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# An Explicit Solution to Harvesting Behaviors in a Predator-Prey System

Guillaume BATAILLE

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# An Explicit Solution to Harvesting Behaviors in a Predator-Prey System[⋆]

Guillaume BATAILLE[1]

*Aix Marseille Univ, CNRS, AMSE, Marseille, France.*

## Abstract

This paper derives closed-form solutions for a *strategic*, *simultaneous* harvesting in a predator-prey system. Using a parametric constraint, it establishes the existence and uniqueness of a linear feedback-Nash equilibrium involving two specialized fleets and allow for continuous time results for a class of payoffs that have constant elasticity of the marginal utility. Theses results contribute to the scarce literature on analytically tractable predator-prey models with endogenous harvesting. A discussion based on industry size effects is provided to highlight the role played by biological versus strategic interactions in the multi-species context.

*Keywords:* Fisheries, Dynamic games, Common-pool Resource, Predator-prey relationship
*JEL classification:* Q22, Q57, C61, C73

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[*] Corresponding author

*Email address:* `guillaume.bataille@univ-amu.fr` (Guillaume BATAILLE)

[1] Address: AMSE, Maison de l'économie et de la gestion d'Aix, 424 chemin du viaduc, 13080 Aix-en-Provence, France

## 1. Introduction

While extensive research exists on single-species harvesting, there has been a shift towards more ecosystem-based fishery management (EBFM). Addressing economic questions and informing policymakers require models with a broader perspective of fish resources and presenting technical challenges such as increased state variables and nonlinear species growth functions. This study addresses these challenges by presenting a workable parametric example considering *strategic, simultaneous* and *selective* harvesting in a two-species predator-prey system. This serves as a basis for exploring policy recommendations and inefficiencies in dynamic models of endogenous fishing.

Species exist within complex ecosystems and cannot be seen as separate resources. Nevertheless, modeling interacting species presents challenges owing to nonlinear species dynamics. Despite this, biologists and mathematicians explored multi-species models, especially predator-prey models, owing to their interesting dynamic properties such as periodic solutions, limit cycles, and stability. Economic studies on multi-species models are scarce, partly because the introduction of endogenous harvesting increases complexity. Early works that integrated harvesting behaviors in two-species models seeking long-term solutions, include works by Hannesson [14], Flaaten [12], May et al. [18], and Querou and Tomini [20], among others. This study provides the first insights into multispecies management challenges, the influence of fish market prices, and the trade-offs in ecosystem management, as discussed very recently by Bataille et al. [2]. However, few studies analytically characterize strategic and optimal harvesting paths in multi-species models. A significant study by Fischer and Mirman [10] extends Levhari and Mirman [17] to include a biological externality in which two players fish for two distinct interacting species, offering initial insights into the exploitation mechanisms in such settings. Fischer and Mirman [11] further develop their model by adding direct strategic interactions, which they call ‘’compleat fish wars’. These foundational discrete-time analyses spurred extensive economic studies in this area. Extensions include models with $n$ players by Okuguchi [19], the consideration of asymmetries by Rettieva [21], the exploration of cooperative behavior and coalition stability by Breton and Keoula [3], and the introduction of uncertainty by Antoniadou et al. [1].

The literature relies on parametric examples because of the challenge of obtaining general results. However, the choices made by modelers, such as solution concepts and functional forms, are subject to criticism. Traditionally, the fishery literature focused primarily on examining open-loop Nash equilibria, often referred to as commitment strategies, because of their relative ease of implementation (see, e.g., Clark and Munro [5]). A significant drawback of this equilibrium concept is its vulnerability to perturbations, which is a particularly pertinent issue in fisheries where disturbances are common. In addition, whether fishermen can consistently commit to a fixed temporal trajectory remains unanswered. In this study, we did not aim to affirm that feedback strategies inherently offer superior predictive capabilities for real-life fishery problems. Rather, we present a specific illustrative example in which fishers use decision rule strategies that can be derived analytically, and the constraints imposed on the primitives of the problem are comparatively less stringent than those in the existing literature.

