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Competitive equilibrium cycles for small discounting in discrete-time two-sector optimal growth models*

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Abstract: *We study the existence of endogenous competitive equilibrium cycles under small discounting in a two-sector discrete-time optimal growth model. We provide precise concavity conditions on the indirect utility function leading to the existence of period-two cycles with a critical value for the discount factor that can be arbitrarily close to one. Contrary to the continuous-time case where the existence of periodic-cycles is obtained if the degree of concavity is close to zero, we show that in a discrete-time setting the driving condition does not require a close to zero degree of concavity but a symmetry of the indirect utility function's concavity properties with respect to its two arguments.*

Keywords: *Two-sector optimal growth model, small discounting, period-two cycles, strong and weak concavity.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

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1 Introduction

Contrary to the aggregate Ramsey [18] formulation, it is well-known that multisector optimal growth models are characterized by the turnpike property, i.e. an equilibrium path that converges to a steady state, if discounting is sufficiently small.[1] Moreover, Brock and Scheinkman [9], Cass and Shell [11], Magill [12] and Rockafellar [18] have shown that there exists a trade-off between the level of discounting ensuring the turnpike property and the curvature of the indirect utility function that summarizes the properties of preferences and technologies. The main conclusions show that increasing the curvature of the indirect utility function increases the set of values of the discount rate for which the turnpike property holds. Strong and weak concavity properties, respectively called α-concavity and concavity-γ, providing through the parameters α and γ precise measures of lower and upper bounds for the curvature of the indirect utility function, are used.

Focusing on the cases where the steady state is unstable, seminal contributions by Benhabib and Nishimura [2, 4] have proved that if discounting is large enough, there exist persistent endogenous fluctuations through a Hopf bifurcation in continuous-time models and a flip bifurcation in discrete-time models. When appraised with respect to the earlier contributions, one may naturally guess that periodic cycles could be compatible with low discounting if the curvature of the indirect utility function is low enough. Such a conclusion has been confirmed for continuous-time models. Venditti [20] proves indeed that if the indirect utility function is concave-γ in a neighborhood of the steady state, the Hopf bifurcation value of the discount rate is bounded from above by a function of γ that converges towards zero as γ is chosen closer to zero. Put differently, the instability of the steady state and the existence of endogenous fluctuations under small discounting are obtained when the indirect utility function has a degree of curvature close to zero. Such a conclusion is therefore consistent with the results of Brock and Scheinkman [9], Cass and Shell [11], Magill [12] and Rockafellar [18].

Up to our knowledge, similar results for discrete-time models are not available in the literature. In this paper, we then focus on a two-sector discrete-

[1]See McKenzie [13] for an almost complete bibliography.

time model and we prove that the intuition derived from earlier contributions does not apply in a discrete-time setting. Of course, obtaining period-two cycles under small discounting requires to focus on a non-strongly concave indirect utility function characterized by a singular Hessian matrix. But assuming a low value for the coefficient γ of weak concavity does not help. We prove indeed that under the assumption of an indirect utility function $V(x, y)$ concave-(γ_1, γ_2), i.e. that has degrees of curvature a priori different across its two arguments, period-two cycles can arise under small discounting if γ_2 converges to γ_1 from below. Such a condition means that the existence of endogenous fluctuations under small discounting is obtained under a symmetric concavity property of the indirect utility function with respect to its two arguments and does not require any small value for the coefficients γ_1 and γ_2.

The rest of the paper is organized as follows. In Section 2 we present the two-sector optimal growth model and a precise description of the indirect utility function derived from the fundamentals. We provide in Section 3 the definitions of strong and weak concavity and we prove some useful properties for our main results. Section 4 is devoted to the analysis of persistent endogenous fluctuations under small discounting. We first provide some turnpike results linking the robustness of the stability property of the steady state with respect to the discount factor and we provide a simpler proof than Benhabib and Nishimura [4] of the existence of optimal period-two cycles with a precise characterization of the discount factor bifurcation value. We then provide sufficient conditions in terms of weak concavity of the indirect utility function that allows to get a bifurcation value of the discount factor arbitrarily close to one. Some well-known examples illustrating our main conclusions are discussed in Section 5. Section 6 provides concluding comments and all the proofs are gathered in a final Appendix.

2 The two-sector model

We consider a two-sector competitive economy with a pure consumption good and a capital good. Total labor is normalized to one, and the production side is defined by the following equations:

$$
\begin{aligned}
&y_0 \le f^0(k_0, l_0), \quad y \le f^1(k_1, l_1) \\
&1 = l_0 + l_1, \qquad\quad k = k_0 + k_1
\end{aligned} \tag{1}
$$

where y_0 is the consumption good output, y the capital good output, k_0, k_1, and l_0, l_1 the amounts of capital good and labor used in each sector, and k the total stock of capital.

Assumption 1. *The functions $f^i : \mathbb{R}^2_+ \to \mathbb{R}_+$, $i = 0, 1$, are time-invariant, C^2, strictly increasing in each argument and concave.*

Assuming a growth rate of labour force $n \ge 0$ and a capital depreciation rate $\mu \in [0, 1]$, we obtain the capital accumulation equation:

$$
y_t = (1 + n)k_{t+1} - (1 - \mu)k_t \tag{2}
$$

Let us maximize the production of the consumption good y_0 subject to the technological constraints, namely:

$$
\max_{(k_0, l_0)} \quad f^0(k_0, l_0) \tag{3}
$$

$$
s.t. \quad y \le f^1(k_1, l_1) \tag{4}
$$

$$
1 = l_0 + l_1, \; k = k_0 + k_1 \tag{5}
$$

$$
k_i \ge 0, l_i \ge 0, \; i = 0, 1 \tag{6}
$$

The optimal solution gives the maximal level of consumption as a function of the capital stock k and the capital good output y, i.e.:

$$
y_0^* = c = T(k, y) \tag{7}
$$

The *social production function* T is defined over a convex set $\mathcal{K} = K \times K \subseteq \mathbb{R}^2_+$, and gives the frontier of the production possibility set. Considering equation (2), we get

$$
T(k_t, y_t) = (T_o B)(k_t, k_{t+1}) \tag{8}
$$

where $B : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map defined by the matrix

$$
B = \begin{pmatrix} 1 & 0 \\ \mu - 1 & 1 + n \end{pmatrix} \tag{9}
$$

Labor supply is inelastic and the preferences of the representative agent are described by some utility function $u(c)$ such that:

Assumption 2. *$u : \mathbb{R}_+ \to \mathbb{R}$ is time-invariant, C^2, increasing and concave.*

Let us introduce the set

$$\mathcal{D} = \{(k_t, k_{t+1}) \in \mathbb{R} \times \mathbb{R} / B(k_t, k_{t+1}) = (k_t, y_t) \in \mathcal{K}\} \tag{10}$$

The indirect utility function is finally defined as $V : \mathcal{D} \to \mathbb{R}$ with

$$V(k_t, k_{t+1}) \equiv (u_o T_o B)(k_t, k_{t+1}) \tag{11}$$

It is obvious from Assumptions 1 and 2 that V is a C^2 concave function such that $V_1(x, y) > 0$ and $V_2(x, y) < 0$.

We then derive the following reduced form formulation of a two-sector optimal growth model in discrete time with discounting:

$$\begin{array}{ll} \max\limits_{\{k_t\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} \delta^t V(k_t, k_{t+1}) \\ s.t. & (k_t, k_{t+1}) \in D \\ & k_0 \ \ given \end{array} \tag{12}$$

where $\delta \in (0, 1]$ is the discount factor.[2]

An interior optimal path $(k_t, k_{t+1}) \in int(D)$ necessarily satisfies the following Euler-Lagrange equation for all $t \geq 0$:

$$V_2(k_t, k_{t+1}) + \delta V_1(k_{t+1}, k_{t+2}) = 0 \tag{13}$$

Definition 1. *A path $\{k_t\}_{t=0}^{\infty}$ such that $k_t = k_\delta^* \in K$ for all $t \geq 0$ which satisfies equation (13) is called an optimal steady state.*

As in the standard aggregate Ramsey [18] model, we know that:

Proposition 1. *Under Assumptions 1-2, there exists a unique optimal steady state k_δ^* solution of equation* (13).

Proof. See Theorem 3.1 in Becker and Tsyganov [1]. □

Let us denote $V_{ij}^\delta = V_{ij}(k_\delta^*, k_\delta^*)$, $i, j = 1, 2$, the second derivatives of the indirect utility function evaluated at the steady state k_δ^*. The linearization

[2] In the case $\delta = 1$, the infinite sum into the optimization program (12) may not converge. In such a case we can apply the definition of optimality as provided by Ramsey [18].

of the Euler-Lagrange equation around k^*_δ gives the following characteristic polynomial:

$$P(\lambda) = \lambda^2 \delta V^\delta_{12} + \lambda(\delta V^\delta_{11} + V^\delta_{22}) + V^\delta_{12} = 0 \tag{14}$$

Before analyzing the existence of period-two cycles for low discounting, we need to introduce more precise measures of concavity for the indirect utility function.

3 Preliminary results on strong and weak concavity

Concavity assumptions used in economics do not in general provide precise restrictions on the degree of curvature of a function. We consider here at the same time the concepts of strong and weak concavity which provide respectively lower and upper bounds for the curvature.

3.1 Strong concavity

Let us introduce the following definition:

Definition 2. *Let $\mathbb{R}^n$ be endowed with the Euclidean norm $\|.\|$, and $D \subset \mathbb{R}^n$ be a non-empty convex set. Let $f : D \to \mathbb{R}$ be a real-valued concave function. Let α be the least upper bound of the set of real numbers a such that the function $f(x) + (1/2)a\|x\|^2$ is concave over D, i.e.*

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2) + (1/2)at(1-t)\|x_1 - x_2\|^2$$

for all $x_1, x_2 \in D$ and all $t \in [0, 1]$. If $\alpha \geq 0$, f is called α-concave, and if $\alpha > 0$, f is said to be strongly concave.[3]

The parameter α can be seen as a measure of the lower curvature of f. The following Lemma shows that for twice differentiable functions, α-concavity can be expressed by means of a condition on the Hessian matrix $D^2 f(x)$:

[3] See Bougeard and Penot [8], Vial [24].

Lemma 1. *Let $D \subset \mathbb{R}^n$ be a non-empty convex set, and $f : D \to \mathbb{R}$ be a real-valued, twice differentiable concave function.*

i) f is α-concave if and only if there exists a real number $\alpha \geq 0$ such that for all $x \in D$ and all $\nu \in \mathbb{R}^n$

$$\nu^t D^2 f(x)\nu + \alpha\|\nu\|^2 \leq 0$$

ii) Let $|\lambda_i(x)|$, i=1,...,n, be the eigenvalues in absolute value of $D^2 f(x)$ and $\lambda^*(x) = \min_{i=1,\dots,n} |\lambda_i(x)|$. Then $\alpha = \inf_{x \in D} \lambda^*(x)$.

Proof. See Appendix 7.1. □

We easily derive from this result:

Corollary 1. *A quadratic form is α-concave with α equal to the smaller eigenvalue in absolute value of the Hessian matrix.*

Note that $\lambda^*(x)$ is the index of local strong concavity of f around x. Applying this concept to the indirect utility function characterizing our two-sector optimal growth model will allow to measure locally, i.e. in a neighborhood of the steady state, its degree of strong concavity through the consideration of the lowest eigenvalue of its Hessian matrix.

Let us now consider the case of a function depending on two groups of variables. The previous concepts can be generalized.

Definition 3. *Let $\mathbb{R}^n$ be endowed with the Euclidean norm $\|.\|$, and $D = X \times Y \subset \mathbb{R}^n \times \mathbb{R}^n$ be a non-empty convex set. Let $U : D \to \mathbb{R}$ be a real-valued concave function. Let α and β be the least upper bounds of the set of real numbers a and b such that the function $U(x, y) + (1/2)a\|x\|^2 + (1/2)b\|y\|^2$ is concave over D, i.e.*

$$\begin{aligned} U(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \geq\ & tU(x_1, y_1) + (1-t)U(x_2, y_2) \\ +\ & (1/2)at(1-t)\|x_1 - x_2\|^2 \\ +\ & (1/2)bt(1-t)\|y_1 - y_2\|^2 \end{aligned} \tag{15}$$

for all $(x_1, y_1), (x_2, y_2) \in D$ and all $t \in [0, 1]$. If $\alpha \geq 0$ and $\beta \geq 0$, U is called (α, β)-concave, and if $\alpha > 0$ or $\beta > 0$, U is said to be strongly concave.

As a consequence of Lemma 1, if $U(x, y)$ is (α, β)-concave, then the Hessian matrix of the function $U(x, y) + (1/2)\alpha\|x\|^2 + (1/2)\beta\|y\|^2$ is negative semi-definite, i.e.

$$\begin{pmatrix} \nu^t & \omega^t \end{pmatrix} \begin{pmatrix} D_{11}^2 U(x,y) + \alpha I & D_{12}^2 U(x,y) \\ D_{21}^2 U(x,y) & D_{22}^2 U(x,y) + \beta I \end{pmatrix} \begin{pmatrix} \nu \\ \omega \end{pmatrix} \leq 0$$

for all $(x, y) \in D$ and all $(\nu, \omega) \in \mathbb{R}^n \times \mathbb{R}^n$. This result implies that the matrices $[D_{11}^2 U(x, y)+\alpha I]$ and $[D_{22}^2 U(x, y)+\beta I]$ are negative semi-definite for all $(x, y) \in D$. Assume now that the function $U(., y)$ is α-concave for all given $y \in Y$. This means that for each given $y \in Y$, the matrix $[D_{11}^2 U(x, y) + \alpha I]$ is negative semi-definite for all $x \in X$. Therefore, if for instance $U(x, y)$ is $(\alpha, 0)$-concave, then $U(., y)$ is α-concave for all given $y \in Y$. All these properties can be applied to to characterize the lower curvature of the indirect utility function $V(x, y)$.

3.2 Weak concavity

Let us now focus on weak concavity with the following definition:

Definition 4. *Let $\mathbb{R}^n$ be endowed with the Euclidean norm $\|.\|$, and $D \subset \mathbb{R}^n$ be a non-empty convex set. Let $f : D \to \mathbb{R}$ be a real-valued concave function. Let γ be the greatest lower bound of the set of real numbers g such that the function $f(x) + (1/2)g\|x\|^2$ is convex over D, i.e.*

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) + (1/2)gt(1-t)\|x_1 - x_2\|^2$$

for all $x_1, x_2 \in D$ and all $t \in [0, 1]$. If $+\infty > \gamma > 0$, f is called concave-γ, or equivalently weakly concave.

The parameter γ can be seen as a measure of the upper curvature of f. Note also that a function $f(x)$ may be simultaneously α-concave and concave-γ. This means therefore that $0 < \alpha \leq \gamma$.

As in the case of strong concavity, for twice differentiable functions, concavity-γ can be expressed by means of a condition on the Hessian matrix $D^2 f(x)$:

Lemma 2. *Let $D \subset \mathbb{R}^n$ be a non-empty convex set, and $f : D \to \mathbb{R}$ be a real-valued, twice differentiable concave function.*

i) f is concave-γ if and only if there exists a real number $+\infty > \gamma > 0$ such that for all $x \in D$ and all $\nu \in \mathbb{R}^n$

$$\nu D^2 f(x)\nu + \gamma\|\nu\|^2 \geq 0$$

ii) Let $|\lambda_i(x)|$, i=1,...,n, be the eigenvalues in absolute value of $D^2 f(x)$ and $\lambda^*(x) = \max_{i=1,\ldots,n} |\lambda_i(x)|$. Then $\gamma = \sup_{x \in D} \lambda^*(x)$.

Proof. See Appendix 7.2. □

We easily derive from this result:

Corollary 2. *A quadratic form is concave-γ with γ equal to the greater eigenvalue in absolute value of the Hessian matrix.*

Note that $\lambda^*(x)$ is the index of local weak concavity of f around x. As in the case of strong concavity, applying this concept to the indirect utility function characterizing our two-sector optimal growth model will allow to measure locally, i.e. in a neighborhood of the steady state, its degree of weak concavity through the consideration of the highest eigenvalue of its Hessian matrix.

Let us now consider the case of a function depending on two groups of variables. The previous concepts can be generalized.

Definition 5. *Let $\mathbb{R}^n$ be endowed with the Euclidean norm $\|.\|$, and $D = X \times Y \subset \mathbb{R}^n \times \mathbb{R}^n$ be a non-empty convex set. Let $U : D \to \mathbb{R}$ be a real-valued concave function. Let γ and η be the greatest lower bounds of the set of real numbers g and h such that the function $U(x, y) + (1/2)g\|x\|^2 + (1/2)h\|y\|^2$ is convex over D, i.e.*

$$\begin{aligned} U(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) &\leq tU(x_1, y_1) + (1-t)U(x_2, y_2) \\ &+ (1/2)gt(1-t)\|x_1 - x_2\|^2 \\ &+ (1/2)ht(1-t)\|y_1 - y_2\|^2 \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in D$ and all $t \in [0, 1]$. If $+\infty > \gamma > 0$ and $+\infty > \eta > 0$, U is called concave-(γ, η), or equivalently weakly concave.

As a consequence of Lemma 2, if $U(x,y)$ is concave-(γ,η), then the Hessian matrix of the function $U(x,y) + (1/2)\gamma\|x\|^2 + (1/2)\eta\|y\|^2$ is positive semi-definite, i.e.

$$\begin{pmatrix} \nu^t & \omega^t \end{pmatrix} \begin{pmatrix} D^2_{11}U(x,y) + \gamma I & D^2_{12}U(x,y) \\ D^2_{21}U(x,y) & D^2_{22}U(x,y) + \eta I \end{pmatrix} \begin{pmatrix} \nu \\ \omega \end{pmatrix} \geq 0$$

for all $(x,y) \in D$ and all $(\nu,\omega) \in \mathbb{R}^n \times \Re^n$. This result implies that the matrices $[D^2_{11}U(x,y)+\gamma I]$ and $[D^2_{22}U(x,y)+\eta I]$ are positive semi-definite for all $(x,y) \in D$. Assume now that the function $U(.,y)$ is concave-γ for all given $y \in Y$. This means that for each given $y \in Y$, the matrix $[D^2_{11}U(x,y) + \gamma I]$ is positive semi-definite for all $x \in X$. Therefore, if for instance $U(x,y)$ is concave-$(\gamma,0)$, then $U(.,y)$ is concave-γ for all given $y \in Y$. As in the case of strong concavity, all these properties can be applied to to characterize the upper curvature of the indirect utility function $V(x,y)$.

4 Competitive equilibrium cycles for small discounting

Our strategy here is to use precise measures of curvature as given by the strong and weak concavity properties to study the local stability properties of the steady state and the possible existence of endogenous fluctuations for small discounting. Let us first focus on the turnpike property and introduce the following curvature assumptions:

Assumption 3. *The indirect utility function is such that*
i) $V(x,y)$ is (α,β)-concave in a neighborhood of the steady state (k^*_δ, k^*_δ) with $\alpha,\beta \geq 0$ and $\alpha+\beta > 0$;
ii) $V(.,k^*_\delta)$ is concave-γ in the neighborhood of k^*_δ.

We then get:

Proposition 2. *Under Assumptions 1, 2 and 3, if $1 \geq \delta > 1-(\alpha+\beta)/\gamma$, then the steady state k^*_δ is a saddle-point.*

Proof. See Appendix 7.3. □

Proposition 2 provides a simpler proof of a result initially formulated in Montrucchio [16] (see also Cartigny and Venditti [10]). It shows that increasing the curvature properties of the indirect utility function allows to reinforce the saddle-point property. This result is the discrete-time analog of the conclusions provided for continuous-time models by Rockafellar [19].[4] There is however a notable difference. Indeed, Rockafellar does not introduce any weak concavity condition and shows basically that the larger the coefficients of strong concavity the more robust is the saddle-point property with respect to the value of the discount factor.[5] In a discrete-time setting, Proposition 2 requires both strong and weak concavity properties. Moreover, even if the coefficients of strong concavity are quite low, a low coefficient of weak concavity can still ensure a large robustness of the saddle-point property.

The (α, β)-concavity of $V(x, y)$ can be rationalized through specific assumptions on the fundamentals. It is proved indeed in Venditti [22] that both coefficients α and β can be positive if the production function of the consumption good $f^0(k_0, l_0)$ is strongly concave, the production function of the investment good $f^1(k_1, l_1)$ is Lipschitz-continuous and the marginal utility $u'(c)$ is bounded from below by a strictly positive number. It is worth noting however that the strong concavity of $f^0(k_0, l_0)$ implies strictly decreasing returns,[6] while the condition on $u'(c)$ is not compatible with the standard Inada conditions.

The concavity-γ of $V(., k_\delta^*)$ in the neighborhood of k_δ^* can also be rationalized through specific assumptions on the fundamentals. It is proved indeed

[4] See also Montrucchio [17].

[5] Actually, Rockafellar [19] solves the model using optimal control and the Hamiltonian formulation. Starting from the indirect utility function in continuous time $U(k(t), \dot{k}(t))$ and assuming that the Hamiltonian $H(k(t), p(t)) = \max_{\dot{k}(t)}\{U(k(t), \dot{k}(t)) + p(t)\dot{k}(t)\}$ is α-concave in k and β-convex in p, he shows that the saddle-point property holds if the discount rate satisfies $\rho^2 < 4\alpha\beta$.

[6] Benhabib and Nishimura [3] study the concavity of the social production function $T(k, y)$ and provide two results depending on the returns to scale of the consumption and capital goods technologies. They show that if each good is produced under decreasing returns to scale, then the Hessian matrix of $T(k, y)$ has full rank, i.e. T is strictly concave. On the contrary, if the consumption good and one capital good at least are produced under constant returns to scale, then the Hessian matrix of $T(k, y)$ cannot have full rank and may not be strictly concave.

in Venditti [23] that γ can have a finite value if the production function of the consumption good $f^0(k_0, l_0)$ and the utility function $u(c)$ are weakly concave, and the marginal utility $u'(c)$ is bounded from above by a finite positive number. Again, the condition on $u'(c)$ is not compatible with the standard Inada conditions. However, the weak concavity of $f^0(k_0, l_0)$ is compatible with constant returns.

As initially proved in the seminal contribution of Benhabib and Nishimura [4], persistent endogenous fluctuation through the occurrence of period-two cycles can arise in an optimal growth model if the cross derivative of the indirect utility function satisfies $V_{12}^{\bar{\delta}} < 0$.[7] We can indeed easily prove the following Proposition:

Proposition 3. *Under Assumptions 1 and 2, let $V_{12}(x, x) < 0$. Assume also that there is a value $\bar{\delta} \in (0, 1)$ such that*

$$\bar{\delta} V_{11}^{\bar{\delta}} + V_{22}^{\bar{\delta}} - (1 + \bar{\delta}) V_{12}^{\bar{\delta}} > 0$$

Then there exists $\delta^* \in (\bar{\delta}, 1)$ such that when δ crosses δ^*, the optimal growth model has optimal solutions which are period-two cycles. Moreover, the bifurcation value δ^* is implicitly defined by

$$\delta^* = \frac{V_{22}^{\delta^*} - V_{12}^{\delta^*}}{V_{12}^{\delta^*} - V_{11}^{\delta^*}} \tag{16}$$

Proof. See Appendix 7.4. □

We derive therefore from Proposition 2 that the bifurcation value must satisfy $\delta^* < 1 - (\alpha + \beta)/\gamma$. It follows that δ^* cannot be arbitrarily close to 1 if $\alpha + \beta > 0$, i.e. if the indirect utility function is strongly concave. We thus need to assume that $\alpha + \beta = 0$, and more precisely that $\alpha = \beta = 0$. In other words, the Hessian matrix of the indirect utility function needs to be singular at the steady state. Let us introduce the following properties:

Assumption 4. *$V(x, y)$ is $(0, 0)$-concave and concave-(γ_1, γ_2) in the neighborhood of the steady state k_δ^* with $\gamma_2 < \gamma_1$.*

[7] As shown in Benhabib and Nishimura [4], such a property requires that the consumption good sector is more capital intensive than the investment good sector.

The $(0,0)$-concavity in the neighborhood of the steady state k_δ^* implies that the Hessian matrix of the indirect utility function is singular. The Assumption of concavity-(γ_1, γ_2) allows to prove the following result:

Proposition 4. *Under Assumptions 1, 2 and 4, let* $V_{12}(x,x) < 0$ *and the condition of Proposition 3 be satisfied. Then the bifurcation value* δ^* *is such that* $\lim_{\gamma_2 \to \gamma_1} \delta^* = 1$.

Proof. See Appendix 7.5. □

Proposition 4 shows that the existence of period-two cycles for small discounting does not depend per se on a low curvature of the indirect utility function. Of course, the indirect utility function must be non-strictly concave, but even if the coefficients of weak concavity γ_1 and γ_2 are quite large, the bifurcation value δ^* can still be (arbitrarily) close to one if γ_2 converges to γ_1 from below. This means that the existence of endogenous fluctuations under small discounting is coming from a symmetric concavity property of the indirect utility function with respect to its two arguments.

In a continuous-time setting, Benhabib and Rustichini [5] and Venditti [20] have studied the existence of endogenous fluctuations in multisector optimal growth models with small discounting. Their results appear to be quite different than in our discrete-time setting. Starting from the seminal contribution of Benhabib and Nishimura [2] showing the existence of periodic orbits through a Hopf bifurcation, Benhabib and Rustichini [5] prove that for any positive discount rate, even arbitrarily small, there exists a large family of standard Cobb-Douglas technologies with three sectors which have optimal growth paths of persistent cycles. However, they do not discuss their conditions in terms of the concavity properties of the technologies nor the indirect utility function.

Assuming that the indirect utility function $U(., \dot{k}(t))$ is concave-γ in a neighborhood of the steady state, Venditti [20] shows that the Hopf bifurcation value of the discount rate ρ is bounded from above by a function of γ that converges towards 0 as γ is chosen closer to 0. Put differently, the instability of the steady state and the existence of endogenous fluctuations under small discounting are obtained when the indirect utility function has a degree of curvature close to 0 with respect to the capital stock. Such a conclusion is therefore consistent with the results of Rockafellar [19]. Venditti also uses the Cobb-Douglas example of Benhabib and Rustichini [5] to interpret their conditions in terms of the curvature properties of the technologies. Using the fact that $U(., \dot{k}(t))$ can be weakly concave if the production function of the consumption good is itself weakly concave, and that the degree of weak concavity is locally measured by the smaller eigenvalue in absolute value of the Hessian matrix of the Cobb-Douglas function, he thus shows that when the discount rate compatible with a Hopf bifurcation is chosen closer to zero, the technological conditions provided by Benhabib and Rustichini imply a lower degree of weak concavity.

One may then wonder why such a difference between continuous and discrete-time models occurs. A first point worth to mention is that while endogenous fluctuations through period-two cycles can occur in a discrete-time two-sector optimal growth model, periodic-cycles only occur in a continuous-time setting if the economy is at least composed of three sectors with two investment goods.[8] A second difference relies on the fact that the existence of endogenous cycles is derived from conditions related to the capital stocks and their impact on the indirect utility function. But in a continuous-time setting, the indirect utility function depends on both the current capital stock and its growth rate, while in a discrete-time setting it depends on both the current and next period capital stocks and not on the variation of the stock. This explains why the technical conditions affect differently the indirect utility functions. However, beside these apparent differences, there are similar implications of the conditions.

In the continuous-time case, when the parameter γ of weak concavity converges towards zero, the indirect utility function $U(k(t), \dot{k}(t))$ becomes linear with respect to the capital stock in the neighborhood of the steady state, and, as a result, the matrix of second order derivatives with respect to the capital stocks $U_{kk}^{\delta *}$ is characterized by quadratic forms on the eigenspace

[8]In a two-sector optimal growth model, the optimal path is determined by the one-dimensional stable manifold. Therefore, in a continuous-time setting, endogenous cycles cannot occur as we know from the Poincaré-Bendixon theorem that a two-dimensional space is required. This configuration can thus be obtained only in a $n+1$-sector economy with $n \geq 2$ investment goods, in which the stable manifold has dimension n.

that converge to zero. But under the standard (non-strict) concavity property of $U(k(t), \dot{k}(t))$, this implies that the quadratic forms on the eigenspace of the two matrices $U^{\delta^*}_{k\dot{k}} + U^{\delta^*}_{\dot{k}k}$ and $U^{\delta^*}_{\dot{k}\dot{k}}$ also converge to zero, i.e. the Hessian matrix of the indirect utility function admits in the limit symmetric sub-matrices. In our discrete-time framework, while the condition on the weak concavity appears drastically different, a similar property occurs. Indeed, when γ_2 converges to γ_1 from below, Assumption 4 implies that $V^{\delta^*}_{22} \to V^{\delta^*}_{12} \to V^{\delta^*}_{11}$ and thus the Hessian matrix of the indirect utility function $V(k_t, k_{t+1})$ also admits in the limit symmetric terms.

In order to have a better appraisal of the conditions provided by Proposition 4, let us discuss some popular examples.

5 Discussion of examples

Let us consider first the well-known Weitzman example in which the indirect utility function is given by:

$$V(k_t, k_{t+1}) = k_t^\theta (1 - k_{t+1})^\rho$$

with $\theta, \rho > 0$ and $\theta + \rho \leq 1$. It is easy to note that when $\theta + \rho < 1$, $V(k_t, k_{t+1})$ is locally strongly concave. The Euler equation is

$$-\rho k_t^\theta (1 - k_{t+1})^{\rho - 1} + \delta\theta k_{t+1}^{\theta - 1} (1 - k_{t+2})^\rho = 0$$

and the associated steady state is

$$k^*_\delta = \tfrac{\delta\theta}{\rho + \delta\theta}$$

Straightforward computations give the characteristic polynomial

$$P(\lambda) = \lambda^2 + \lambda \left(\tfrac{1-\theta}{\delta\theta} + \tfrac{1-\rho}{\rho} \right) + \tfrac{1}{\delta}$$

It follows that when $\delta = 1$, the steady state is saddle-point stable as $P(-1) < 0$, and $\lim_{\delta \to 0} P(-1) > 0$ if and only if $\theta > 1/2$. Therefore, there exists a bifurcation value such that:

$$\delta^* = \tfrac{\rho(2\theta - 1)}{\theta(1 - 2\rho)}$$

leading to the existence of period-two cycles through a flip bifurcation. It clearly appears that if $\theta + \rho < 1$, the bifurcation value δ^* cannot be made arbitrarily close to one. On the contrary, if $\theta + \rho = 1$, δ^* can be arbitrarily close to 1 when θ ($\rho = 1-\theta$) tends to 1/2 from above (below). Using Lemma 2 and Definition 5, the coefficients γ_1 and γ_2 of weak concavity are given at the steady state by the second-order derivatives $V_{11}(k_\delta^*, k_\delta^*)$ and $V_{22}(k_\delta^*, k_\delta^*)$. It can then be checked that, as shown by Proposition 4, $V_{11}(k_\delta^*, k_\delta^*)$ converges towards $V_{22}(k_\delta^*, k_\delta^*)$ as θ tends to 1/2.

Let us consider now the example of Boldrin and Deneckere [6] where the consumption and capital goods are respectively produced with Cobb-Douglas and Leontief functions such that

$$\begin{aligned} f^0(k_0, l_0) &= k_0^{1-\alpha} l_0^\alpha \\ f^1(k_1, l_1) &= \min\{k_1/\gamma, l_1\} \end{aligned} \tag{17}$$

with $\alpha \in (0,1)$ and $\gamma \in (0,1)$. Under a linear utility function, and assuming for simplification a constant population ($n = 0$) and a complete depreciation of capital in each period ($\mu = 1$), the indirect utility function is easily derived as

$$V(k_t, k_{t+1}) = (1 - k_{t+1})^\alpha (k_t - \gamma k_{t+1})^{1-\alpha}$$

Since this function is homogeneous of degree one, it is non-strictly concave and has a singular Hessian matrix. As shown in Boldrin and Deneckere [6], the steady state is given by the following expression

$$k_\delta^* = \frac{(\delta-\gamma)(1-\alpha)}{(\delta-\gamma)(1-\alpha)+\alpha(1-\gamma)}$$

and $k_\delta^* > 0$ requires $\gamma < \delta$. Moreover, straightforward computations show that $V_{12}(k_\delta^*, k_\delta^*) < 0$ if and only if $\gamma > \delta(1-\alpha)$. We assume therefore that $\gamma \in (\delta(1-\alpha), \delta)$. The characteristic polynomial reduces to

$$P(\lambda) = \lambda^2 - \lambda \frac{\delta + \left(\frac{\gamma-\delta(1-\alpha)}{\alpha}\right)^2}{\frac{\delta[\gamma-\delta(1-\alpha)]}{\alpha}} + \frac{1}{\delta}$$

It follows that

$$P(-1) = \frac{[\gamma-\delta(1-2\alpha)][\gamma+\alpha-\delta(1-\alpha)]}{\alpha\delta[\gamma-\delta(1-\alpha)]} = 0$$

if and only if δ is equal to one of the following values:

$$\delta_1^* = \frac{\gamma+\alpha}{1-\alpha}, \quad \delta_2^* = \frac{\gamma}{1-2\alpha}$$

with $\delta_2^* < \delta_1^* < 1$ if and only if $\gamma < 1-2\alpha$. We consider therefore $\delta_1^* = \delta^*$ as the flip bifurcation value and we conclude obviously that δ^* can be arbitrarily close to 1 as γ tends to $1-2\alpha$ from below. But at the same time, obvious computations show again that $V_{11}(k_\delta^*, k_\delta^*)$ converges towards $V_{22}(k_\delta^*, k_\delta^*)$.

6 Concluding comments

We have studied the existence of endogenous competitive equilibrium cycles under small discounting in a two-sector discrete-time optimal growth model. Assuming an indirect utility function $V(x, y)$ concave-(γ_1, γ_2), i.e. that has degrees of curvature a priori different across its two arguments, period-two cycles can arise through a flip bifurcation under a discount factor arbitrarily close to one if γ_2 converges to γ_1 from below. Such a condition means that the existence of endogenous fluctuations under small discounting is obtained under a symmetric concavity property of the indirect utility function with respect to its two arguments. Contrary to the continuous-time case where the existence of periodic-cycles is obtained if the degree of concavity is close to zero, we show that in a discrete-time setting the driving condition does not require any small value for the coefficients γ_1 and γ_2.

Our analysis concerns two-sector models. The question of the robustness of our conclusions therefore arises for multi-sector models. Moreover, we have discussed the existence of period-two cycles through a flip bifurcation as in a two-sector model only real characteristic roots can occur. In a multi-sector framework, complex characteristic roots and thus the possible existence of a Hopf bifurcation can be also considered. Such analyses could be done using the techniques developed in Cartigny and Venditti [10] and Venditti [21]. This is left for future researches.

7 Appendix

7.1 Proof of Lemma 1

i) This is a standard result (see Rockafellar [19], Montrucchio [14]).
ii) From i) we get

$$\nu^t \left[D^2 f(x) + \alpha I\right] \nu \leq 0$$

which means that the matrix $[D^2 f(x)\nu + \alpha I]$ should be negative semi-definite for all $x \in D$, i.e. there must exist a $\tilde{\nu} \neq 0$ such that

$$\tilde{\nu}^t \left[D^2 f(x) + \alpha I\right] \tilde{\nu} = 0$$

This means therefore that

$$Det\left[D^2 f(x) + \alpha I\right] \equiv Det\left[D^2 f(x) - (-\alpha) I\right] = 0$$

It follows that $-\alpha$ is an eigenvalue of the matrix $D^2 f(x)$. Considering $\lambda_i(x)$ an eigenvalue of $D^2 f(x)$, we may define for all $x \in D$

$$\lambda^*(x) = \min_{i=1,\ldots,n} \{|\lambda_i(x)|\}$$

and we conclude therefore that $\alpha = \inf_{x \in D} \lambda^*(x)$. If f is α-concave over D then $\alpha \neq 0$ since the Hessian matrix is non-singular for all $x \in D$. □

7.2 Proof of Lemma 2

The arguments are the same as in the proof of Lemma 1. □

7.3 Proof of Proposition 2

Consider first the case $V_{12}^\delta > 0$. We know from (14) that $P(0) = V_{12}^\delta > 0$ and $\lim_{\lambda \to \pm\infty} P(\lambda) = +\infty$. It follows that the steady state is a saddle-point if and only if

$$P(1) = \delta V_{11}^\delta + V_{22}^\delta + (1+\delta) V_{12}^\delta < 0$$

If $V(x, y)$ is (α, β)-concave, we derive from Lemma 1 that the following quadratic form holds

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} V_{11}^\delta + \alpha & V_{12}^\delta \\ V_{12}^\delta & V_{22}^\delta + \beta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq 0$$

which implies

$$\alpha + \beta \leq -2V_{12}^\delta - V_{11}^\delta - V_{22}^\delta$$

If $V(., k_\delta^*)$ is concave-γ, we derive from Lemma 2 that $\gamma \geq -V_{11}^\delta$. We then derive

$$\tfrac{\alpha+\beta}{\gamma} \leq -\tfrac{2V_{12}^\delta + V_{11}^\delta + V_{22}^\delta}{V_{11}^\delta}$$

Assume now that $1 \geq \delta > 1 - (\alpha + \beta)/\gamma$. We derive from the previous inequality

$$\delta > \tfrac{2V_{12}^\delta + V_{22}^\delta}{V_{11}^\delta}$$

which implies

$$\delta V_{11}^\delta + V_{22}^\delta + 2V_{12}^\delta < 0$$

Since $2V_{12}^\delta \geq (1+\delta) V_{12}^\delta$ we conclude that

$$P(1) = \delta V_{11}^{\delta} + V_{22}^{\delta} + (1+\delta)V_{12}^{\delta} < 0$$

and the steady state is a saddle-point.

Consider now the case $V_{12}^{\delta} < 0$. We know from (14) that $P(0) = V_{12}^{\delta} < 0$ and $\lim_{\lambda \to \pm\infty} P(\lambda) = -\infty$. It follows that the steady state is a saddle-point if and only if

$$P(-1) = -\left[\delta V_{11}^{\delta} + V_{22}^{\delta} - (1+\delta)V_{12}^{\delta}\right] > 0$$

If $V(x, y)$ is (α, β)-concave, we derive from Lemma 1 that the following quadratic form holds

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} V_{11}^{\delta} + \alpha & V_{12}^{\delta} \\ V_{12}^{\delta} & V_{22}^{\delta} + \beta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \leq 0$$

which implies

$$\alpha + \beta \leq 2V_{12}^{\delta} - V_{11}^{\delta} - V_{22}^{\delta}$$

If $V(., k_{\delta}^{*})$ is concave-γ, we derive from Lemma 2 that $\gamma \geq -V_{11}^{\delta}$. We then derive

$$\tfrac{\alpha+\beta}{\gamma} \leq -\tfrac{2V_{12}^{\delta} + V_{11}^{\delta} + V_{22}^{\delta}}{V_{11}^{\delta}}$$

Assume now that $1 \geq \delta > 1 - (\alpha + \beta)/\gamma$. We derive from the previous inequality

$$\delta > \tfrac{2V_{12}^{\delta} - V_{22}^{\delta}}{V_{11}^{\delta}}$$

which implies

$$\delta V_{11}^{\delta} + V_{22}^{\delta} - 2V_{12}^{\delta} < 0$$

Since $2V_{12}^{\delta} \leq (1+\delta)V_{12}^{\delta}$ we conclude that

$$P(-1) = -\left[\delta V_{11}^{\delta} + V_{22}^{\delta} - (1+\delta)V_{12}^{\delta}\right] > 0$$

and the steady state is a saddle-point. The results of the Proposition follow.
□

7.4 Proof of Proposition 3

Consider in a first step the case $\delta = 1$. Assumption 4 implies that $V_{11}^{1} \neq V_{22}^{1}$. Since we assume $V_{12}^{\delta} < 0$, we derive from the proof of Proposition 2 that

$$P(-1) = V_{11}^{1} + V_{22}^{1} - 2V_{12}^{1}$$

The singularity of the Hessian matrix of V at the steady state implies then $V_{11}^{1}V_{22}^{1} - (V_{12}^{1})^2 = 0$ or equivalently $V_{12}^{1} = -\sqrt{V_{11}^{1}V_{22}^{1}}$. Substituting this into $P(-1)$ yields

$$P(-1) = -\left(\sqrt{|V_{11}^1|} - \sqrt{|V_{22}^1|}\right)^2 < 0$$

It follows that when $\delta = 1$ the steady state is a saddle-point. Let us then assume that there is a value $\bar{\delta} \in (0, 1)$ such that

$$\bar{\delta} V_{11}^{\bar{\delta}} + V_{22}^{\bar{\delta}} - (1 + \bar{\delta}) V_{12}^{\bar{\delta}} > 0$$

Then there exists $\delta^* \in (\bar{\delta}, 1)$ such that

$$P(-1) = -\left[\delta^* V_{11}^{\delta^*} + V_{22}^{\delta^*} - (1 + \delta^*) V_{12}^{\delta^*}\right] = 0$$

and there exists an eigenvalue $\lambda(\delta^*) = -1$ leading to the existence of a flip bifurcation, i.e. period-two cycles in a neighborhood of δ^*.

□

7.5 Proof of Proposition 4

Using Lemma 2 and Definition 5, we have that the following matrix is positive semi-definite:

$$\begin{pmatrix} V_{11}^{\delta^*} + \gamma_1 & V_{12}^{\delta^*} \\ V_{12}^{\delta^*} & V_{22}^{\delta^*} + \gamma_2 \end{pmatrix}$$

This means in particular that its determinant is equal to zero:

$$(V_{11}^{\delta^*} + \gamma_1)(V_{22}^{\delta^*} + \gamma_2) - (V_{12}^{\delta^*})^2 = 0$$

Using also the fact that $V_{11}^{\delta^*} V_{22}^{\delta^*} - (V_{12}^{\delta^*})^2 = 0$ we obtain that:

$$\gamma_1 = -2V_{11}^{\delta^*}, \quad \gamma_2 = -2V_{22}^{\delta^*}$$

Let us now consider the bifurcation value given by equation (16). We then have:

$$\delta^* = \frac{\sqrt{\gamma_1\gamma_2} - \gamma_2}{\gamma_1 - \sqrt{\gamma_1\gamma_2}} = \sqrt{\frac{\gamma_2}{\gamma_1}} \qquad (18)$$

We first note that for the bifurcation value to be less than 1, we need to assume that $\gamma_2 < \gamma_1$. Second, we conclude that if $(\gamma_1 - \gamma_2) \to 0$ then the bifurcation value $\delta^* \to 1$.

□

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