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HJB equations in infinite dimension and optimal control of stochastic evolution equations via generalized Fukushima decomposition

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Abstract

A stochastic optimal control problem driven by an abstract evolution equation in a separable Hilbert space is considered. Thanks to the identification of the mild solution of the state equation as ν-weak Dirichlet process, the value processes is proved to be a real weak Dirichlet process. The uniqueness of the corresponding decomposition is used to prove a verification theorem. Through that technique several of the required assumptions are milder than those employed in previous contributions about non-regular solutions of Hamilton-Jacobi-Bellman equations.

KEY WORDS AND PHRASES: Weak Dirichlet processes in infinite dimension; Stochastic evolution equations; Generalized Fukushima decomposition; Stochastic optimal control in Hilbert spaces.

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1 Introduction

The goal of this paper is to show that, if we carefully exploit some recent developments in stochastic calculus in infinite dimension, we can weaken some of the hypotheses typically demanded in the literature of non-regular solutions of Hamilton-Jacobi-Bellman (HJB) equations to prove verification theorems and optimal syntheses of stochastic optimal control problems in Hilbert spaces.

As well-known, the study of a dynamic optimization problem can be linked, via the dynamic programming to the analysis of the related HJB equation, that is, in the context we are interested in, a second order infinite dimension PDE. When this approach can be successfully applied, one can prove a verification theorem and express the optimal control in feedback form (that is, at any time, as a function of the state) using the solution of the HJB equation. In this case the latter can be identified with the value function of the problem.

In the regular case (i.e. when the value function is $C^{1,2}$, see for instance Chapter 2 of [12]) the standard proof of the verification theorem is based on the Itô formula. In this paper we show that some recent results in stochastic calculus, in particular Fukushima-type decompositions explicitly suited for the infinite dimensional context, can be used to prove the same kind of result for less regular solutions of the HJB equation.

The idea is the following. In a previous paper ([13]) the authors introduced the class of *ν-weak Dirichlet processes* (the definition is recalled in Section 2, ν is a Banach space strictly associated with a suitable subspace ν_0 of H) and showed that convolution type processes, and in particular mild solutions of infinite dimensional stochastic evolution equations (see e.g. [6], Chapter 4), belong to this class. By applying this result to the solution of the state equation of a class of stochastic optimal control problems in infinite dimension we are able to show that the value process, that is the value of any given solution of the HJB equation computed on the trajectory taken into account [1], is a (real-valued) *weak Dirichlet processes* (with respect to a given filtration), a notion introduced in [11] and subsequently analyzed in [26]. Such a process can be written as the sum of a local martingale and a martingale orthogonal process, i.e. having zero covariation with every continuous local martingale. Such decomposition is unique and in Theorem 3.7, we exploit that uniqueness property to characterize the martingale part of the value process as a suitable stochastic integral with respect to a Girsanov-transformed Wiener process which allows to obtain a substitute of the Itô-Dynkin formula for solutions of the Hamilton-Jacobi-Bellman equation. This is possible when the value process associated to the optimal control problem can be expressed by a $C^{0,1}([0,T[\times H)$ function of the state process, with however a stronger regularity on the first derivative. We finally use this expression to prove the verification result stated in Theorem 4.1[2].

[1] The expression *value process* is sometime used for the value of the value function computed on the trajectory, often the two definition equal but it not always the case.

[2] A similar approach is used, when H is finite-dimensional, in [25]. In that case things are

We think the interest of our contribution is twofold. On the one hand we show that recent developments in stochastic calculus in Banach spaces, see for instance [9, 10], from which we adopt the framework related to generalized covariations and Itô-Fukushima formulae, but also other approaches as [5, 28, 35] may have important control theory counterpart applications. On the other hand the method we present allows to improve some previous verification results weakening a series of hypotheses.

We discuss here this second point in detail. There are several ways to introduce non-regular solutions of second order HJB equations in Hilbert spaces. They are more precisely surveyed in [12] but they essentially are viscosity solutions, strong solutions and the study of the HJB equation through backward SDEs. *Viscosity solutions* are defined, as in the finite-dimensional case, using test functions that locally "touch" the candidate solution. The viscosity solution approach was first adapted to the second order Hamilton Jacobi equation in Hilbert space in [29, 30, 31] and then, for the "unbounded" case (i.e. including a possibly unbounded generator of a strongly continuous semigroup in the state equation, see e.g. equation (5)) in [34]. Several improvements of those pioneering studies have been published, including extensions to several specific equations but, differently from what happens in the finite-dimensional case, there are no verification theorems available at the moment for stochastic problems in infinite-dimension that use the notion of viscosity solution. The *backward SDE approach* can be applied when the mild solution of the HJB equation can be represented using the solution of a forward-backward system. It was introduced in [32] in the finite dimensional setting and developed in several works, among them [7, 15, 16, 17, 18]. This method only allows to find optimal feedbacks in classes of problems satisfying a specific "structural condition", imposing, roughly speaking, that the control acts within the image of the noise. The same limitation concerns the L^2_μ *approach* introduced and developed in [1] and [20].

In the *strong solutions* approach, first introduced in [2], the solution is defined as a proper limit of solutions of regularized problems. Verification results in this framework are given in [21, 22, 23, 24]. They are collected and refined in Chapter 4 of [12]. The results obtained using *strong solutions* are the main term of comparison for ours both because in this context the verification results are more developed and because we partially work in the same framework by approximating the solution of the HJB equation using solutions of regularized problems. With reference to them our method has some advantages [3]: (i) the assumptions on the cost structure are milder, notably they do not include any continuity assumption on the running cost that is only asked to be a measur-

simpler and there is not need to use the notion of ν-weak Dirichlet processes and and results that are specifically suited for the infinite dimensional case. In that case ν_0 will be isomorphic to the full space H.

[3] Results for specific cases, as boundary control problems and reaction-diffusion equation (see [3, 4]) cannot be treated at the moment with the method we present here.

able function; moreover the admissible controls are only asked to verify, together with the related trajectories, a quasi-integrability condition of the functional, see Hypothesis 3.3 and the subsequent paragraph; (ii) we work with a bigger set of approximating functions because we do not require the approximating functions and their derivatives to be uniformly bounded; (iii) the convergence of the derivatives of the approximating solution is not necessary and it is replaced by the weaker condition (16). This convergence, in different possible forms, is unavoidable in the standard structure of the strong solutions approach and it is avoided here only thanks to the use of Fukushima decomposition in the proof. In terms of the last just mentioned two points, our notion of solution is weaker than those used in the mentioned works, we need nevertheless to assume that the gradient of the solution of the HJB equation is continuous as an $D(A^*)$-valued function.

The paper proceeds as follows. Section 2 is devoted to some preliminary notions, notably the definition of ν-weak-Dirichlet process and some related results. Section 3 focuses on the optimal control problem and the related HJB equation. It includes the key decomposition Theorem 3.7. Section 4 concerns the verification theorem.

2 Some preliminary definitions and result

Consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Fix $T > 0$ and $s \in [0, T[$. Let $\{\mathscr{F}_t^s\}_{t \geq s}$ be a filtration satisfying the usual conditions. Each time we use expressions as "adapted", "martingale", etc... we always mean "with respect to the filtration $\{\mathscr{F}_t^s\}_{t \geq s}$".

Given a metric space S we denote by $\mathscr{B}(S)$ the Borel σ-field on S. Consider two real Hilbert spaces H and G. By default we assume that all the processes $\mathbb{X} \colon [s,T] \times \Omega \to H$ are measurable functions with respect to the product σ-algebra $\mathscr{B}([s,T]) \otimes \mathscr{F}$ with values in $(H, \mathscr{B}(H))$. Similar conventions are done for G-valued processes. We denote by $H \hat{\otimes}_\pi G$ the projective tensor product of H and G, see [33] for details.

Definition 2.1. *A real process $X \colon [s,T] \times \Omega \to \mathbb{R}$ is called weak Dirichlet process if it can be written as $X = M + A$, where M is a local martingale and A is a process such that $A(s) = 0$ and $[A, N] = 0$ for every continuous local martingale N.*

The following result is proved in Remarks 3.5 and 3.2 of [26].

Theorem 2.2. *1. The decomposition described in Definition 2.1 is unique.*

2. A semimartingale is a weak Dirichlet process.

We recall that, following [8, 10], a *Chi-subspace* (of $(H \hat{\otimes}_\pi G)^*$) is defined as any Banach subspace $(\chi, |\cdot|_\chi)$ which is continuously embedded into $(H \hat{\otimes}_\pi G)^*$

and, following [13], given a Chi-subspace χ we introduce the notion of χ-covariation as follows.

Definition 2.3. *Given two process $\mathbb{X}\colon [s,T] \to H$ and $\mathbb{X}\colon [s,T] \to G$, we say that $(\mathbb{X},\mathbb{Y})$ admits a χ-covariation if the two following conditions are satisfied.*

H1 *For any sequence of positive real numbers $\epsilon_n \searrow 0$ there exists a subsequence ϵ_{n_k} such that*

$$\sup_k \int_s^T \frac{|(J\left(\mathbb{X}(r+\epsilon_{n_k}) - \mathbb{X}(r)\right) \otimes \left(\mathbb{Y}(r+\epsilon_{n_k}) - \mathbb{Y}(r)\right))|_{\chi^*}}{\epsilon_{n_k}} dr \ < \infty \ a.s., \tag{1}$$

where $J : H\hat{\otimes}_\pi G \longrightarrow (H\hat{\otimes}_\pi G)^{**}$ is the canonical injection between a space and its bidual.

H2 *If we denote by $[\mathbb{X},\mathbb{Y}]^\epsilon_\chi$ the application*

$$\begin{cases} [\mathbb{X},\mathbb{Y}]^\epsilon_\chi : \chi \longrightarrow \mathcal{C}([s,T]) \\ \phi \mapsto \displaystyle\int_s^\cdot \ {}_\chi\!\left\langle \phi, \frac{J\left((\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r))\right)}{\epsilon} \right\rangle_{\chi^*} dr, \end{cases} \tag{2}$$

the following two properties hold.

(i) There exists an application, denoted by $[\mathbb{X},\mathbb{Y}]_\chi$, defined on χ with values in $\mathcal{C}([s,T])$, satisfying

$$[\mathbb{X},\mathbb{Y}]^\epsilon_\chi(\phi) \xrightarrow[\epsilon \longrightarrow 0_+]{ucp} [\mathbb{X},\mathbb{Y}]_\chi(\phi), \tag{3}$$

for every $\phi \in \chi \subset (H\hat{\otimes}_\pi G)^*$.

(ii) There exists a Bochner measurable process $\widetilde{[\mathbb{X},\mathbb{Y}]}_\chi : \Omega \times [s,T] \longrightarrow \chi^*$, such that

- *for almost all $\omega \in \Omega$, $\widetilde{[\mathbb{X},\mathbb{Y}]}_\chi(\omega,\cdot)$ is a (càdlàg) bounded variation process,*
- *$\widetilde{[\mathbb{X},\mathbb{Y}]}_\chi(\cdot,t)(\phi) = [\mathbb{X},\mathbb{Y}]_\chi(\phi)(\cdot,t)$ a.s. for all $\phi \in \chi$, $t \in [s,T]$.*

If $(\mathbb{X},\mathbb{Y})$ admits a χ-covariation we call $\widetilde{[\mathbb{X},\mathbb{Y}]}$ χ-covariation of $(\mathbb{X},\mathbb{Y})$. If $\widetilde{[\mathbb{X},\mathbb{Y}]}$ vanishes we also write that $[\mathbb{X},Y]_\chi = 0$. We say that a process $\mathbb{X}$ admits a χ-quadratic variation if $(\mathbb{X},\mathbb{X})$ admits a χ-covariation. In that case $\widetilde{[\mathbb{X},\mathbb{X}]}$ is called χ-quadratic variation of $\mathbb{X}$.

Definition 2.4. *Let H and G be two separable Hilbert spaces. Let $\nu \subseteq (H\hat{\otimes}_\pi G)^*$ be a Chi-subspace. A continuous adapted H-valued process $\mathbb{A}\colon [s,T]\times\Omega \to H$ is said to be* ν-martingale-orthogonal *if $[\mathbb{A},\mathbb{N}]_\nu = 0$, for any G-valued continuous local martingale $\mathbb{N}$.*

Lemma 2.5. *Let H and G be two separable Hilbert spaces, $\mathbb{V}\colon [s,T]\times\Omega\to H$ a bounded variation process. Then the two items below hold.*

1. *Given any continuous process $\mathbb{Z}\colon [s,T]\times\Omega\to G$ and any Chi-subspace $\nu\subseteq (H\hat{\otimes}_\pi G)^*$, we have $[\mathbb{V},\mathbb{Z}]_\nu = 0$.*
2. *In particular, for any any Chi-subspace $\nu\subseteq (H\hat{\otimes}_\pi G)^*$, $\mathbb{V}$ is ν-martingale-orthogonal.*

Proof. By Lemma 3.2 of [13] it is enough to show that

$$A(\varepsilon) := \int_s^T \sup_{\substack{\Phi\in\nu,\\ \|\Phi\|_\nu\le 1}} \left|\left\langle J\Big((\mathbb{V}(t+\varepsilon)-\mathbb{V}(t))\otimes(\mathbb{Z}(t+\varepsilon)-\mathbb{Z}(t))\Big),\Phi\right\rangle\right| \,\mathrm{d}t \xrightarrow{\varepsilon\to 0} 0$$

in probability (the processes are extended on $]T,T+\varepsilon]$ by defining, for instance, $\mathbb{Z}(t)=\mathbb{Z}(T)$ for any $t\in]T,T+\varepsilon]$). Now, since ν is continuously embedded in $(H\hat{\otimes}_\pi G)^*$, there exists a constant C such that $\|\cdot\|_{(H\hat{\otimes}_\pi G)^*}\le C\|\cdot\|_\nu$ so that

$$\begin{aligned}
A(\varepsilon) &\le C\int_s^T \sup_{\substack{\Phi\in\nu,\\ \|\Phi\|_{(H\hat{\otimes}_\pi G)^*}\le 1}} \left|\left\langle J\Big((\mathbb{V}(t+\varepsilon)-\mathbb{V}(t))\otimes(\mathbb{Z}(t+\varepsilon)-\mathbb{Z}(t))\Big),\Phi\right\rangle\right| \,\mathrm{d}t\\
&\le C\int_s^T \left\| J\Big((\mathbb{V}(t+\varepsilon)-\mathbb{V}(t))\otimes(\mathbb{Z}(t+\varepsilon)-\mathbb{Z}(t))\Big)\right\|_{(H\hat{\otimes}_\pi G)^{**}} \mathrm{d}t\\
&= C\int_s^T \left\| \Big((\mathbb{V}(t+\varepsilon)-\mathbb{V}(t))\otimes(\mathbb{Z}(t+\varepsilon)-\mathbb{Z}(t))\Big)\right\|_{(H\hat{\otimes}_\pi G)} \mathrm{d}t\\
&= C\int_s^T \|(\mathbb{V}(t+\varepsilon)-\mathbb{V}(t))\|_H \,\|(\mathbb{Z}(t+\varepsilon)-\mathbb{Z}(t))\|_G \,\mathrm{d}t, \qquad (4)
\end{aligned}$$

where the last step follows by Proposition 2.1 page 16 of [33]. Now, denoting $t\mapsto |||\mathbb{Y}|||(t)$ the real total variation function of an H-valued bounded variation function $\mathbb{Y}$ defined on the interval $[s,T]$ we get

$$\|\mathbb{Y}(t+\varepsilon)-\mathbb{Y}(t)\| = \left\|\int_t^{t+\varepsilon}\mathrm{d}Y(r)\right\| \le \int_t^{t+\varepsilon)} \mathrm{d}|||Y|||(r).$$

So, by using Fubini's theorem in (4),

$$A(\varepsilon)\le C\delta(\mathbb{Z};\varepsilon)\int_s^{T+\varepsilon}\mathrm{d}|||\mathbb{V}|||(r),$$

where $\delta(\mathbb{Z};\varepsilon)$ is the modulus of continuity of $\mathbb{Z}$. Finally this converges to zero almost surely and then in probability. □

Definition 2.6. *Let H and G be two separable Hilbert spaces. Let $\nu\subseteq(H\hat{\otimes}_\pi G)^*$ be a Chi-subspace. A continuous H-valued process $\mathbb{X}\colon[s,T]\times\Omega\to H$ is called* ν-weak-Dirichlet process *if it is adapted and there exists a decomposition $\mathbb{X}=\mathbb{M}+\mathbb{A}$ where* *(i)* $\mathbb{M}$ *is an H-valued continuous local martingale,*

(ii) $\mathbb{A}$ *is an ν-martingale-orthogonal process with $\mathbb{A}(s) = 0$.*

The theorem below was the object of Theorem 3.19 of [13].

Theorem 2.7. *Let ν_0 be a Banach subspace continuously embedded in H. Define $\nu := \nu_0 \hat{\otimes}_\pi \mathbb{R}$ and $\chi := \nu_0 \hat{\otimes}_\pi \nu_0$. Let $F: [s, T] \times H \to \mathbb{R}$ be a $C^{0,1}$-function. Denote with $\partial_x F$ the Fréchet derivative of F with respect to x and assume that the mapping $(t, x) \mapsto \partial_x F(t, x)$ is continuous from $[s, T] \times H$ to ν_0. Let $\mathbb{X}(t) = \mathbb{M}(t) + \mathbb{A}(t)$ for $t \in [s, T]$ be an ν-weak-Dirichlet process with finite χ-quadratic variation. Then $Y(t) := F(t, \mathbb{X}(t))$ is a real weak Dirichlet process with local martingale part*

$$R(t) = F(s, \mathbb{X}(s)) + \int_s^t \langle \partial_x F(r, \mathbb{X}(r)), \mathrm{d}\mathbb{M}(r) \rangle, \qquad t \in [s, T].$$

3 The setting of the problem and HJB equation

In this section we introduce a class of infinite dimensional optimal control problems and we prove a decomposition result for the strong solutions of the related Hamilton-Jacobi-Bellman equation. We refer the reader to [36] and [6] respectively for the classical notions of functional analysis and stochastic calculus in infinite dimension we use.

3.1 The optimal control problem

Assume from now that H and U are real separable Hilbert spaces, $Q \in \mathcal{L}(U)$, $U_0 := Q^{1/2}(U)$. Assume that $\mathbb{W}_Q = \{\mathbb{W}_Q(t) : s \leq t \leq T\}$ is an U-valued $\mathscr{F}_s^t$-Q-Wiener process (with $\mathbb{W}_Q(s) = 0$, $\mathbb{P}$ a.s.) and denote by $\mathcal{L}_2(U_0, H)$ the Hilbert space of the Hilbert-Schmidt operators from U_0 to H.

We denote by $A: D(A) \subseteq H \to H$ the generator of the C_0-semigroup e^{tA} (for $t \geq 0$) on H. A^* denotes the adjoint of A. Recall that $D(A)$ and $D(A^*)$ are Banach spaces when endowed with the graph norm. Let Λ be a Polish space.

We formulate the following standard assumptions that will be needed to ensure the existence and the uniqueness of the solution of the state equation.

Hypothesis 3.1. *$b: [0, T] \times H \times \Lambda \to H$ is a continuous function and satisfies, for some $C > 0$,*

$$|b(s, x, a) - b(s, y, a)| \leq C|x - y|,$$
$$|b(s, x, a)| \leq C(1 + |x|),$$

for all $x, y \in H$, $s \in [0, T]$, $a \in \Lambda$. $\sigma: [0, T] \times H \to \mathcal{L}_2(U_0, H)$ is continuous and, for some $C > 0$, satisfies,

$$\|\sigma(s, x) - \sigma(s, y)\|_{\mathcal{L}_2(U_0, H)} \leq C|x - y|,$$
$$\|\sigma(s, x)\|_{\mathcal{L}_2(U_0, H)} \leq C(1 + |x|),$$

for all $x, y \in H$*,* $s \in [0, T]$.

Given an adapted process $a = a(\cdot) : [s, T] \times \Omega \to \Lambda$, we consider the state equation

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t, \mathbb{X}(t), a(t)))\ \mathrm{d}t + \sigma(t, \mathbb{X}(t))\,\mathrm{d}\mathbb{W}_Q(t) \\ \mathbb{X}(s) = x. \end{cases} \tag{5}$$

The solution of (5) is understood in the mild sense: an H-valued adapted process $\mathbb{X}(\cdot)$ is a solution if

$$\mathbb{P}\left\{\int_s^T \Big(|\mathbb{X}(r)| + |b(r, \mathbb{X}(r), a(r))| + \|\sigma(r, \mathbb{X}(r))\|^2_{\mathcal{L}_2(U_0,H)}\Big)\ \mathrm{d}r < +\infty\right\} = 1$$

and

$$\mathbb{X}(t) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} b(r, \mathbb{X}(r), a(r))\,\mathrm{d}r + \int_s^t e^{(t-r)A}\sigma(r, \mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r) \tag{6}$$

$\mathbb{P}$-a.s. for every $t \in [s, T]$. Thanks to Theorem 3.3 of [19], given Hypothesis 3.1, there exists a unique (up to modifications) continuous (mild) solution $\mathbb{X}(\cdot; s, x, a(\cdot))$ of (5).

Proposition 3.2. *Set* $\bar{\nu}_0 = D(A^*)$*,* $\nu = \bar{\nu}_0 \hat{\otimes}_\pi \mathbb{R}$*,* $\bar{\chi} = \bar{\nu}_0 \hat{\otimes}_\pi \bar{\nu}_0$*. The process* $\mathbb{X}(\cdot; s, x, a(\cdot))$ *is* ν*-weak-Dirichlet process admitting a* $\bar{\chi}$*-quadratic variation with decomposition* $\mathbb{M} + \mathbb{A}$ *where* $\mathbb{M}$ *is the local martingale defined by* $\mathbb{M}(t) = x + \int_s^t \sigma(r, \mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)$ *and* $\mathbb{A}$ *is a* ν*-martingale-orthogonal process.*

Proof. See Corollary 4.6 of [13]. □

Hypothesis 3.3. *Let* $l\colon [0, T] \times H \times \Lambda \to \mathbb{R}$ *(the running cost) be a measurable function and* $g\colon H \to \mathbb{R}$ *(the terminal cost) a continuous function.*

We consider the class $\mathcal{U}_s$ of admissible controls constituted by the adapted processes $a : [s, T] \times \Omega \to \Lambda$ such that $(r, \omega) \mapsto l(r, \mathbb{X}(r, s, x, a(\cdot)), a(r)) + g(\mathbb{X}(T, s, x, a(\cdot)))$ is $\mathrm{d}r \otimes \mathrm{d}\mathbb{P}$- is quasi-integrable. This means that, either its positive or negative part are integrable.

We consider the problem of minimizing, over all $a(\cdot) \in \mathcal{U}_s$, the cost functional

$$J(s, x; a(\cdot)) = \mathbb{E}\left[\int_s^T l(r, \mathbb{X}(r; s, x, a(\cdot)), a(r))\,\mathrm{d}r + g(\mathbb{X}(T; s, x, a(\cdot)))\right]. \tag{7}$$

The value function of this problem is defined, as usual, as

$$V(s, x) = \inf_{a(\cdot)\in\mathcal{U}_s} J(s, x; a(\cdot)). \tag{8}$$

As usual we say that the control $a^*(\cdot) \in \mathcal{U}_s$ is *optimal* at (s, x) if $a^*(\cdot)$ minimizes (7) among the controls in $\mathcal{U}_s$, i.e. if $J(s, x; a^*(\cdot)) = V(s, x)$. In this case the pair $(a^*(\cdot), \mathbb{X}^*(\cdot))$, where $\mathbb{X}^*(\cdot) := \mathbb{X}(\cdot; s, x, a^*(\cdot))$, is called an *optimal couple* at (s, x).

3.2 The HJB equation

The HJB equation associated with the minimization problem above is

$$\begin{cases} \partial_s v + \langle A^* \partial_x v, x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial_{xx}^2 v\right] \\ \qquad\qquad\qquad + \inf_{a\in\Lambda} \Big\{ \langle \partial_x v, b(s,x,a)\rangle + l(s,x,a) \Big\} = 0, \\ v(T,x) = g(x). \end{cases} \tag{9}$$

In the above equation $\partial_x v$ (respectively $\partial_{xx}^2 v$) is the first (respectively second) Fréchet derivatives of v with respect to the x variable; it is identified (via Riesz Representation Theorem, see [36], Theorem III.3) with elements of H, respectively (see [14], statement 3.5.7, page 192) with a symmetric bounded operator on H; $\partial_s v$ is the derivative with respect to the time variable. The function

$$F_{CV}(s,x,p,a) := \langle p, b(s,x,a)\rangle + l(s,x,a), \quad (s,x,p,a) \in [0,T]\times H\times H\times\Lambda, \tag{10}$$

is called the *current value Hamiltonian* of the system and its infimum over $a \in \Lambda$

$$F(s,x,p) := \inf_{a\in\Lambda} \{\langle p, b(s,x,a)\rangle + l(s,x,a)\} \tag{11}$$

is called the *Hamiltonian*. Using this notation the HJB equation (9) can be rewritten as

$$\begin{cases} \partial_s v + \langle A^* \partial_x v, x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial_x^2 v\right] + F(s,x,\partial_x v) = 0, \\ v(T,x) = g(x). \end{cases} \tag{12}$$

The hypothesis below will be used in the sequel.

Hypothesis 3.4. *The Hamiltonian $F(s,x,p)$ is well-defined and finite for all $(s,x,p) \in [0,T]\times H\times H$ and it is continuous in the three variables.*

We introduce the operator $\mathscr{L}_0$ on $C([0,T]\times H)$ defined as

$$\begin{cases} D(\mathscr{L}_0) := \left\{\varphi \in C^{1,2}([0,T]\times H) \;:\; \partial_x\varphi \in C([0,T]\times H; D(A^*))\right\} \\ \mathscr{L}_0(\varphi)(s,x) := \partial_s\varphi(s,x) + \langle A^*\partial_x\varphi(s,x), x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial_{xx}^2\varphi(s,x)\right], \end{cases} \tag{13}$$

so that the HJB equation (12) can be formally rewritten as

$$\begin{cases} \mathscr{L}_0(v)(s,x) = -F(s,x,\partial_x v(s,x)) \\ v(T,x) = g(x). \end{cases} \tag{14}$$

Recalling that we suppose the validity of Hypotheses 3.3 and 3.4, we consider the two following definitions of solution of the HJB equation.

Definition 3.5. *We say that $v \in C([0,T]\times H)$ is a* strict solution *of (14) if $v \in D(\mathscr{L}_0)$ and (14) is satisfied.*

Definition 3.6. *Given $h \in C([0,T] \times H)$ and $g \in C(H)$ we say that $v \in C^{0,1}([0,T[\times H) \cap C^0([0,T] \times H)$ with $\partial_x v \in UC([0,T[\times H; D(A^*))$ is a* strong solution *of (14) if there exist three sequences: $\{v_n\} \subseteq D(\mathscr{L}_0)$, $\{h_n\} \subseteq C([0,T]\times H)$ and $\{g_n\} \subseteq C(H)$ fulfilling the following.*

(i) For any $n \in \mathbb{N}$, v_n is a strict solution of the problem

$$\begin{cases} \mathscr{L}_0(v_n)(s,x) = h_n(s,x) \\ v_n(T,x) = g_n(x). \end{cases} \tag{15}$$

(ii) The following convergences hold:

$$\begin{cases} v_n \to v & \text{in } C([0,T] \times H) \\ h_n \to -F(\cdot,\cdot,\partial_x v(\cdot,\cdot)) & \text{in } C([0,T] \times H) \\ g_n \to g & \text{in } C(H), \end{cases}$$

where the convergences in $C([0,T] \times H)$ and $C(H)$ are meant in the sense of uniform convergence on compact sets.

3.3 Decomposition for solutions of the HJB equation

Theorem 3.7. *Suppose Hypotheses 3.1 and 3.4 are satisfied. Suppose that $v \in C^{0,1}([0,T[\times H) \cap C^0([0,T] \times H)$ with $\partial_x v \in UC([0,T[\times H; D(A^*))$ is a strong solution of (14). Let $\mathbb{X}(\cdot) := \mathbb{X}(\cdot; t, x, a(\cdot))$ be the solution of (5) starting at time s at some $x \in H$ and driven by some control $a(\cdot) \in \mathcal{U}_s$. Assume that b is of the form*

$$b(t,x,a) = b_g(t,x,a) + b_i(t,x,a), \tag{16}$$

where b_g and b_i satisfy the following conditions.

(i) $\sigma(t,\mathbb{X}(t))^{-1} b_g(t,\mathbb{X}(t),a(t))$ is bounded (being $\sigma(t,\mathbb{X}(t))^{-1}$ the pseudo-inverse of σ);

(ii) b_i satisfies

$$\lim_{n\to\infty} \int_s^{\cdot} \langle \partial_x v_n(r,\mathbb{X}(r)) - \partial_x v(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r)) \rangle \, \mathrm{d}r = 0 \quad \textit{ucp on } [s,T_0], \tag{17}$$

for each $s < T_0 < T$.

Then

$$v(t,\mathbb{X}(t)) - v(s,\mathbb{X}(s)) = v(t,\mathbb{X}(t)) - v(s,x) = -\int_s^t F(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r)))\, \mathrm{d}r$$
$$+ \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), b(r,\mathbb{X}(r),a(r)) \rangle \, \mathrm{d}r + \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle, \ t \in [s,T[. \tag{18}$$

Proof. We fix T_0 in $]s,T[$. We denote by v_n the sequence of smooth solutions of the approximating problems prescribed by Definition 3.6, which converges to v. Thanks to Itô formula for convolution type processes (see e.g. Corollary 4.10 in [13]), every v_n verifies

$$\begin{aligned} v_n(t,\mathbb{X}(t)) &= v_n(s,x) + \int_s^t \partial_r v_n(r,\mathbb{X}(r))\,\mathrm{d}r \\ &+ \int_s^t \langle A^*\partial_x v_n(r,\mathbb{X}(r)),\mathbb{X}(r)\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r \\ &+ \frac{1}{2}\int_s^t \mathrm{Tr}\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^*\partial^2_{xx}v_n(r,\mathbb{X}(r))\right]\mathrm{d}r \\ &+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)),\sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle\,,\ t\in[s,T]. \qquad \mathbb{P}-\text{a.s.} \end{aligned} \tag{19}$$

Using Girsanov's Theorem (see [6] Theorem 10.14) we can observe that

$$\beta_Q(t) := W_Q(t) + \int_s^t \sigma(r,\mathbb{X}(r))^{-1} b_g(r,\mathbb{X}(r),a(r))\,\mathrm{d}r,$$

is a Q-Wiener process with respect to a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ on the whole interval $[s,T]$. We can rewrite (19) as

$$\begin{aligned} v_n(t,\mathbb{X}(t)) &= v_n(s,x) + \int_s^t \partial_r v_n(r,\mathbb{X}(r))\,\mathrm{d}r \\ &+ \int_s^t \langle A^*\partial_x v_n(r,\mathbb{X}(r)),\mathbb{X}(r)\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r, \\ &+ \frac{1}{2}\int_s^t \mathrm{Tr}\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^*\partial^2_{xx}v_n(r,\mathbb{X}(r))\right]\mathrm{d}r \\ &+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)),\sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)\rangle\,. \qquad \mathbb{P}-a.s. \end{aligned} \tag{20}$$

Since v_n is a strict solution of (15), the expression above gives

$$\begin{aligned} v_n(t,\mathbb{X}(t)) &= v_n(s,x) + \int_s^t h_n(r,\mathbb{X}(r))\,\mathrm{d}r \\ &+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)),\sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)\rangle\,. \end{aligned} \tag{21}$$

Since we wish to take the limit for $n\to\infty$, we define

$$\begin{aligned} M_n(t) := v_n(t,\mathbb{X}(t)) - v_n(s,x) &- \int_s^t h_n(r,\mathbb{X}(r))\,\mathrm{d}r \\ &- \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r. \end{aligned} \tag{22}$$

$\{M_n\}_{n\in\mathbb{N}}$ is a sequence of real $\mathbb{Q}$-local martingales converging ucp, thanks to the definition of strong solution and Hypothesis (17), to

$$M(t) := v(t,\mathbb{X}(t)) - v(s,x) + \int_s^t F(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r)))\,\mathrm{d}r \\ - \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle \,\mathrm{d}r,\ t\in[s,T_0]. \quad (23)$$

Since the space of real continuous local martingales equipped with the ucp topology is closed (see e.g. Proposition 4.4 of [26]) then M is a continuous $\mathbb{Q}$-local martingale indexed by $t\in[s,T_0]$.

We have now gathered all the ingredients to conclude the proof. We set $\bar\nu_0 = D(A^*)$, $\nu = \bar\nu_0\hat\otimes_\pi\mathbb{R}, \bar\chi = \bar\nu_0\hat\otimes_\pi\bar\nu_0$. Proposition 3.2 ensures that $\mathbb{X}(\cdot)$ is a ν-weak Dirichlet process admitting a $\bar\chi$-quadratic variation with decomposition $\mathbb{M}+\mathbb{A}$ where $\mathbb{M}$ is the local martingale (with respect to $\mathbb{P}$) defined by $\mathbb{M}(t) = x + \int_s^t \sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)$ and $\mathbb{A}$ is a ν-martingale-orthogonal process. Now

$$\mathbb{X}(t) = \tilde{\mathbb{M}}(t) + \mathbb{V}(t) + \mathbb{A}(t), t\in[s,T_0],$$

where $\tilde{\mathbb{M}}(t) = x + \int_s^t \sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)$ and $\mathbb{V}(t) = -\int_s^t b_g(r,\mathbb{X}(r),a(r))dr$, $t\in[s,T_0]$ is a bounded variation process. Thanks to [27] Theorem 2.14 page 14-15, $\tilde{\mathbb{M}}$ is a $\mathbb{Q}$-local martingale. Moreover $\mathbb{V}$ is a bounded variation process and then, thanks to Lemma 2.5, it is a $\mathbb{Q}-\nu$-martingale orthogonal process. So $\mathbb{V}+\mathbb{A}$ is a again (one can easily verify that the sum of two ν-martingale-orthogonal processes is again a ν-martingale-orthogonal process) a $\mathbb{Q}-\nu$-martingale orthogonal process and $\mathbb{X}$ is a ν-weak Dirichlet process with local martingale part $\tilde{\mathbb{M}}$, with respect to $\mathbb{Q}$. Still under $\mathbb{Q}$, since $v\in C^{0,1}([0,T_0]\times H)$, Theorem 2.7 ensures that the process $v(\cdot,\mathbb{X}(\cdot))$ is a real weak Dirichlet process on $[s,T_0]$, whose local martingale part being equal to

$$N(t) = \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)\rangle, t\in[s,T_0].$$

On the other hand, with respect to $\mathbb{Q}$, (23) implies that

$$v(t,\mathbb{X}(t)) = \Big[v(s,x) - \int_s^t F(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r)))\,\mathrm{d}r \\ + \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle \,\mathrm{d}r\Big] + N(t),\ t\in[s,T_0], \quad (24)$$

is a decomposition of $v(\cdot,\mathbb{X}(\cdot))$ as $\mathbb{Q}$- semimartingale, which is also in particular, a $\mathbb{Q}$-weak Dirichlet process. By Theorem 2.2 such a decomposition is unique on $[s,T_0]$ and so $M(t) = N(t), t\in[s,T_0]$, so $M(t)=N(t), t\in[s,T[$.

Consequently

$$M(t) = \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\beta_Q(r)\rangle$$
$$= \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), b_g(r, \mathbb{X}(r), a(r))\, \mathrm{d}r\rangle$$
$$+ \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r)\rangle\,,\ t \in [s, T]. \quad (25)$$

□

Example 3.8. *The decomposition* (16) *with validity of Hypotheses* (i) *and* (ii) *in Theorem 3.7 are satisfied if v is a strong solution of the HJB equation in the sense of Definition 3.6 and, moreover the sequence of corresponding functions $\partial_x v_n$ converge to $\partial_x v$ in $C([0,T] \times H)$. In that case we simply set $b_g = 0$ and $b = b_i$. This is the typical assumption required in the standard strong solutions literature.*

Example 3.9. *Again the decomposition* (16) *with validity of Hypotheses* (i) *and* (ii) *in Theorem 3.7 is fulfilled if the following assumption is satisfied.*

$$\sigma(t, \mathbb{X}(t))^{-1} b(t, \mathbb{X}(t), a(t)) \text{ is bounded,}$$

for all choice of admissible controls $a(\cdot)$. In this case we apply Theorem 3.7 with $b_i = 0$ and $b = b_g$.

4 Verification Theorem

In this section, as anticipated in the introduction, we use the decomposition result of Theorem 3.7 to prove a verification theorem.

Theorem 4.1. *Assume that Hypotheses 3.1, 3.3 and 3.4 are satisfied and that the value function is finite for any $(s,x) \in [0,T] \times H$. Let $v \in C^{0,1}([0,T[\times H) \cap C^0([0,T] \times H)$ with $\partial_x v \in UC([0,T[\times H; D(A^*))$ be a strong solution of (9) such that $\partial_x v$ has most polynomial growth in the x variable.*
Assume that for all initial data $(s,x) \in [0,T] \times H$ and every control $a(\cdot) \in \mathcal{U}_s$ b can be written as $b(t,x,a) = b_g(t,x,a) + b_i(t,x,a)$ with b_i and b_g satisfying hypotheses (i) and (ii) of Theorem 3.7. Then we have the following.

(i) $v \le V$ *on* $[0,T] \times H$.

(ii) Suppose that, for some $s \in [0,T)$, there exists a predictable process $a(\cdot) = a^*(\cdot) \in \mathcal{U}_s$ such that, denoting $\mathbb{X}(\cdot; s, x, a^*(\cdot))$ simply by $\mathbb{X}^*(\cdot)$, we have

$$F(t, \mathbb{X}^*(t), \partial_x v(t, \mathbb{X}^*(t))) = F_{CV}(t, \mathbb{X}^*(t), \partial_x v(t, \mathbb{X}^*(t)); a^*(t)), \quad (26)$$

$dt \otimes \mathrm{d}\mathbb{P}$ a.e. Then $a^*(\cdot)$ is optimal at (s,x); moreover $v(s,x) = V(s,x)$.

Proof. We choose a control $a(\cdot) \in \mathcal{U}_s$ and call $\mathbb{X}$ the related trajectory. We make use of (18) in Theorem 3.7. Then we need to extend (18) to the case when $t \in [s, T]$. This is possible since v is continuous, $\partial_x v$ is locally bounded and F is uniformly continuous on compact sets, also using Hypothesis 3.1 for b and σ. At this point, setting $t = T$ we can write

$$g(\mathbb{X}(T)) = v(T, \mathbb{X}(T)) = v(s, x) - \int_s^T F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))\, \mathrm{d}r$$
$$+ \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), b(r, \mathbb{X}(r), a(r)) \rangle\, \mathrm{d}r + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle\,. \tag{27}$$

Since both sides of (27) are a. s. finite, we can add $\int_s^T l(r, \mathbb{X}(r), a(r))\, \mathrm{d}r$ to them, obtaining

$$g(\mathbb{X}(T)) + \int_s^T l(r, \mathbb{X}(r), a(r))\, \mathrm{d}r = v(s, x) + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle$$
$$+ \int_s^T \left(-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) + F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) \right) \mathrm{d}r. \tag{28}$$

Observe now that, by definition of F and F_{CV} we know that

$$-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) + F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))$$

is always positive. So its expectation always exists even if it could be $+\infty$, but not $-\infty$ on an event of positive probability. This shows a posteriori that $\int_s^T l(r, \mathbb{X}(r), a(r))\, \mathrm{d}r$ cannot be $-\infty$ on a set of positive probability.
By Proposition 7.4 in [6], all the momenta of $\sup_{r \in [s,T]} |\mathbb{X}(r)|$ are finite. On the other hand, σ is Lipschitz-continuous, $v(s, x)$ is deterministic and, since $\partial_x v$ has polynomial growth, then

$$\mathbb{E} \int_s^T \left\langle \partial_x v(r, \mathbb{X}(r)), \left(\sigma(r, \mathbb{X}(r)) Q^{1/2} \right) \left(\sigma(r, \mathbb{X}(r)) Q^{1/2} \right)^* \partial_x v(r, \mathbb{X}(r)) \right\rangle \mathrm{d}r$$

is finite. Consequently (see [6] Sections 4.3, in particular Theorem 4.27 and 4.7),

$$\int_s^{\cdot} \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle\,,$$

is a true martingale vanishing at s. Consequently, its expectation is zero. So the expectation of the right-hand side of (28) exists even if it could be $+\infty$; consequently the same holds for the left-hand side.

By definition of J, we have

$$J(s,x,a(\cdot)) = \mathbb{E}\Big[g(\mathbb{X}(T)) + \int_s^T l(r,\mathbb{X}(r),a(r))\,\mathrm{d}r\Big] = v(s,x)$$
$$+\,\mathbb{E}\int_s^T \Big(-F(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r))) + F_{CV}(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r)),a(r))\Big)\,\mathrm{d}r. \tag{29}$$

So minimizing $J(s,x,a(\cdot))$ over $a(\cdot)$ is equivalent to minimize

$$\mathbb{E}\int_s^T \Big(-F(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r))) + F_{CV}(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r)),a(r))\Big)\,\mathrm{d}r, \tag{30}$$

which is a non-negative quantity. As mentioned above, the integrand of such an expression is always nonnegative and then a lower bound for (30) is 0. If the conditions of point (ii) are satisfied such a bound is attained by the control $a^*(\cdot)$, that in this way is proved to be optimal.

Concerning the proof of (i), since the integrand in (30) is nonnegative, (29) gives

$$J(s,x,a(\cdot)) \geq v(s,x).$$

Taking the inf over $a(\cdot)$ we get $V(s,x) \geq v(s,x)$, which concludes the proof. □

Remark 4.2. *1. The first part of the proof does not make use that a belongs to $\mathcal{U}_s$, but only that $r \mapsto l(r,\mathbb{X}(\cdot,s,x,a(\cdot)),a(\cdot))$ is a.s. strictly bigger then $-\infty$. Under that only assumption, $a(\cdot)$ is forced to be admissible, i.e. to belong to $\mathcal{U}_s$.*

2. Let v be a strong solution of HJB equation. Observe that the condition (26) can be rewritten as

$$a^*(t) \in \arg\min_{a\in\Lambda}\Big[F_{CV}\left(t,\mathbb{X}^*(t),\partial_x v\left(t,\mathbb{X}^*(t)\right);a\right)\Big].$$

Suppose that for any $(t,y) \in [0,T]\times H$, $\phi(t,y) = \arg\min_{a\in\Lambda}\left(F_{CV}\left(t,y,\partial_x v(t,y);a\right)\right)$ is measurable and single-valued. Suppose moreover that

$$\int_s^T l(r,\mathbb{X}^*(r),a^*(r))dr > -\infty \text{ a.s.} \tag{31}$$

Suppose that the equation

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t,\mathbb{X}(t),\phi(t,\mathbb{X}(t))\ \mathrm{d}t + \sigma(t,\mathbb{X}(t))\,\mathrm{d}\mathbb{W}_Q(t) \\ \mathbb{X}(s) = x, \end{cases} \tag{32}$$

admits a unique mild solution $\mathbb{X}^*$. Now (31) and Remark 4.2 1. imply that $a^*(\cdot)$ is admissible. Then $\mathbb{X}^*$ is the optimal trajectory of the state variable *and $a^*(t) = \phi(t, \mathbb{X}^*(t)), t \in [0, T]$ is the optimal control. The function ϕ is the* optimal feedback *of the system since it gives the optimal control as a function of the state.*

Remark 4.3. *Observe that, using exactly the same arguments we used in this section one could treat the (slightly) more general case in which b has the form:*

$$b(t, x, a) = b_0(t, x) + b_g(t, x, a) + b_i(t, x, a).$$

where b_g and b_i satisfy condition of Theorem 3.7 and $b_0 : [0, T] \times H \to H$ is continuous. In this case the addendum b_0 can be included in the expression of $\mathscr{L}_0$ that becomes the following

$$\begin{cases} D(\mathscr{L}_0^{b_0}) := \left\{\varphi \in C^{1,2}([0, T] \times H) \ : \ \partial_x \varphi \in C([0, T] \times H; D(A^*))\right\} \\ \mathscr{L}_0^{b_0}(\varphi) := \partial_s \varphi + \langle A^* \partial_x \varphi, x\rangle + \langle \partial_x \varphi, b_0(t, x)\rangle + \frac{1}{2} Tr\left[\sigma(s, x)\sigma^*(s, x)\partial^2_{xx}\varphi\right]. \end{cases} \tag{33}$$

Consequently in the definition of regular solution the operator $\mathscr{L}_0^{b_0}$ appears instead $\mathscr{L}_0$.

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