Even in single-species models, the existing literature demonstrated that obtaining analytical solutions for feedback strategies can be challenging, with some exceptions for specific game classes (refer to Dockner et al. [7] for a comprehensive review). Notably, analytical tractability is facilitated when the problem is linearized in state, assuming that the objective and natural growth functions of the species share at least one common parameter (see Van Long [22] and Gaudet and Lohoues [13] for more details). This requirement connects the payoff function structure to the reproductive function of the species. In the multispecies context, several studies implicitly adopted this assumption by employing logarithmic utility functions and Cobb-Douglas species growth functions (e.g., Fischer and Mirman [10]; Doyen et al. [9]; Breton et al. [4]). Expanding beyond this specific framework requires explicitly imposing constraints on player payoffs and species growth functions. Koulovatianos [16] recently addresses the issue by examining corner solutions and introducing uncertainty, a departure from the standard log-utility and Cobb-Douglas growth models. While the author does not explicitly provide justification for these constraints, they possess the notable quality of enabling (*i*) continuous-time results within a class of constant elasticity of marginal utility, facilitating the full characterization of feedback strategies, and (*ii*) precise examination of the consequences of simultaneous harvesting across ecosystems with multiple species, along with associated policy implications that remain insufficiently understood.

Our aim is to enrich the existing literature by presenting a tractable model for simultaneous exploitation in a predator-prey system. We diverge from recent works such as Doyen et al. [9] and Breton et al. [4], which mainly use the same model as in Fischer and Mirman [10], that is, a discrete-time analysis with logarithmic payoffs and Cobb-Douglas species growth functions. Rather, we build on Koulovatianos [16] by opting for a continuous-time framework with payoff functions with a constant elasticity of marginal utility, following Gaudet and Lohoues [13]'s recommendations for the choice of species dynamics. Our model introduces two key extensions from Koulovatianos [16]: (*i*) we consider equilibria involving simultaneous harvesting across the ecosystem, and (*ii*) we contrast strategic solutions with a centralized system accounting for all externalities. This model not only stands as a valuable example in itself but also aids researchers in exploring species harvesting behaviors in a less restrictive context and may guide future research toward more general model formulations.

Using Koulovatianos [16]'s approach, we model an infinite-horizon strategic dynamic setting in which two specialized fleets target either the prey or predator species and are allowed to catch them simultaneously. We apply a parametric constraint linking fishers' payoffs to the biological growth of both species, enabling analytical solutions for harvesting and stock flows. Subsequently, we introduce a sole-owner management scenario for ecosystem efficiency and compare its outcomes with those of a decentralized solution. The key findings are as follows. We identify the existence of a unique feedback–Nash equilibrium with species-specific linear harvesting strategies. This study also offers an optimal extraction approach involving catching a fixed proportion of each stock. Furthermore, we compare the decentralized and centralized systems, indicating the potential for over-or underfishing of predators. Finally, we provide a numerical illustration that elucidates the influence of industry size on stock levels and harvesting flows in both regimes.

The remainder of this paper is structured as follows. The next section introduces the model's core components. Section 3 outlines the regimes decentralized and centralized regimes. Section 4 discusses the existence and uniqueness of the feedback-Nash equilibrium with linear species-specific harvesting strategies. In Section 5, we compute and compare the outcomes of the centralized solution with those of the decentralized regime, focusing on comparative statics to assess fleet size effects on fishing pressure. Section 6 provides stylized examples of stock and harvesting flows for various industry sizes. All the proofs are provided in the Appendix.

## 2. The model

The theoretical framework of this study is primarily based on that of Koulovatianos [16]. It centers on a two-species system consisting of a prey population denoted by $x(t)$ and a predator population denoted by $y(t)$. The dynamics of these populations are described by the following system of first-order non-linear differential equations:

$$\begin{aligned} \dot{x}(t) &= A_x x(t)^{\theta} - \delta_x x(t) - b_x y(t)^{1-\theta} x(t)^{\theta}, \\ \dot{y}(t) &= A_y y(t)^{\theta} - \delta_y y(t) + b_y x(t)^{1-\theta} y(t)^{\theta}, \end{aligned} \tag{1}$$

where $\dot{x}(t)$ and $\dot{y}(t)$ represent the time variation of prey and predator stock, respectively. The parameters in the model have the following interpretations for each species $s \in \{x, y\}$: $A_s$ is the intrinsic reproductive rate, $0 < \theta < 1$ scales the reproductive rate with the population size, and $\delta_s$ represents the natural mortality rate. The parameters $b_x$ and $b_y$ denote the rates at which predators consume and convert prey, respectively, and they depend on the predator-to-prey population ratios. Therefore, the natural growth function of species $A_s s(t)^{\theta} - \delta_s s(t)$ is strictly concave and has a typical inverted U shape. This model incorporates the Holling Type II response, which accounts for the limited prey-processing ability of predators, whereby prey availability relative to population size constrains the predator's consumption rate.

The initial assumption constrains the parameter space to ensure a positive steady-state without human intervention.

**Assumption 1.** $A_x \delta_y > A_y b_x$.

This restriction sets a lower bound on the relative implicit growth rates of the prey and predators, ensuring a relatively large prey growth rate for coexistence. This guarantees a positive interior steady state $(\bar{x}, \bar{y})$ in system (1), maintaining non-negative populations for both species.

We next introduce the economic activities. We focus on prey and predator harvesting, represented by $h_s(t)$ for species $s \in \{x, y\}$. These populations are harvested by two distinct groups of specialized harvesters, each comprising $n_s$ individuals who derive utility from fishing. We denote the common discount factor as $\rho > 0$ and consider a prey fisher denoted by $i \in \{1, 2, \ldots, n_x\}$ and a predator fisher denoted by $j \in \{1, 2, \ldots, n_y\}$. These individuals derive utility over an infinite time horizon through harvesting activities. The payoff functions associated with these agents exhibit standard properties and can be described as follows:

$$\left[\mathcal{J}^{s,k}\left(h_{s,k}(\cdot)\right)\right]_{s=x,y}^{k=1,\ldots,n_s} = \int_0^{\infty} \frac{h_{s,k}(t)^{1-\nu}}{1-\nu} \exp^{-\rho t} dt, \tag{2}$$

where the parameter $\nu$ represents the inter-temporal smoothness preference in harvesting and is assumed to be identical across all players and industries[1]. Therefore, the species dynamics under simultaneous and specialized harvesting are given as follows:

$$\begin{aligned} \dot{x}(t) &= A_x x(t)^{\theta} - \delta_x x(t) - b_x y(t)^{1-\theta} x(t)^{\theta} - \textstyle\sum_{i=1}^{n_x} h_{x,i}(t), \\ \dot{y}(t) &= A_y y(t)^{\theta} - \delta_y y(t) + b_y x(t)^{1-\theta} y(t)^{\theta} - \textstyle\sum_{j=1}^{n_y} h_{y,j}(t). \end{aligned} \tag{3}$$

For tractability in this context, a connection between the objective function and species dynamics is essential, as Gaudet and Lohoues [13] and Van Long [22] note. Considering payoffs with constant marginal utility elasticity $\nu$, we equalize this elasticity to the concavity level $\theta$ of the growth function of the species.

**Assumption 2.** $\theta = \nu$

Here, we unify $\theta$ and $\nu$ as a single parameter. While this may limit empirical analysis, it facilitates analytical results in predator-prey models with simultaneous harvesting, which is our primary focus. Future work could extend these findings by relaxing this assumption through numerical simulations. However, this analysis is beyond the scope of this study.

## 3. Decentralized and centralized maximization problems

### *3.1. The differential game*

The strategic framework for this problem is conventional. It involves two distinct groups of specialized players, or fishers, who make decisions regarding their harvesting

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[1] Theses payoffs functions belong to the class utility with constant elasticity of marginal utility. Logarithmic preference are only a limiting case of (2) when $\nu$ approaches 1. Using L'Hôpital's Rule, we have $\lim_{\nu\to 1}(1-\theta)h^{-\theta}/\lim_{\nu\to 1}(-1) \Leftrightarrow \lim_{\nu\to 1}\frac{d(1-\nu)log(h)}{d\nu}/\lim_{\nu\to 1}\frac{d(1-\nu)}{d\nu} \Leftrightarrow \lim_{\nu\to 1}(-1)log(h)/\lim_{\nu\to 1}(-1) = log(h)$.

plans for specific species. Specifically, given $x(0), y(0) > 0$ a prey fisher $i$ choose his harvesting paths, $h_{x,i}(t)$, that maximizes (2) while considering species dynamics (3) and feedback strategies from other prey fishers, $\phi_{x,k}(x(t), y(t))$ $\forall k \in \{1, \ldots, n_x - 1\}$ and predator fishers' feedback strategies, $\phi_{y,j}(x(t), y(t)) \forall j \in \{1, \ldots, n_y\}$. The maximization problem is

$$\max_{h_{x,i}(\cdot)} \mathcal{J}^{x,i}\Big(h_{x,i}(\cdot)\Big)$$

$$\text{s.t.} \quad \begin{aligned} &\dot{x}(t) = A_x x(t)^\theta - \delta_x x(t) - b_x y(t)^{1-\theta} x(t)^\theta - h_{x,i}(t) - \sum_{k=1, k\neq i}^{n_x} \phi_{x,k}(x(t), y(t)) \\ &\dot{y}(t) = A_y y(t)^\theta - \delta_y y(t) + b_y x(t)^{1-\theta} y(t)^\theta - \sum_{j=1}^{n_y} \phi_{y,j}(x(t), y(t)) \\ &x(0), y(0) > 0. \end{aligned} \tag{4}$$

Likewise, given $x(0), y(0) > 0$ a predator fisher $j$ chooses the flow of harvest, $h_{y,j}(t)$, such that (2) is maximized knowing the species dynamics (3) and the feedback strategies of other predator fishers $\phi_{y,k}(x(t), y(t))$ $\forall k \in \{1, \ldots, n_y - 1\}$, and prey fishers' feedback strategies $\phi_{x,i}(x(t), y(t)) \forall i \in \{1, \ldots, n_x\}$. The maximization problem can be formally written as

$$\max_{h_{y,j}(\cdot)\geq 0} \mathcal{J}^{x,i}\Big(h_{x,i}(\cdot)\Big)$$

$$\text{s.t.} \quad \begin{aligned} &\dot{x}(t) = A_x x(t)^\theta - \delta_x x(t) - b_x y(t)^{1-\theta} x(t)^\theta - \sum_{i=1}^{n_x} \phi_{x,i}(x(t), y(t)) \\ &\dot{y}(t) = A_y y(t)^\theta - \delta_y y(t) + b_y x(t)^{1-\theta} y(t)^\theta - h_{y,j}(t) - \sum_{k=1, k\neq j}^{n_y} \phi_{y,k}(x(t), y(t)) \\ &x(0), y(0) > 0. \end{aligned} \tag{5}$$

This dynamic game involves two externalities: strategic interactions within industries competing for common stock, and cross-industry effects due to predator-prey relationships among species. In this context, decision rules are inherently history-independent, allowing the application of dynamic programming techniques. The assumption is that fishers can observe and react to stock levels at a given time independent of past events. It is reasonable to assume that because our problem is autonomous, we consider only time-independent strategies. The timing of the decision does not alter the game's infinite nature or fundamentals. Therefore, decision rules are influenced only by time through stock levels.

Solving this problem, particularly in the context of two interacting species, remains challenging and potentially yields infinite solutions, as Dockner and Sorger [8] discuss.

To address this complexity, we restrict the strategy space of the players. Specifically, building on the common property natural resource game literature, including Van Long and Shimomura [23], Clemhout and Wan [6], and Fischer and Mirman [10], we seek decision rules that are linear functions of stock size. Each fisher is presumed to believe that others employ species-specific linear harvesting strategies, allowing adaptive adjustments based on the current stock relevant to their specialization:

$$\left(\phi_{s,k}(s(t))\right)_{s=x,y}^{k=1,\dots,n_s} = \omega_{s,k} s(t), \tag{6}$$

where $\omega_{s,k}$ represents the individual catch rates for a player in industry $s$. This assumption implies that fishers targeting prey or predators can adapt their strategies based solely on the current stock levels of their target species. Consequently, the concept of equilibrium was introduced.

**Definition 1.** *$(\omega_{x,1}^{NE}x(t), \omega_{x,2}^{NE}x(t), \dots, \omega_{x,n_x}^{NE}x(t))$ and $(\omega_{y,1}^{NE}y(t), \omega_{y,2}^{NE}y(t), \dots, \omega_{y,n_y}^{NE}y(t))$ constitute a linear Feedback Nash Equilibrium $(FBNE)$ if and only if*

- *For each $i \in \{1, \dots, n_x\}$, there exists an optimal control path $h_{x,i}^{NE}(t)$ solution to (4), given by the feedback strategy $h_{x,i}^{NE}(t) = \omega_{x,i}^{NE}x(t)$ and,*
- *For each $j \in \{1, \dots, n_y\}$, there exists an optimal control path $h_{y,j}^{NE}(t)$ solution to (5), given by the feedback strategy $h_{y,j}^{NE}(t) = \omega_{y,j}^{NE}y(t)$.*

### *3.2. The sole-owner problem*

We address the scenario of a sole owner with exclusive rights and the requisite technology to harvest both species simultaneously. This simplified hypothetical problem aims to formulate the optimal policies for managers or institutions to exploit and manage an ecosystem over time. The decision maker selects individual harvesting paths for the prey and predators, denoted by $h_{x,i}(t)$ for $i \in \{1, \dots, n_x\}$ and $h_{y,j}(t)$ for $j \in \{1, \dots, n_y\}$, respectively, with the objective of maximizing the sum of the discounted payoffs. Thus, the optimal control problem is

$$\max_{(h_{s,k}(\cdot))_{s=x,y}^{k=1,\dots,n_s} \geq 0} \sum_{k=1}^{n_s} \mathcal{J}^{s,k}\left(h_{s,k}(\cdot)\right)_{s=x,y}^{k=1,\dots,n_s}$$

$$\text{s.t.} \quad \begin{aligned} &\dot{x}(t) = A_x x(t)^\theta - \delta_x x(t) - b_x y(t)^{1-\theta} x(t)^\theta - \sum_{i=1}^{n_x} h_{x,i}(t) \\ &\dot{y}(t) = A_y y(t)^\theta - \delta_y y(t) + b_y x(t)^{1-\theta} y(t)^\theta - \sum_{j=1}^{n_y} h_{y,j}(t) \\ &x(0), y(0) > 0 \end{aligned} \tag{7}$$

We seek to identify the two control vectors that yield the highest average utility across both industries. Compensation may occur, allowing the sole owner to implement harvesting in the industry, which contributes to an overall higher utility. This concept of compensation is crucial in our context, considering the interdependence of species and potential system-wide repercussions of harvesting.

**Definition 2.** *An optimal solution is defined as a specific set of harvesting strategies,* $(\hat{h}_{x,1}(t),\ldots,\hat{h}_{x,n_x}(t))$ *and* $(\hat{h}_{y,1}(t),\ldots,\hat{h}_{y,n_y}(t))$ *that solve (7).*

## 4. Species-specific linear feedback-Nash equilibrium

To establish the equilibrium existence of the differential game given by Definition 1, our analysis relies on dynamic programming techniques, and more specifically, Theorem 4.1 of Dockner et al. [7]. This theorem specifies three essential conditions for the existence of a feedback-Nash equilibrium: $(i)$ the existence of a value function satisfying the Hamilton-Jacobi-Bellman equation for each player, $(ii)$ a unique, continuous solution for the system dynamics driven by feedback strategies, $(iii)$ convergence of the value function. This section is structured around the three aforementioned conditions.

Our problem involves two industries, each with symmetrical players, exploiting one of the two interacting species. Rather than finding two sets of value functions for each player in both industries, the symmetry property within industries allows us to focus on finding only a pair of value functions, $V^x(x,y)$ and $V^y(x,y)$, which represent individual prey and predator fishers, respectively. The Hamilton-Jacobi-Bellman $(HJB)$ equations for the individual prey and predator fishers, respectively, are given by

$$\begin{aligned}\rho V^x(x,y) = \max_{h_x\geq 0}\Bigg\{&\frac{h_x^{1-\theta}}{1-\theta} + \partial_x V^x(x,y)\Bigg(A_x x^\theta - \delta_x x - b_x y^{1-\theta}x^\theta - h_x - (n_x-1)\omega_x x\Bigg) \\ &+ \partial_y V^x(x,y)\Bigg(A_y y^\theta - \delta_y y + b_y x^{1-\theta}y^\theta - n_y\omega_y y\Bigg)\Bigg\}\end{aligned} \tag{8}$$

and,

$$\begin{aligned}\rho V^y(x,y) = \max_{h_y\geq 0}\Bigg\{&\frac{h_y^{1-\theta}}{1-\theta} + \partial_x V^y(x,y)\Bigg(A_x x^\theta - \delta_x x - b_x y^{1-\theta}x^\theta - n_x\omega_x x\Bigg) \\ &+ \partial_y V^y(x,y)\Bigg(A_y y^\theta - \delta_y y + b_y x^{1-\theta}y^\theta - h_y - (n_y-1)\omega_y y\Bigg)\Bigg\}.\end{aligned} \tag{9}$$

In each case, the left-hand side denotes the present value of the objective in state $(x,y)$. This value should be equal to the immediate reward of choosing harvesting optimally, in addition to the expected future rewards resulting from state changes and knowing that opponent fishers use linear species-specific strategies. We construct $V^x(x,y)$ and $V^y(x,y)$ such that they have the following functional forms:

$$V^x(x,y) = \alpha_x + \frac{\beta_x x^{1-\theta}}{1-\theta} + \frac{\gamma_x y^{1-\theta}}{1-\theta} \quad \text{and} \quad V^y(x,y) = \alpha_x + \frac{\beta_y x^{1-\theta}}{1-\theta} + \frac{\gamma_y y^{1-\theta}}{1-\theta}, \tag{10}$$

where $\alpha_x, \beta_x, \gamma_x, \alpha_y, \beta_y, \gamma_y$ must be defined by identification. The optimal condition of the $HJB$ equations given in (8) and (9) implies that a prey and predator fisher will choose their harvesting such that they exploit a fixed fraction of the stock of fish they target.

$$h_x = \beta_x^{-1/\theta} x = \omega_x x \quad \text{and} \quad h_y = \gamma_y^{-1/\theta} y = \omega_y x, \tag{11}$$

where $\beta_x^{-1/\theta} = \omega_x$ and $\gamma_y^{-1/\theta} = \omega_y$ indicate the individual catch rates. Substituting Equations (10) and (11) into (8) and (9) leads to a system of equations determining the values of the six coefficients $\alpha_x$, $\beta_x$, $\gamma_x$, $\alpha_y$, $\beta_y$, and $\gamma_y$ simultaneously. The following lemma provides a solution for individual catch rates within industries.

**Lemma 1.** *By identification, there exist a unique set of parameters $\alpha_x$, $\beta_x$, $\gamma_x$, $\alpha_y$, $\beta_y$, and $\gamma_y$ such that $V^x(x,y)$ and $V^y(x,y)$ in (10) satisfy (8) and (9) for all $(x,y)$. