{L}_I(t)$ of capital and labor used in the investment sector, such that:

$$A(t) = \bar{K}_I(t)^{\alpha\Theta} \bar{L}_I(t)^{(1-\alpha)\Theta} \tag{3}$$

with $\Theta \geq 0$.[2] These economy-wide averages are taken as given by individual firms. Assuming that factor markets are perfectly competitive and that capital and labor inputs are perfectly mobile across the two sectors, the first order conditions for profit maximization of the representative firm in each sector are:

$$r(t) = \frac{\alpha Y_c(t)}{K_c(t)} = p(t) \frac{\alpha Y_I(t)}{K_I(t)}, \tag{4}$$

$$w(t) = \frac{(1-\alpha) Y_c(t)}{L_c(t)} = p(t) \frac{(1-\alpha) Y_I(t)}{L_I(t)} \tag{5}$$

where r, p and w are respectively the rental rate of capital, the price of the investment good and the real wage rate at time t, all in terms of the price of the consumption good.

[2] We do not consider externalities in the consumption good sector as they do not play any crucial role in the existence of multiple equilibria.

2.2 Households' behavior

We consider an economy populated by a continuum of unit mass of identical infinitely-lived agents. The representative agent enters each period t with a capital stock $k(t)$ inherited from the past. He then supplies elastically an amount $l(t) \in [0, \bar{l})$ of labor (with $\bar{l} > 0$ his exogenous time endowment), rents its capital stock $k(t)$ to the representative firms in the consumption and investment sectors, consumes $c(t)$, and invests $i(t)$ in order to accumulate capital.

Denoting by $y(t)$ the GDP, the budget constraint faced by the representative household is

$$c(t) + p(t)i(t) = r(t)k(t) + w(t)l(t) \equiv y(t) \tag{6}$$

Assuming that capital depreciates at rate $\delta \in (0, 1)$ in each period, the law of motion of the capital stock is:

$$\dot{k}(t) = i(t) - \delta k(t) \tag{7}$$

The intertemporal optimization problem of the representative household is then given by:

$$\begin{aligned}
\max_{\{c(t),i(t),l(t)\}} \quad & \int_0^{+\infty} U(c(t), (\bar{l} - l(t)))e^{-\rho t}dt \\
s.t. \quad & c(t) + p(t)i(t) \equiv y(t) = r(t)k(t) + w(t)l(t) \\
& \dot{k}(t) = i(t) - \delta k(t) \\
& k(0) \ given
\end{aligned} \tag{8}$$

where $\rho \geq 0$ is the discount rate.

The Hamiltonian in current value is given by:

$$\mathcal{H} = U(c(t), (\bar{l} - l(t))) + \lambda(t)\left[r(t)k(t) + w(t)l(t) - c(t) - p(t)i(t)\right] + q(t)\left[i(t) - \delta k(t)\right]$$

with $q(t)$ the co-state variable which corresponds to the utility price of the capital good in current value and $\lambda(t)$ the Lagrange multiplier associated with the budget constraint. The first order conditions of problem (8) are given by the following equations:

$$U_1(c(t), (\bar{l} - l(t))) = \lambda(t) \tag{9}$$

$$U_2(c(t), (\bar{l} - l(t))) = w(t)\lambda(t) \tag{10}$$

$$q(t) = p(t)\lambda(t) \tag{11}$$

$$\dot{q}(t) = (\delta + \rho)q(t) - r(t)\lambda(t) \tag{12}$$

An equilibrium path also satisfies the transversality condition

$$\lim_{t \to +\infty} e^{-\rho t} U_1(c(t), (\bar{l} - l(t)))p(t)k(t) = 0. \tag{13}$$

Following Jaimovich [13] and Jaimovich and Rebelo [14], we assume a JR-type utility function which is flexible enough to encompass varying degrees of income effect. Denoting leisure as $\mathcal{L} = \bar{l} - l$, let

$$U(c, \mathcal{L}) = \frac{\left[c - \frac{(\bar{l}-\mathcal{L})^{1+\chi}}{1+\chi}c^{\gamma}\right]^{1-\sigma} - 1}{1-\sigma} \tag{14}$$

with $\sigma \geq 0$, $\chi \geq 0$ and $\gamma \in [0, 1]$. This utility function satisfies the standard normality condition between consumption and leisure. In the following, we will also introduce some parameter restrictions ensuring that concavity holds at the steady-state.[3] This specification nests as particular cases the Greenwood-Hercovitz-Huffman [9] (GHH) formulation (obtained when $\gamma = 0$), characterized by the lack of any income effect on labor supply, and the King-Plosser-Rebelo [16] (KPR) formulation (obtained when $\gamma = 1$), characterized by a large income effect compatible with endogenous growth. We are then able to control the magnitude of the income effect by varying the calibration for γ between these two extremes.

Remark 1: Jaimovich [13] and Jaimovich and Rebelo [14] actually consider discrete-time models with a slightly different specification such that

$$U(c_t, \mathcal{L}_t, X_t) = \frac{\left[c_t - \frac{(\bar{l}-\mathcal{L}_t)^{1+\chi}}{1+\chi} X_t\right]^{1-\sigma} - 1}{1-\sigma} \tag{15}$$

with $X_t = c_t^{\gamma} X_{t-1}^{1-\gamma}$. When $\gamma \in (0,1)$, the income effect depends on the dynamics of this additional state variable X_t. Such a formulation allows to get more persistence of income effects during the transition, but focusing on such a property is out of the scope of this paper.

Remark 2: Using this specification for the utility function, from equations (9)-(10) we can write the first order condition that drives the trade-off between consumption and leisure as follows

$$\frac{(1+\chi)l^{\chi}c^{\gamma}}{1+\chi-\gamma l^{1+\chi}c^{\gamma-1}} = w \tag{16}$$

Denoting $\mathcal{I}$ the total income of the representative agent and normalizing the price of consumption to 1, we consider the static budget constraint

$$c + w\mathcal{L} = \mathcal{I} \tag{17}$$

Considering that $\mathcal{L} = \bar{l} - l$, solving equations (16)-(17) gives demand functions for consumption and leisure, namely $c = c(w, \mathcal{I})$ and $\mathcal{L} = \mathcal{L}(w, \mathcal{I})$. Assuming a constant wage, considering that $d\mathcal{L} = -dl$ and deriving the ratio wl/c from (16), we then get the following derivatives that describe the income effect for any $\gamma \in [0,1]$:

$$\begin{aligned} \varepsilon_{c\mathcal{I}} \equiv \frac{dc}{d\mathcal{I}} &= \left[1 + \gamma \frac{(1+\chi)l^{1+\chi}c^{\gamma-1}}{1+\chi-\gamma l^{1+\chi}c^{\gamma-1}} \frac{1+\chi-l^{1+\chi}c^{\gamma-1}}{(1+\chi)\chi+\gamma l^{1+\chi}c^{\gamma-1}}\right]^{-1} \\ \varepsilon_{l\mathcal{I}} \equiv \frac{dl}{d\mathcal{I}} = -\frac{d\mathcal{L}}{d\mathcal{I}} &= -\gamma \frac{1+\chi-l^{1+\chi}c^{\gamma-1}}{(1+\chi)\chi+\gamma l^{1+\chi}c^{\gamma-1}} \left[1 + \gamma \frac{(1+\chi)l^{1+\chi}c^{\gamma-1}}{1+\chi-\gamma l^{1+\chi}c^{\gamma-1}} \frac{1+\chi-l^{1+\chi}c^{\gamma-1}}{(1+\chi)\chi+\gamma l^{1+\chi}c^{\gamma-1}}\right]^{-1} \end{aligned} \tag{18}$$

These expressions clearly show that in the GHH case with $\gamma = 0$ there is no income effect as $\varepsilon_{l\mathcal{I}} = 0$ and $\varepsilon_{c\mathcal{I}} = 1$ while in the KPR case with $\gamma = 1$ we get some income effect with $\varepsilon_{l\mathcal{I}} \in (-1, 0)$ and $\varepsilon_{c\mathcal{I}} \in (0,1)$. In the intermediary case with $\gamma \in (0,1)$, the income effect lies in between these two extremes.

[3] It is important to note that when $\gamma \neq 0$, this utility function may not be concave. This characteristic is well-known for the KPR specification with $\gamma = 1$ for which additional restrictions on σ and χ are required to guarantee concavity (see for instance Hintermaier [12]). However, in order to avoid technical and cumbersome assumptions, we will only focus with Lemma 1 below on the conditions for local concavity properties around the steady state. Precise general conditions for global concavity can be provided upon request.

2.3 Intertemporal equilibrium

We consider symmetric equilibria which consist of prices $\{r(t), p(t), w(t)\}_{t\geq 0}$ and quantities $\{c(t), l(t), i(t), k(t), Y_c(t), Y_I(t), K_c(t), K_I(t), L_c(t), L_I(t)\}_{t\geq 0}$ that satisfy the household's and the firms' first-order conditions as given by (4)–(5) and (9)-(12), the technological and budget constraints (1)-(3) and (6)-(7), the good market equilibrium conditions

$$c = Y_c,\ i = Y_I,$$

the market clearing conditions for capital and labor

$$K_c + K_I = k, \quad L_c + L_I = l$$

and the transversality condition (13).[4]

All firms in the investment sector being identical, we have $\bar{K}_I = K_I$ and $\bar{L}_I = L_I$. At the equilibrium, the production function in the investment good sector is then given by

$$Y_I = K_I^{\alpha(1+\Theta)} L_I^{(1-\alpha)(1+\Theta)} \tag{19}$$

We thus have increasing social returns which size is measured by Θ.

3 Steady state and characteristic polynomial

After a few manipulations, the intertemporal equilibrium described above can be reduced to a dynamic system of two equations in two variables, k and p. From the firms' first-order conditions (4)-(5), we derive that the equilibrium capital-labor ratios in the consumption and investment sectors are identical and equal to $a \equiv k/l = K_c/L_c = K_I/L_I = \alpha w/\left((1-\alpha)r\right)$, with $w = (1-\alpha)a^\alpha$ and $r = \alpha a^{\alpha-1}$. Combining these results with (1)-(2), we get $pA = 1$ with $A = K_I(k/l)^{-(1-\alpha)\Theta}$ and thus

$$K_I = (k/l)^{1-\alpha} p^{-1/\Theta} \tag{20}$$

Moreover, substituting these expressions into the production functions (1)-(2), we also derive:

$$\begin{aligned} i &= Y_I = p^{-\frac{1+\Theta}{\Theta}} \equiv Y_I(p) && (21)\\ c &= Y_c = \left(\frac{k}{l}\right)^{\alpha-1}\left[k - \left(\frac{k}{l}\right)^{1-\alpha} p^{-1/\Theta}\right] && (22)\end{aligned}$$

Combining equations (9)-(10), describing the labor-leisure trade-off at the equilibrium, with (22) allows to write consumption and labor as functions of the capital stock k and the price of the investment good p, namely $c = c(k,p)$, and $l = l(k,p)$. It follows therefore that

$$\begin{aligned} a &= k/l(k,p) \equiv a(k,p)\\ w &= (1-\alpha)(a(k,p))^\alpha \equiv w(k,p)\\ r &= \alpha(a(k,p))^{\alpha-1} \equiv r(k,p)\end{aligned} \tag{23}$$

Let us introduce the following elasticities:

$$\epsilon_{cc} = -\frac{U_1(c,\mathcal{L})}{U_{11}(c,\mathcal{L})c}, \quad \epsilon_{lc} = -\frac{U_2(c,\mathcal{L})}{U_{21}(c,\mathcal{L})c}, \quad \epsilon_{cl} = -\frac{U_1(c,\mathcal{L})}{U_{12}(c,\mathcal{L})l}, \quad \epsilon_{ll} = -\frac{U_2(c,\mathcal{L})}{U_{22}(c,\mathcal{L})l} \tag{24}$$

[4]When there is no possible confusion, the time index (t) is not mentioned.

Note that ϵ_{cc} corresponds to the elasticity of intertemporal substitution in consumption while the Frisch elasticity of the labor supply is given by

$$\epsilon_{lw} = \left[\frac{1}{\epsilon_{ll}} - \frac{1}{\epsilon_{cl}}\right]^{-1} \tag{25}$$

Combining (9)-(12) with (20)-(23), the equations of motion are finally derived as

$$\begin{aligned} \dot{k} &= Y_I(p) - \delta k \\ \dot{p} &= \frac{(\delta+\rho)p - r(k,p) + \left[\frac{1}{\epsilon_{cc}}\frac{\partial c}{\partial k}\frac{p}{c(k,p)} - \frac{1}{\epsilon_{cl}}\frac{\partial l}{\partial k}\frac{p}{l(k,p)}\right](Y_I(p)-\delta k)}{E(k,p)} \end{aligned} \tag{26}$$

with

$$E(k,p) = 1 - \left[\frac{1}{\epsilon_{cc}}\frac{\partial c}{\partial p}\frac{p}{c(k,p)} - \frac{1}{\epsilon_{cl}}\frac{\partial l}{\partial p}\frac{p}{l(k,p)}\right] \tag{27}$$

Any solution $\{k(t), p(t)\}_{t\geq 0}$, with $k(0)$ given, that also satisfies the transversality condition (13) is called an equilibrium path.

A steady state of the dynamical system (26) is defined by a pair (k^*, p^*) solution of

$$Y_I(p) = \delta k, \quad r(k,p) = (\delta+\rho)p \tag{28}$$

We then derive:

Proposition 1. *Assume that* $\chi[1-\alpha(1+\Theta)] + \gamma(1-\alpha) - \alpha\Theta \neq 0$. *Then there exists a unique steady state* (k^*, p^*) *such that* $Y_I(p^*) = \delta k^*$ *and* $r(k^*, p^*) = (\delta+\rho)p^*$.

Proof: See Appendix 8.1. □

Remark 3: Using a continuity argument we derive from Proposition 1 that there exists an intertemporal equilibrium for any initial capital stock $k(0)$ in the neighborhood of k^*. Moreover, any solution of (26) that converges to the steady state satisfies the transversality condition (13) and is an equilibrium. Therefore, given $k(0)$, if there is more than one initial price $p(0)$ in the stable manifold of the steady state, the equilibrium path from $k(0)$ is not unique and we have local indeterminacy.

Remind also from footnote 3 that the JR-type utility function as given by (14) may not be concave. Since we focus on the local stability properties of equilibria around the steady state, we provide a local condition for concavity.

Lemma 1. *The JR-type utility function as given by (14) is concave in a neighborhood of the steady state if and only if*

$$\sigma \geq \sigma_c(\gamma) \equiv \frac{\gamma\mathcal{C}(\gamma+\chi)[1+\chi-(1-\gamma)\mathcal{C}]}{(1+\chi)^2\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]} \tag{29}$$

with $\mathcal{C} = [(1-\alpha)(\delta+\rho)]/[\rho+\delta(1-\alpha)](<1)$.

Remark 4: When evaluated at the steady state, the income effect (18), the elasticity of intertemporal substitution in consumption as defined in (24) and the Frisch elasticity of labor (25) become:[5]

[5]See Appendix 8.2.

$$\varepsilon_{c\mathcal{I}} = \frac{\chi+\gamma\mathcal{C}}{\chi+\gamma\mathcal{C}[2-\mathcal{C}(1-\gamma)]}, \quad \varepsilon_{l\mathcal{I}} = -\frac{\gamma[1-\mathcal{C}(1-\gamma)]}{\chi+\gamma\mathcal{C}[2-\mathcal{C}(1-\gamma)]} \tag{30}$$

$$\epsilon_{cc} = \left[\sigma\frac{1+\chi}{1+\chi-\mathcal{C}(1-\gamma)} - \gamma(1-\gamma)\frac{\mathcal{C}}{1+\chi}\right]^{-1}, \quad \epsilon_{lw} = \frac{1}{\chi+\gamma\mathcal{C}} \tag{31}$$

Considering (21) and linearizing the dynamical system (26) around the steady state leads to the characteristic polynomial

$$\mathcal{P}(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D} \tag{32}$$

with

$$\begin{aligned} \mathcal{D} &= \frac{-\delta\left(\delta+\rho-\frac{\partial r}{\partial p}\right)-\frac{\delta(1+\Theta)}{\Theta}\frac{\partial r}{\partial k}\frac{k^*}{p^*}}{E(k^*,p^*)} \\ \mathcal{T} &= \frac{\rho+\delta\left(\frac{1}{\epsilon_{cc}}\frac{\partial c}{\partial p}\frac{p^*}{c^*}-\frac{1}{\epsilon_{cl}}\frac{\partial l}{\partial p}\frac{p^*}{l^*}\right)-\frac{\partial r}{\partial p}-\frac{\delta(1+\Theta)}{\Theta}\left(\frac{1}{\epsilon_{cc}}\frac{\partial c}{\partial k}\frac{k^*}{c^*}-\frac{1}{\epsilon_{cl}}\frac{\partial l}{\partial k}\frac{k^*}{l^*}\right)}{E(k^*,p^*)} \end{aligned} \tag{33}$$

Most of these partial derivatives are functions of ϵ_{cc}, ϵ_{cl}, ϵ_{lc} and ϵ_{ll}. The role of ϵ_{lc} and ϵ_{ll} occurs through the presence of endogenous labor but remains implicit at this stage.

System (26) has one state variable and one control variable. As is well known, if (32) has two roots with negative real parts, there is a continuum of converging paths and thus a continuum of equilibria: the steady state is locally indeterminate and there exist expectation-driven endogenous fluctuations. Local indeterminacy therefore requires that $\mathcal{D} > 0$ and $\mathcal{T} < 0$. Obviously saddle-point stability is obtained when $\mathcal{D} < 0$, while total instability holds (with both eigenvalues having positive real parts) if $\mathcal{D} > 0$ and $\mathcal{T} > 0$.[6] In the following, we will focus on locally indeterminate equilibria and we will also look for the existence of a Hopf bifurcation, occurring when $\mathcal{T} = 0$ while $\mathcal{D} > 0$, which leads to periodic cycles.

4 Local indeterminacy with variable income effects

Deriving the local stability properties of system (26) in the most general case (without additional parameter restrictions) is very cumbersome, as a lot of different configurations may arise. In order to reduce the number of possible configurations, we now introduce the following parameter restrictions:

Assumption 1. *$\alpha < 1/2$, $\delta = 0.025$, $\rho > 0.005$, $\chi \leq 3$ and $\Theta \in (0, \bar{\Theta})$ with $\bar{\Theta} = (1-\alpha)/\alpha$.*

The calibration for δ is common to many studies in the DSGE literature and corresponds to an annual capital depreciation rate of 10%. The restriction on α is innocuous as capital shares are typically less than 50% of GDP in industrialized economies. Likewise, the assumption on the rate of time preference ρ is not very restrictive as the standard calibration for this parameter is $\rho = 0.01$. The restriction on χ allows to consider realistic values for the Frisch elasticity of labor ϵ_{lw} as given in (31) (see Section 5). Finally, using a benchmark calibration for the US economy at quarterly frequency, namely $(\alpha, \rho, \delta) = (0.3, 0.01, 0.025)$, Assumption 1 implies $\bar{\Theta} \approx 2.33$. This bound defines an interval for Θ which largely covers the range of available estimates for the degree of IRS in the investment sector, since empirical studies typically conclude for values around 0.3.[7] We obtain:

[6] We will show in this case that there exists a Hopf bifurcation leading to the existence of periodic cycles.

[7] For example, Basu and Fernald [3] obtain a point estimate for the degree of IRS in the durable manufacturing sector in the US economy of 0.33, with standard deviation 0.11.

Proposition 2. *Under Assumption 1, consider the following critical values of σ, Θ and χ:*

$$\begin{array}{rcl}
\sigma^{sup}(\gamma) & \equiv & \frac{[1+\chi-\mathcal{C}(1-\gamma)]\left\{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\Theta[\rho+\delta(1-\alpha)]+\frac{\gamma\mathcal{C}\alpha\delta}{1+\chi}[\alpha+\chi+\gamma(1-\alpha)]\right\}}{(1+\chi)\alpha\delta\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]} \\
\sigma^{H}(\gamma) & \equiv & \frac{[1+\chi-\mathcal{C}(1-\gamma)]\left\{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\rho\Theta[\rho+\delta(1-\alpha)]+\gamma\mathcal{C}\alpha\delta\left[\frac{\rho[\alpha+\chi+\gamma(1-\alpha)]}{1+\chi}+\alpha\delta\Theta\right]\right\}}{(1+\chi)\alpha\delta[\rho+\Theta(\delta+\rho)]\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]} \\
\tilde{\Theta}(\gamma) & \equiv & \frac{\gamma^2\mathcal{C}\alpha^2\delta\left[1-(1-\gamma)\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]}{\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right][\rho+\delta(1-\alpha)]} \in (0,\bar{\Theta}) \\
\underline{\chi}(\gamma) & \equiv & \frac{\alpha\Theta-\gamma(1-\alpha)}{1-\alpha-\alpha\Theta} \in (0,3)
\end{array} \tag{34}$$

with $\sigma^{sup}(\gamma) > \sigma^H(\gamma)$. Let $\sigma^{inf}(\gamma) = \max\{\sigma^H(\gamma), \sigma_c(\gamma)\}$. Then the steady state (k^*, p^*) is locally indeterminate if and only if $\chi > \underline{\chi}(\gamma)$, $\Theta \in (\tilde{\Theta}(\gamma), \bar{\Theta})$ and $\sigma \in (\sigma^{inf}(\gamma), \sigma^{sup}(\gamma))$, while saddle-point stability holds if $\Theta \in (\tilde{\Theta}(\gamma), \bar{\Theta})$ and $\sigma > \sigma^{sup}(\gamma)$ or $\Theta < \tilde{\Theta}(\gamma)$.

Proof: See Appendix 8.3. □

Remark 5: Some comments on the occurrence of saddle-point stability are in order here. As shown in Appendix 8.3, under Assumption 1, $\mathcal{D}$ is positive and local indeterminacy may arise if and only if $\sigma < \sigma^{sup}(\gamma)$. But Lemma 1 shows that the JR utility function is locally concave if and only if $\sigma \geq \sigma_c(\gamma)$. The compatibility of these two conditions is ensured if and only if $\Theta > \tilde{\Theta}(\gamma)$. Therefore, $\mathcal{D}$ is positive and saddle-point stability holds in two cases: i) when $\Theta \in (\tilde{\Theta}(\gamma), \bar{\Theta})$ and $\sigma > \sigma^{sup}(\gamma)$, or ii) when $\Theta < \tilde{\Theta}(\gamma)$ which implies $\sigma > \sigma^{sup}(\gamma)$ under the concavity condition.

Proposition 2 shows that for any intensity $\gamma \in [0, 1]$ of income effects, there is a non-empty range of values for the parameter σ such that indeterminacy occurs. This conclusion is in sharp contrast with the results obtained in one-sector models. For example, Nishimura *et al.* [19] show that indeterminacy is ruled out in such models under both GHH ($\gamma = 0$) and KPR ($\gamma = 1$) preferences, as long as realistic parameter values are considered. Likewise, Jaimovich [13] shows in a calibrated version of the aggregate infinite-horizon model with increasing returns that local indeterminacy arises for *intermediary values* of γ, while it is ruled out when the income effect is too low (γ close to 0) or too large (γ close to 1).

Proposition 2 also implies that a Hopf bifurcation exists in the parameter space, provided that $\sigma^H(\gamma) > \sigma_c(\gamma)$. One can complete the proposition by deriving conditions under which this inequality is satisfied:

Corollary 1. *Under Assumption 1, let $\chi > \underline{\chi}(\gamma)$ and consider the critical values as given by (34) together with the following one:*

$$\hat{\Theta}(\gamma) \equiv \frac{\gamma^2\mathcal{C}\alpha^2\delta\rho\left[1-\frac{(1-\gamma)\mathcal{C}}{1+\chi}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]}{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\left[\rho[\rho+\delta(1-\alpha)]\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]-\frac{\gamma\mathcal{C}\alpha\delta(\gamma+\chi)(\delta+\rho)}{1+\chi}\right]+\gamma\mathcal{C}(\alpha\delta)^2\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]} \in (0,\bar{\Theta})$$

Denote $\underline{\Theta}(\gamma) = \max\{\hat{\Theta}(\gamma), \tilde{\Theta}(\gamma)\}$. If $\Theta \in (\underline{\Theta}(\gamma), \bar{\Theta})$, the steady state (k^*, p^*) is saddle-point stable when $\sigma > \sigma^{sup}(\gamma)$, locally indeterminate when $\sigma \in (\sigma^H(\gamma), \sigma^{sup}(\gamma))$ and totally unstable *when $\sigma \in (\sigma_c(\gamma), \sigma^H(\gamma))$. When σ crosses $\sigma^H(\gamma)$ from above a Hopf bifurcation generically occurs and gives rise to the existence of locally indeterminate (totally unstable) periodic cycles in a left (right) neighborhood of $\sigma^H(\gamma)$.*

Proof: See Appendix 8.4. □

Remark 6: Corollary 1 shows that local indeterminacy arises when $\sigma \in (\sigma^H(\gamma), \sigma^{sup}(\gamma))$ with the occurrence of a pair of purely imaginary complex eigenvalues when $\sigma = \sigma^H(\gamma)$. The Hopf bifurcation Theorem (see Grandmont [8]) then implies that there exist periodic cycles for σ in a left or right neighborhood of $\sigma^H(\gamma)$ depending on whether the bifurcation is super or sub-critical. In the super-critical case, the periodic cycles occur when the steady-state is totally unstable which implies that the periodic cycles are stable, i.e. locally indeterminate. On the contrary, in the sub-critical case, the periodic cycles occur when the steady-state is locally indeterminate. This means that the periodic cycles are totally unstable and define a corridor of stability for the steady-state and thus for the existence of an equilibrium. Indeed, any path starting from the outside of the area defined by a periodic cycle is a divergent one that will violate the transversality condition and cannot be an equilibrium.

Remind that σ and χ are the crucial parameters influencing the degree of intertemporal substitution in consumption and the Frisch elasticity of labor supply. Proposition 2 and Corollary 1 then provide clear-cut conclusions about the conditions required for local indeterminacy and the existence of sunspot-driven fluctuations in canonical two-sector models. Local indeterminacy occurs, for any given degree $\gamma \in [0, 1]$ of income effects, provided that the degree of IRS is not too small, the wage elasticity of labor supply is not too large, and the EIS in consumption is in an intermediary range. Note that the interval of values for the amount of externalities Θ compatible with local indeterminacy is quite large under the benchmark calibration $(\alpha, \rho, \delta) = (0.3, 0.01, 0.025)$ as $\underline{\Theta}(\gamma) \in [0, 0.0323)$ for $\gamma \in [0, 1]$ and $\chi \geq 0$ while $\bar{\Theta} \approx 2.33$.

As an illustration to Proposition 2 , Figure 1 plots the relevant bifurcation loci and the local indeterminacy areas in the three-dimensional space with axes given by (χ, γ, σ) in panel (a), and by $(\chi, \gamma, \epsilon_{cc})$ in panel (b). The critical values obtained for the EIS in consumption ϵ_{cc} in panel (b) are derived from the analytical expression relating σ to ϵ_{cc} at the steady-state as given by (31). Moreover, panels (c) and (d) in Figure 2 display, for each pair (χ, γ), the corresponding values for the Frisch elasticity of labor supply (panel (c)), and the income effect on labor supply (panel (d)), both evaluated at the steady state (see (30) and (31)). All these graphs are computed using the benchmark calibration $(\alpha, \rho, \delta) = (0.3, 0.01, 0.025)$ and a degree of IRS in the investment sector of $\Theta = 0.33$, the point estimate obtained by Basu and Fernald [3].

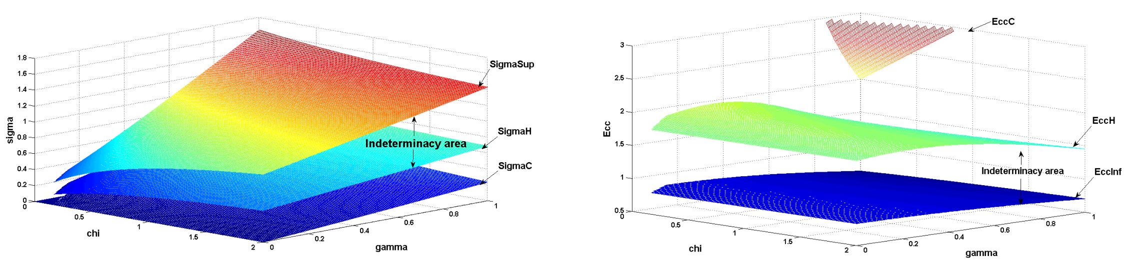

Figure 1: (a) bifurcations in the (χ, γ, σ) plane; (b) bifurcations in the $(\chi, \gamma, \epsilon_{cc})$ plane.

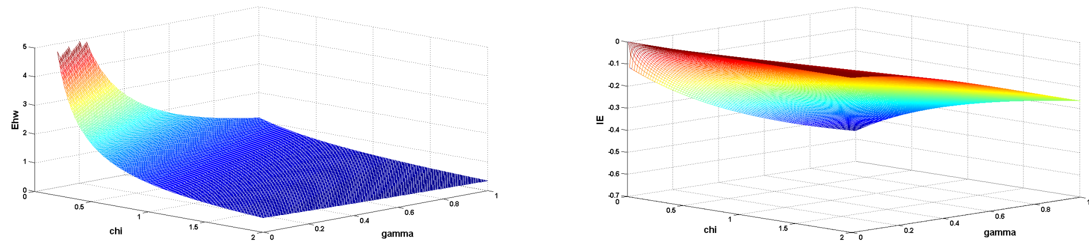

Figure 2: (c) wage elasticity of labor supply ϵ_{lw}; (d) income effect on labor supply $\varepsilon_{l\mathcal{I}}$.

As can be seen, local indeterminacy occurs for a wide range of values for the EIS in consumption ϵ_{cc}, typically ranging between 0.61 and 1.95 using our benchmark calibration. This is in line with the empirical literature, which provides estimates typically ranging between 0 and 2. Moreover, combining the results displayed in panel (b) with panels (c) and (d), it can be observed that when γ is small (close to 0), indeterminacy emerges for a wide range of values for the Frisch labor supply elasticity (ranging between 0.5 and more than 5), a wide range of values for the EIS in consumption (ranging between 0.7 and 1.7), but a moderate intensity of income effects on labor supply (ranging between 0 and -0.15). Conversely, when γ tends to 1, indeterminacy can emerge under much more significant income effects (up to a value of -0.55, obtained when χ is close to its minimum value $\underline{\chi}(1)$ consistent with indeterminacy). Yet, the maximal value for the wage-elasticity of labor is now relatively small (with a maximum value given by $\epsilon_{lw} = 1.2$).

However, it is worth pointing out that the interval of values for the structural parameter σ given in Proposition 2 and Corollary 1 varies with the size of the income effect γ. But usually we consider a constant value for σ, e.g. $\sigma = \boldsymbol{\sigma}$ for any $\gamma \leq 1$. In such a case, one may wonder whether local indeterminacy may arise for any size of the income effect $\gamma \in [0, 1]$. The answer to this question depends on the values of $\sigma^{sup}(0)$ and $\sigma^H(1)$. Clearly, a positive answer requires $\sigma^{sup}(0) > \sigma^H(1)$. We need also to satisfy the necessary condition for local indeterminacy exhibited in Proposition 2 and Corollary 1, namely $\chi > \underline{\chi}(\gamma)$ and $\Theta > \underline{\Theta}(\gamma)$, for any $\gamma \in [0, 1]$. Noting that the maximal values of $\underline{\chi}(\gamma)$ and $\underline{\Theta}(\gamma)$ are respectively $\underline{\chi}(0) = \alpha\Theta/(1 - \alpha - \alpha\Theta)$ and $\underline{\Theta}(1)$,[8] this property is satisfied if $\chi > \underline{\chi}(0)$ and $\Theta > \underline{\Theta}(1)$. We then get the following Lemma:

Lemma 2. *Under Assumption 1, let $\chi > \underline{\chi}(0)$. Then there exist $\bar{\bar{\Theta}} \in (0, \bar{\Theta})$ and $\bar{\chi} > \underline{\chi}(0)$ such that when $\Theta \in (0, \bar{\bar{\Theta}})$, $\sigma^{sup}(0) - \sigma^H(1) \lesseqgtr 0$ if and only if $\chi \lesseqgtr \bar{\chi}$.*

Proof: See Appendix 8.5. □

Let us introduce an additional technical assumption:

Assumption 2. $\bar{\bar{\Theta}} > \underline{\Theta}(1)$ *and* $\bar{\chi} < 3$.

Considering again $(\alpha, \rho, \delta) = (0.3, 0.01, 0.025)$, this Assumption easily holds as $\bar{\bar{\Theta}} \approx 0.67$, $\underline{\Theta}(1) < 0.0323$ and $\bar{\chi} \approx 1.39$. We can then finally derive the following Corollary:

[8] It can be shown indeed that $\underline{\Theta}(\gamma)$ is an increasing function of γ while $\underline{\chi}(\gamma)$ is a decreasing function.

Corollary 2. *Under Assumptions 1-2, let* $\Theta \in (\underline{\Theta}(1), \bar{\bar{\Theta}})$. *Then the following cases occur:*

a) If $\chi \in (\underline{\chi}(0), \bar{\chi})$, *there exist* $0 < \underline{\gamma} < \bar{\gamma}$ *such that the steady state is locally indeterminate in the following cases:*

i) when $\sigma \in (\sigma^H(0), \sigma^{sup}(0))$ *and* $\gamma \in [0, \underline{\gamma})$*;*

ii) when $\sigma \in (\sigma^{sup}(0), \sigma^H(1))$ *and* $\gamma \in (\underline{\gamma}, \bar{\gamma})$*;*

iii) when $\sigma \in (\sigma^H(1), \sigma^{sup}(1))$ *and* $\gamma \in (\bar{\gamma}, 1]$.

b) If $\chi > \bar{\chi}$, *the steady state is locally indeterminate for any* $\gamma \in [0,1]$ *when* $\sigma \in (\sigma^H(1), \sigma^{sup}(0))$. *Moreover, there exist* $0 < \underline{\gamma} < \bar{\gamma}$ *such that local indeterminacy also holds in the following cases:*

i) when $\sigma \in (\sigma^H(0), \sigma^H(1))$ *and* $\gamma \in [0, \underline{\gamma})$*;*

ii) when $\sigma \in (\sigma^{sup}(0), \sigma^{sup}(1))$ *and* $\gamma \in (\bar{\gamma}, 1]$.

Corollary 2 shows that there is a trade-off between the values of χ, σ and γ for the existence of local indeterminacy. When χ is low, i.e. the Frisch elasticity of labor is large, the lower (higher) the values of σ, the lower (higher) the values of γ must be for local indeterminacy to arise. The same type of results partially arises when χ is large enough, i.e. the Frisch elasticity of labor is low enough, as low (high) values of σ still require low (high) values of γ. However, local indeterminacy may also arise for any $\gamma \in [0,1]$ as long as σ admits intermediary values. As σ is inversely related to the EIS in consumption, we conclude that the size of the income effect necessary for the existence of self-fulfilling expectations strongly depends on the way the representative agent adjusts his intertemporal consumption profile. Such a conclusion is important as there is no clear evidence of the empirically realistic values of the size γ. Khan and Tsoukalas [15] provide some estimates in favor of a large income effect with $\gamma > 0.5$, while Schmitt-Grohé and Uribe [23] conclude for evidences in favor of a low income effect with values of γ close to zero. It is therefore necessary to explore our main results on a numerical basis in order to evaluate the magnitude of each structural parameter that affects the occurrence of expectations-driven fluctuations.

5 Numerical illustrations

We have shown in Proposition 2 and Corollary 1 that local indeterminacy arises under different scenarios for the values of the Frisch elasticity of labor ϵ_{lw}, the EIS in consumption ϵ_{cc} and the size of income effect. There is no consensus in the literature about ϵ_{lw} and ϵ_{cc}. Concerning ϵ_{lw}, Rogerson and Wallenius [22] and Prescott and Wallenius [21] recommend values around 3 to calibrate business cycle models, based on both theoretical considerations and cross-country tax analysis.[9] More recently, Chetty *et al.* [5] recommend on the contrary an aggregate Frisch elasticity of 0.5 on the intensive margin for labor supply. Concerning ϵ_{cc}, while early studies suggest quite low values, e.g. Campbell [4] and Kocherlakota [17], more recent estimates provide a much more contrasted view. Indeed, Mulligan [18] and Vissing-Jorgensen and Attanasio [24] repeatedly obtained estimates above unity, typically in the range $1.1 - 2.1$.

[9]See Prescott and Wallenius [21] for a discussion of the factors that make the wage elasticity of aggregate labor supply significantly differ from the corresponding elasticity at the micro level.

Let us now provide some numerical illustrations in order to check whether macroeconomic fluctuations based on self-fulfilling expectations may arise under realistic calibrations for these parameters. When $(\alpha, \rho, \delta) = (0.3, 0.01, 0.025)$, we get $(\underline{\Theta}(1), \underline{\chi}(0), \bar{\chi}) \approx (0.0263, 0.1475, 1.39)$. Let us then assume $\chi = 0.15$ and $\Theta = 0.3$ so that for any given $\gamma \in [0,1]$, $\Theta \in (\underline{\Theta}(\gamma), \bar{\Theta})$ and $\chi \in (\underline{\chi}(\gamma), \bar{\chi})$. In this configuration, we are in case a) of Corollary 2 with $\sigma^{sup}(0) < \sigma^H(1)$. We then get the following Figure covering different possible values of σ:

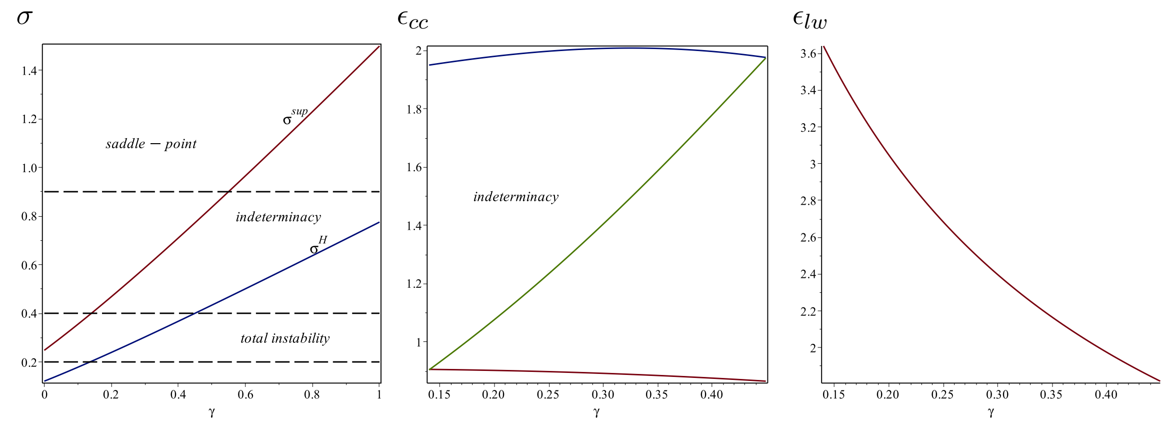

Figure 3: (a) indeterminacy areas for $\chi = 0.15$; (b) and (c) variations of ϵ_{cc} and ϵ_{lw} when $\sigma = 0.4$

It follows that local indeterminacy occurs:

i) when $\sigma = 0.2$ if $\gamma \in [0, 0.136)$,

ii) when $\sigma = 0.4$ if $\gamma \in (0.14, 0.449)$,

iii) when $\sigma = 0.9$ if $\gamma \in (0.549, 1]$.

As an illustration of configuration ii), and according to Figure 1, we find values for the EIS in consumption in line with the more recent estimates provided by Mulligan [18] and Vissing-Jorgensen and Attanasio [24]. Moreover, the values for the Frisch elasticity of labor match the recommendations of Rogerson and Wallenius [22] and Prescott and Wallenius [21].

Considering now $\chi = 1.7$, we are in case b) of Corollary 2 and we get $\sigma^H(1) \approx 0.6774$ and $\sigma^{sup}(0) \approx 0.737$. It follows therefore that if $\sigma = 0.7$, local indeterminacy arises for any $\gamma \in [0,1]$. We have indeed the following Figure:

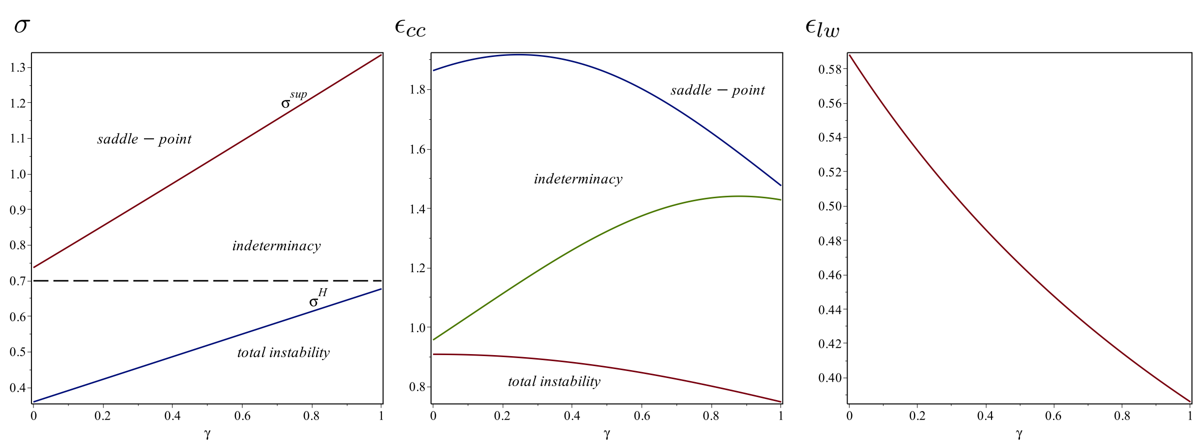

Figure 4: (a) indeterminacy for any $\gamma \in [0,1]$; (b) and (c) variations of ϵ_{cc} and ϵ_{lw} when $\sigma = 0.7$

Moreover, the EIS again belongs to an empirically realistic interval compatible with the estimates of Mulligan [18] and Vissing-Jorgensen and Attanasio [24] and the Frisch elasticity of labor is now in line with the recommendation of Chetty *et al.* [5].

Our results then prove that the existence of sunspot fluctuations can be obtained for any size of income effect as long as the values of the Frisch elasticity of labor and the EIS in consumption are adequately chosen. Moreover, in any cases, these values can be in line with the estimates provided by the recent literature.

6 Economic interpretations

The general intuition for the existence of indeterminacy in a one-sector model is quite simple.[10] Starting from the steady state, let us assume that agents expect a faster rate of accumulation. To be an equilibrium this new path would require a higher return on investment. If higher anticipated stocks of future capital raise the marginal product of capital by drawing labor out of leisure, the expected higher rate of return may be self-fulfilling. When there is a sufficient amount of increasing returns based on externalities and the Frisch elasticity of the labor supply is large enough, the movement of labor into production may be strong enough to boost the rate of return leading to self-fulfilling expectations and multiple equilibria. However, depending on the utility function, if we consider as suggested by the empirical evidence that the labor demand function is decreasing with respect to wage, there is an upper bound for the size of externalities and such a mechanism may not be admissible. As shown by Hintermaier [11, 12], with a KPR utility function ($\gamma = 1$), the concavity restrictions prevent the occurrence of this mechanism. Similarly, Nishimura *et al.* [19] prove the same impossibility result with a GHH utility function ($\gamma = 0$). In this case, the argument is not based on concavity but on the absence of income effect. As externalities are not strong enough and labor is not affected by the increased income ($\varepsilon_{l\mathcal{I}} = 0$), the expected increase of the marginal product of capital does not generate a sufficient adjustment of labor and the expectations cannot be self-fulfilling. This explains why Jaimovich [13] obtains the existence of local indeterminacy for intermediary values of γ.[11]

In two-sector models, the story is different. As shown by Benhabib and Farmer [3], when external effects in each sector depend on the aggregate output of their own sector, factor reallocations across sectors can have strong effects on marginal products. It follows that local indeterminacy can occur with much smaller externalities than those required in the one sector case, a standard positive slope for the labor demand function and under a lower variability of labor. Our main conclusions are of course compatible with a decreasing labor demand function which is obtained as soon as $\Theta < \alpha/(1-\alpha)(< \bar{\Theta})$.

We prove that the existence of sunspot fluctuations depends on a trade-off between the values of the Frisch elasticity of labor, the EIS in consumption and the size of income effect. To understand such a trade-off, let us start with Corollary 2-a). It is shown that for a given low value of χ, i.e. a large value of ϵ_{lw}, the larger the income effect, the lower the EIS in consumption for

[10] See Benhabib and Farmer [2].

[11] Recall however that his utility formulation contains an additional state variable X_t which may play a significant role for these result.

indeterminacy to arise. In order to get an intuition for such conclusions, consider the expressions of $\varepsilon_{c\mathcal{I}}$ and $\varepsilon_{l\mathcal{I}}$ evaluated at the steady state as given in (30). It is easy to check that $\varepsilon_{c\mathcal{I}}$ and $\varepsilon_{l\mathcal{I}}$ are decreasing in γ while increasing in χ.

Starting from the steady state, let us assume as previously that agents expect an increase in the future marginal return on capital leading then to a decrease in current consumption in order to invest more today and at the same time an increase in future income. As $\varepsilon_{c\mathcal{I}}$ is decreasing in γ, the lower the income effect, the larger the increase in future consumption following the larger expected income. This effect therefore generates a large fluctuation of consumption and the expectation can be self-fulfilling provided the EIS is large enough. Since the two-sector structure requires lower external effects, the adjustment of labor is sufficient to get multiple equilibria even under a low income effect. Similarly, when the income effect is larger, the increase of future consumption following a larger expected income is weaker, and the expectations can now be self-fulfilling under a lower EIS in consumption.

Let us now consider Corollary 2-b). We show here that if the value of χ is large enough, i.e. the value of ϵ_{lw} is low enough, local indeterminacy may arise for any size of income effect when the EIS in consumption has intermediary values. Following the same intuition, as $\varepsilon_{c\mathcal{I}}$ is increasing in χ, the larger expected future income implies a significant increase in future consumption that can be compatible with the decrease of present consumption if the EIS is sufficiently high. But now, as the income effect is increased by the large value of χ, this impact can be large enough no matter what is the value of $\gamma \in [0, 1]$.

7 Concluding comments

Although one-sector infinite horizon models are known to require very specific positive amount of income effect for the existence of local indeterminacy,[12] two-sector models have been shown to generate sunspot-driven business cycles under no-income effect preferences. Dufourt *et al.* [6] indeed show that when properly calibrated, the model solves several empirical puzzles traditionally associated with two-sector RBC models.[13] However, there is not yet a complete analysis of the impact of various income effects on the occurrence of local indeterminacy.

This paper provides such an analysis. We have shown that for any given size of income effect, there is a non-empty range of values for the Frisch elasticity of labor and the EIS in consumption such that indeterminacy occurs. This is in contrast to the results obtained for aggregate models both in Hintermaier [11, 12] and Nishimura *et al.* [19], in which it is shown that indeterminacy cannot occur under GHH and KPR preferences, and in Jaimovich [13], in which local indeterminacy only arises for intermediary income effects.

More precisely, we have proved that for a large enough Frisch elasticity of labor, the larger the income effect, the lower the EIS in consumption for indeterminacy to arise. On the contrary, when the Frisch elasticity of labor is low enough, local indeterminacy may arise for any size of income effect when the EIS in consumption has intermediary values. We then exhibit a clear trade-off between all these structural parameters that characterize preferences and that affect

[12]See Jaimovich [13], Nishimura *et al.* [19].

[13]See also Guo and Harrison [10], Nishimura and Venditti [20].

the occurrence of expectations-driven fluctuations. Such a conclusion appears as important as there is yet no clear empirical estimates of the size of income effect.[14]

8 Appendix

8.1 Proof of Proposition 1

Consider the steady state with $Y_I = \delta k$ and $r = (\delta + \rho)p$. Since $r = p\alpha Y_I/K_I$, we get

$$K_I = \frac{\alpha\delta}{\delta+\rho}k \tag{35}$$

Using the production function (19) for the investment good we derive

$$Y_I = \left(\frac{k}{l}\right)^{(\alpha-1)(1+\Theta)}\left(\frac{\alpha\delta}{\delta+\rho}k\right)^{1+\Theta} = \delta k$$

Solving this equation yields

$$k^* = l^{\frac{(1-\alpha)(1+\Theta)}{1-\alpha(1+\Theta)}}\left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1+\Theta}{1-\alpha(1+\Theta)}}\delta^{\frac{\Theta}{1-\alpha(1+\Theta)}} \equiv l^{\frac{(1-\alpha)(1+\Theta)}{1-\alpha(1+\Theta)}}\kappa^* \tag{36}$$

Substituting this expression into (22) we get

$$c^* = l^{\frac{1-\alpha}{1-\alpha(1+\Theta)}}\frac{\delta(1-\alpha)+\rho}{\delta+\rho}\kappa^{*\alpha} \equiv l^{\frac{1-\alpha}{1-\alpha(1+\Theta)}}\psi^* \tag{37}$$

Recall that the trade-off between consumption and leisure is described by

$$\frac{(1+\chi)l^{\chi}c^{\gamma}}{1+\chi-\gamma l^{1+\chi}c^{\gamma-1}} = w \tag{38}$$

Using (23) with (36)-(37) we get

$$(1+\chi)l^{\chi+\frac{\gamma(1-\alpha)}{1-\alpha(1+\Theta)}}\psi^{*\gamma} = (1-\alpha)l^{\frac{\alpha\Theta}{1-\alpha(1+\Theta)}}\kappa^{*\alpha}\left[1+\chi-\gamma l^{1+\chi-\frac{(1-\gamma)(1-\alpha)}{1-\alpha(1+\Theta)}}\psi^{*\gamma-1}\right]$$

If $\chi[1-\alpha(1+\Theta)] + \gamma(1-\alpha) - \alpha\Theta \neq 0$, solving this equation yields

$$l^* = \left\{\frac{(1-\alpha)\kappa^*}{\psi^{*\gamma}}\left[1+\frac{(1-\alpha)\kappa^*\gamma}{(1+\chi)\psi^*}\right]^{-1}\right\}^{\frac{1-\alpha(1+\Theta)}{\chi[1-\alpha(1+\Theta)]+\gamma(1-\alpha)-\alpha\Theta}}$$

We finally derive from (23)

$$p^* = \alpha(k^*/l^*)^{\alpha-1}$$

□

8.2 Proof of Lemma 1

Using (24) and the first order conditions (9)-(10), we get $\epsilon_{cl} = \epsilon_{lc}(c/wl)$. Using the expression of w given in (23) together with the values of k^* and l^* provided in Section 8.1 we find $wl/c = (1-\alpha)(\delta+\rho)/[\delta(1-\alpha)+\rho]$. Then at the steady state we get

$$\epsilon_{cl} = \frac{\delta(1-\alpha)+\rho}{(1-\alpha)(\delta+\rho)}\epsilon_{lc} \tag{39}$$

Using (24), we compute for the utility function as given by (14) the following elasticities:

$$\begin{aligned}
\frac{1}{\epsilon_{cc}} &= \sigma\frac{c-\gamma\frac{l^{1+\chi}}{1+\chi}c^{\gamma}}{c-\frac{l^{1+\chi}}{1+\chi}c^{\gamma}} - \gamma(1-\gamma)\frac{\frac{l^{1+\chi}}{1+\chi}c^{\gamma}}{c-\gamma\frac{l^{1+\chi}}{1+\chi}c^{\gamma}}, \quad && \frac{1}{\epsilon_{lc}} = \sigma\frac{c-\gamma\frac{l^{1+\chi}}{1+\chi}c^{\gamma}}{c-\frac{l^{1+\chi}}{1+\chi}c^{\gamma}} - \gamma \\
\frac{1}{\epsilon_{cl}} &= \frac{l^{1+\chi}c^{\gamma}}{c-\gamma\frac{l^{1+\chi}}{1+\chi}c^{\gamma}}\left[\sigma\frac{c-\gamma\frac{l^{1+\chi}}{1+\chi}c^{\gamma}}{c-\frac{l^{1+\chi}}{1+\chi}c^{\gamma}} - \gamma\right], && \frac{1}{\epsilon_{ll}} = \sigma\frac{l^{1+\chi}c^{\gamma}}{c-\frac{l^{1+\chi}}{1+\chi}c^{\gamma}} + \chi
\end{aligned} \tag{40}$$

[14]See Khan and Tsoukalas [15], Schmitt-Grohé and Uribe [23].

Obviously, normality holds as we derive from these expressions that

$$\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}} \geq 0 \text{ and } \frac{1}{\epsilon_{cl}} - \frac{1}{\epsilon_{ll}} \geq 0 \tag{41}$$

Consider now equation (39) together with the expressions given by (40). We then derive that

$$\frac{l^{1+\chi}c^{\gamma-1}}{1-\gamma\frac{l^{1+\chi}}{1+\chi}c^{\gamma-1}} = \frac{(1-\alpha)(\delta+\rho)}{\delta(1-\alpha)+\rho} \tag{42}$$

Denoting $\mathcal{C} = [(1-\alpha)(\delta+\rho)]/[\delta(1-\alpha)+\rho] < 1$, solving this equation yields

$$l^{1+\chi}c^{\gamma-1} = \frac{\mathcal{C}(1+\chi)}{1+\chi+\gamma\mathcal{C}} \tag{43}$$

and thus

$$\frac{c-\gamma\frac{l^{1+\chi}}{1+\chi}c^{\gamma}}{c-\frac{l^{1+\chi}}{1+\chi}c^{\gamma}} = \frac{1+\chi}{1+\chi-\mathcal{C}(1-\gamma)}, \quad \frac{l^{1+\chi}c^{\gamma}}{c-\frac{l^{1+\chi}}{1+\chi}c^{\gamma}} = \frac{\mathcal{C}(1+\chi)}{1+\chi-\mathcal{C}(1-\gamma)} \tag{44}$$

Using these expressions we then derive from (40):

$$\begin{aligned} \frac{1}{\epsilon_{cc}} &= \sigma\frac{1+\chi}{1+\chi-\mathcal{C}(1-\gamma)} - \gamma(1-\gamma)\frac{\mathcal{C}}{1+\chi}, & \frac{1}{\epsilon_{lc}} &= \sigma\frac{1+\chi}{1+\chi-\mathcal{C}(1-\gamma)} - \gamma \\ \frac{1}{\epsilon_{cl}} &= \frac{\mathcal{C}}{\epsilon_{lc}}, & \frac{1}{\epsilon_{ll}} &= \sigma\frac{(1+\chi)\mathcal{C}}{1+\chi-\mathcal{C}(1-\gamma)} + \chi \end{aligned} \tag{45}$$

Concavity of the utility function requires

$$\frac{1}{\epsilon_{cc}\epsilon_{ll}} - \frac{1}{\epsilon_{lc}\epsilon_{cl}} \geq 0 \quad \text{and} \quad \frac{1}{\epsilon_{cc}} \geq 0$$

Straightforward computations show that these two inequalities are satisfied if and only if

$$\sigma \geq \sigma_c(\gamma) \equiv \frac{\gamma\mathcal{C}(\gamma+\chi)[1+\chi-(1-\gamma)\mathcal{C}]}{(1+\chi)^2\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]}$$

□

8.3 Proof of Proposition 2

We start by the computation of $\mathcal{D}$ and $\mathcal{T}$ using a general formulation for $U(c,\mathcal{L})$. Consider the consumption-labor trade-off as described by (9)-(10) together with the expressions of wage and consumption as given by (22) and (23). We get the following two equations

$$U_2(c,\ell-l)l^{\alpha} = (1-\alpha)k^{\alpha}U_1(c,\ell-l) \tag{46}$$

$$cl^{\alpha-1} = k^{\alpha-1}\left[k - \left(\frac{k}{l}\right)^{1-\alpha}p^{-1/\Theta}\right] \tag{47}$$

Total differentiation of (46) gives

$$\frac{dc}{c}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right) + \frac{dl}{l}\left(\frac{1}{\epsilon_{ll}} - \frac{1}{\epsilon_{cl}} + \alpha\right) = \alpha\frac{dk}{k} \tag{48}$$

Total differentiation of (47) gives

$$\frac{dc}{c} - (1-\alpha)\frac{dl}{l} = -(1-\alpha)\frac{dk}{k} + \frac{k^*}{k^*-K_I^*}\frac{dk}{k} - \frac{K_I^*}{k^*-K_I^*}\left[(1-\alpha)\left(\frac{dk}{k} - \frac{dl}{l}\right) - \frac{1}{\Theta}\frac{dp}{p}\right] \tag{49}$$

At the steady state we know that $(\delta+\rho)p = r$ with $r = p\alpha Y_I/K_I = p\alpha\delta k/K_I$. We then derive $K_I^* = \alpha\delta k^*/(\delta+\rho)$ and thus

$$\frac{k^*}{k^*-K_I^*} = \frac{\delta+\rho}{\rho+\delta(1-\alpha)}, \quad \frac{K_I^*}{k^*-K_I^*} = \frac{\alpha\delta}{\rho+\delta(1-\alpha)}$$

Equation (48) then becomes:

$$[\rho+\delta(1-\alpha)]\frac{dc}{c} - (1-\alpha)(\delta+\rho)\frac{dl}{l} = \alpha(\delta+\rho)\frac{dk}{k} + \frac{\alpha\delta}{\Theta}\frac{dp}{p} \tag{50}$$

From (48) we derive

$$\frac{dl}{l} = -\frac{dc}{c}\frac{\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}}{\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha} + \frac{dk}{k}\frac{\alpha}{\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha} \tag{51}$$

Substituting this expression into (50) gives

$$\begin{aligned}\frac{dc}{c} &= \frac{\alpha(\delta+\rho)\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+1\right)}{[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)}\frac{dk}{k}\\ &+ \frac{\alpha\delta\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)}{\Theta\left[[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)\right]}\frac{dp}{p}\end{aligned} \tag{52}$$

Substituting (52) into (51) finally gives

$$\begin{aligned}\frac{dl}{l} &= -\frac{\alpha\left[(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)-[\rho+\delta(1-\alpha)]\right]}{[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)}\frac{dk}{k}\\ &- \frac{\alpha\delta\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)}{\Theta\left[[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)\right]}\frac{dp}{p}\end{aligned} \tag{53}$$

We then conclude from this

$$\begin{aligned}\frac{\partial c}{\partial k}\frac{k^*}{c^*} &= \frac{\alpha(\delta+\rho)\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+1\right)}{[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)}\\ \frac{\partial c}{\partial p}\frac{p^*}{c^*} &= \frac{\alpha\delta\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)}{\Theta\left[[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)\right]}\\ \frac{\partial l}{\partial k}\frac{k^*}{l^*} &= -\frac{\alpha\left[(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)-[\rho+\delta(1-\alpha)]\right]}{[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)}\\ \frac{\partial l}{\partial p}\frac{p^*}{l^*} &= -\frac{\alpha\delta\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)}{\Theta\left[[\rho+\delta(1-\alpha)]\left(\frac{1}{\epsilon_{ll}}-\frac{1}{\epsilon_{cl}}+\alpha\right)+(1-\alpha)(\delta+\rho)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{lc}}\right)\right]}\end{aligned} \tag{54}$$

Recall now that $r = \alpha(k/l)^{\alpha-1}$ and $Y_I = p^{-(1+\Theta)/\Theta}$. Using again the steady state relationships $Y_I = \delta k$ and $(\delta+\rho)p = r$, we derive

$$\frac{dY_I}{dp}\frac{p^*}{Y_I^*} = -\frac{1+\Theta}{\Theta}, \quad \frac{dr}{dk}\frac{Y_I^*}{p^*} = -\delta(1-\alpha)(\delta+\rho)\left(1-\frac{dl}{dk}\frac{k^*}{l^*}\right), \quad \frac{dr}{dp} = (1-\alpha)(\delta+\rho)\frac{dl}{dp}\frac{p^*}{l^*} \tag{55}$$

Linearizing the dynamical system (26) around the steady state leads to the following Jacobian matrix

$$\mathcal{J} = \begin{pmatrix} -\delta & -\frac{1+\Theta}{\Theta}\frac{Y_I^*}{p^*} \\ -\frac{\frac{\partial r}{\partial k}+\delta\left[\frac{1}{\epsilon_{cc}}\frac{\partial c}{\partial k}\frac{p^*}{c^*}-\frac{1}{\epsilon_{lc}}\frac{\partial l}{\partial k}\frac{p^*}{l^*}\right]}{E(k^*,p^*)} & \frac{\delta+\rho-\frac{\partial r}{\partial p}-\frac{1+\Theta}{\Theta}\frac{Y_I^*}{p^*}\left[\frac{1}{\epsilon_{cc}}\frac{\partial c}{\partial k}\frac{p^*}{c^*}-\frac{1}{\epsilon_{lc}}\frac{\partial l}{\partial k}\frac{p^*}{l^*}\right]}{E(k^*,p^*)} \end{pmatrix}$$

with $E(k,p)$ as given by (27). The associated characteristic polynomial is then given by (32) with the Determinant and Trace of the Jacobian matrix as defined by (33). Using (31), (54) and (55) we finally derive after straightforward simplifications

$$\begin{aligned}\mathcal{D}(\gamma) &= \frac{\delta(\delta+\rho)(1+\chi+\gamma\mathcal{C})[\rho+\delta(1-\alpha)]\left[\frac{(1-\alpha)(\gamma+\chi)}{1+\chi}-\alpha\Theta\right]}{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\left[\Theta[\rho+\delta(1-\alpha)]-\frac{\sigma(1+\chi)\alpha\delta}{1+\chi-\mathcal{C}(1-\gamma)}\right]+\frac{\gamma\mathcal{C}\alpha\delta}{1+\chi}[\alpha+\chi+\gamma(1-\alpha)]}\\ \mathcal{T}(\gamma) &= \frac{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\left[\rho\Theta[\rho+\delta(1-\alpha)]-\frac{\sigma(1+\chi)\alpha\delta}{1+\chi-\mathcal{C}(1-\gamma)}[\rho+\Theta(\delta+\rho)]\right]+\gamma\mathcal{C}\alpha\delta\left[\frac{\rho[\alpha+\chi+\gamma(1-\alpha)]}{1+\chi}+\alpha\delta\Theta\right]}{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\left[\Theta[\rho+\delta(1-\alpha)]-\frac{\sigma(1+\chi)\alpha\delta}{1+\chi-\mathcal{C}(1-\gamma)}\right]+\frac{\gamma\mathcal{C}\alpha\delta}{1+\chi}[\alpha+\chi+\gamma(1-\alpha)]}\end{aligned} \tag{56}$$

Note that if $\Theta = 0$ we conclude under the concavity condition $\sigma \geq \sigma_c(\gamma)$ that $\mathcal{D} < 0$, and the steady state is always saddle-point stable, i.e. locally determinate.

Assume first that

$$\frac{(1-\alpha)(\gamma+\chi)}{1+\chi} - \alpha\Theta > 0 \text{ or equivalently } \chi > \frac{\alpha\Theta-\gamma(1-\alpha)}{1-\alpha-\alpha\Theta} \equiv \underline{\chi}(\gamma) \tag{57}$$

To keep reasonable values for the external effect we assume from here that $\Theta < \bar{\Theta} \equiv (1-\alpha)/\alpha$ and thus $\underline{\chi}(\gamma) > 0$. Then $\mathcal{D} > 0$ if and only if its denominator is positive, namely if and only if

$$\sigma < \sigma^{sup}(\gamma) \equiv \frac{[1+\chi-\mathcal{C}(1-\gamma)]\left\{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\Theta[\rho+\delta(1-\alpha)]+\frac{\gamma\mathcal{C}\alpha\delta}{1+\chi}[\alpha+\chi+\gamma(1-\alpha)]\right\}}{(1+\chi)\alpha\delta\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]} \tag{58}$$

But then local indeterminacy arises if and only if $\mathcal{T}(\gamma) < 0$, namely if and only if its numerator is negative, i.e.

$$\sigma > \sigma^H(\gamma) \equiv \frac{[1+\chi-\mathcal{C}(1-\gamma)]\left\{\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\rho\Theta[\rho+\delta(1-\alpha)]+\gamma\mathcal{C}\alpha\delta\left[\frac{\rho[\alpha+\chi+\gamma(1-\alpha)]}{1+\chi}+\alpha\delta\Theta\right]\right\}}{(1+\chi)\alpha\delta[\rho+\Theta(\delta+\rho)]\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]} \tag{59}$$

Obvious computations show that $\sigma^{sup}(\gamma) > \sigma^H(\gamma)$ for any $\gamma \in [0,1]$. We need however to check that $\sigma^{sup}(\gamma) > \sigma_c(\gamma)$ in order to be able to have a compatibility between the concavity property of the utility function at the steady state $\sigma \geq \sigma_c(\gamma)$ and the condition for local indeterminacy $\sigma < \sigma^{sup}(\gamma)$. Tedious but straightforward computations yield $\sigma^{sup}(\gamma) > \sigma_c(\gamma)$ if and only if

$$\Theta > \tilde{\Theta}(\gamma) \equiv \frac{\gamma^2\mathcal{C}\alpha^2\delta\left[1-\frac{(1-\gamma)\mathcal{C}}{1+\chi}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]}{\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right][\rho+\delta(1-\alpha)]} \tag{60}$$

Under Assumption 1 we have $\tilde{\Theta}'(\gamma) > 0$ and $\tilde{\Theta}(\gamma) < \bar{\Theta}$ for any $\gamma \in [0,1]$.

Denoting $\sigma^{inf}(\gamma) = \max\{\sigma^H(\gamma), \sigma_c(\gamma)\}$, we have proved that under condition (57), for any given $\gamma \in [0,1]$, local indeterminacy occurs if and only if $\Theta \in (\tilde{\Theta}(\gamma), \bar{\Theta})$ and $\sigma \in (\sigma^{inf}(\gamma), \sigma^{sup}(\gamma))$. Obviously, recalling that Lemma 1 shows that the JR utility function is locally concave if and only if $\sigma \geq \sigma_c(\gamma)$, we derive from (58) and (60) that $\mathcal{D}$ is negative and the steady state (k^*, p^*) is saddle-point stable in two cases: i) when $\Theta \in (\tilde{\Theta}(\gamma), \bar{\Theta})$ and $\sigma > \sigma^{sup}(\gamma)$, or ii) when $\Theta < \tilde{\Theta}(\gamma)$ which implies $\sigma(\geq \sigma_c(\gamma)) > \sigma^{sup}(\gamma)$.

Let us consider now the case in which

$$\frac{(1-\alpha)(\gamma+\chi)}{1+\chi} - \alpha\Theta < 0 \text{ or equivalently } \chi < \frac{\alpha\Theta-\gamma(1-\alpha)}{1-\alpha-\alpha\Theta} \equiv \underline{\chi}(\gamma) \tag{61}$$

We need to assume here that $\Theta > \gamma(1-\alpha)/\alpha$ and thus that $\gamma < 1$ to get a compatibility with the assumption $\Theta < \bar{\Theta}$. Following the same argument as previously, we conclude now that local indeterminacy arises if $\sigma > \sigma^{sup}(\gamma)$ and $\sigma < \sigma^H(\gamma)$. But such a configuration is not possible as $\sigma^{sup}(\gamma) > \sigma^H(\gamma)$ for any $\gamma \in [0,1]$. It follows that under condition (61), the steady state (k^*, p^*) is saddle-point stable when $\sigma < \sigma^{sup}(\gamma)$, totally unstable when $\sigma > \sigma^{sup}(\gamma)$ and is ruled out.

We conclude therefore that for any given $\gamma \in [0,1]$, local indeterminacy arises if and only if $\chi > \underline{\chi}(\gamma)$, $\Theta \in (\tilde{\Theta}(\gamma), \bar{\Theta})$ and $\sigma \in (\sigma^{inf}(\gamma), \sigma^{sup}(\gamma))$.

□

8.4 Proof of Corollary 1

Taking into account the concavity condition as given in Lemma 1, the existence of a Hopf bifurcation requires the bound $\sigma^H(\gamma)$ as given in (34) to be larger than $\sigma_c(\gamma)$. We then get

$\sigma^H(\gamma) > \sigma_c(\gamma)$ if and only if

$$\Theta g(\rho,\gamma,\chi) > \gamma^2\mathcal{C}\alpha^2\delta\rho\left[1-\frac{(1-\gamma)\mathcal{C}}{1+\chi}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]$$

with

$$\begin{aligned} g(\rho,\gamma,\chi) = & \left[\alpha+\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]\left[\rho[\rho+\delta(1-\alpha)]\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]-\frac{\gamma\mathcal{C}\alpha\delta(\gamma+\chi)(\delta+\rho)}{1+\chi}\right] \\ & + \gamma\mathcal{C}(\alpha\delta)^2\left[\chi+\gamma\mathcal{C}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right] \end{aligned}$$

Under Assumption 1 we have $g(\rho,\gamma,\chi) > 0$ for any $\gamma \in [0,1]$. It follows that $\sigma^H(\gamma) > \sigma_c(\gamma)$ if and only if

$$\Theta > \hat{\Theta}(\gamma) \equiv \frac{\gamma^2\mathcal{C}\alpha^2\delta\rho\left[1-\frac{(1-\gamma)\mathcal{C}}{1+\chi}\left(2-\frac{\mathcal{C}(1-\gamma)}{1+\chi}\right)\right]}{g(\rho,\gamma,\chi)}$$

Assumption 1 also implies $\hat{\Theta}'(\gamma) > 0$ and $\hat{\Theta}(\gamma) < \bar{\Theta}$ for any $\gamma \in [0,1]$. The result follows from Proposition 2 considering $\underline{\Theta}(\gamma) = \max\{\hat{\Theta}(\gamma), \tilde{\Theta}(\gamma)\}$.

□

8.5 Proof of Lemma 2

The maximal value of $\underline{\chi}(\gamma)$ is $\underline{\chi}(0) = \alpha\Theta/(1-\alpha-\alpha\Theta)$. We then assume $\chi > \underline{\chi}(0)$ in order to ensure $\chi > \underline{\chi}(\gamma)$ for any $\gamma \in [0,1]$. Let us consider the following two critical values

$$\begin{aligned} \sigma^{sup}(0) &\equiv \frac{\Theta\{\alpha\rho+\chi[\rho+\delta(1-\alpha)]\}}{(1+\chi)\alpha\delta} \\ \sigma^H(1) &\equiv \frac{(\alpha+\chi+2\mathcal{C})\rho\Theta[\rho+\delta(1-\alpha)]+\mathcal{C}\alpha\delta(\rho+\alpha\delta\Theta)}{\alpha\delta[\rho+\Theta(\delta+\rho)](\alpha+\chi+2\mathcal{C})} \end{aligned} \tag{62}$$

We easily get

$$\lim_{\chi\to+\infty}\sigma^{sup}(0) = \frac{\Theta[\rho+\delta(1-\alpha)]}{\alpha\delta} > \lim_{\chi\to+\infty}\sigma^H(1) = \frac{\rho\Theta[\rho+\delta(1-\alpha)]}{\alpha\delta[\rho+\Theta(\delta+\rho)]} \tag{63}$$

Similarly, we have

$$\begin{aligned} \sigma^{sup}(0)|_{\chi=\underline{\chi}(0)} &= \frac{\Theta[\rho+\Theta(\delta+\rho)]}{\delta} \\ \sigma^H(1)|_{\chi=\underline{\chi}(0)} &\equiv \frac{\rho\Theta[\rho+\delta(1-\alpha)]}{\alpha\delta[\rho+\Theta(\delta+\rho)]} + \frac{\mathcal{C}(1-\alpha-\alpha\Theta)(\rho+\alpha\delta\Theta)}{[\rho+\Theta(\delta+\rho)][\alpha(1-\alpha)(1+\Theta)+2\mathcal{C}(1-\alpha-\alpha\Theta)]} \end{aligned}$$

It follows obviously that

$$\lim_{\Theta\to 0}\sigma^{sup}(0)|_{\chi=\underline{\chi}(0)} = 0 < \lim_{\Theta\to 0}\sigma^H(1)|_{\chi=\underline{\chi}(0)} \equiv \frac{\mathcal{C}}{\alpha+2\mathcal{C}}$$

while

$$\lim_{\Theta\to\bar{\Theta}}\sigma^{sup}(0)|_{\chi=\underline{\chi}(0)} = \frac{(1-\alpha)[\rho+\delta(1-\alpha)]}{\alpha^2\delta} > \lim_{\Theta\to\bar{\Theta}}\sigma^H(1)|_{\chi=\underline{\chi}(0)} \equiv \frac{(1-\alpha)\rho}{\alpha\delta}$$

Therefore, there exists $\bar{\bar{\Theta}} \in (0,\bar{\Theta})$ such that if $\Theta \in (0,\bar{\bar{\Theta}})$, then $\sigma^{sup}(0)|_{\chi=\underline{\chi}(0)} < \sigma^H(1)|_{\chi=\underline{\chi}(0)}$. Based on this result and using (63), we conclude that there also exists $\bar{\chi} \in (\underline{\chi}(0),+\infty)$ such that when $\Theta \in (0,\bar{\bar{\Theta}})$, $\sigma^{sup}(0) - \sigma^H(1) \lesseqgtr 0$ if and only if $\chi \lesseqgtr \bar{\chi}$.

□

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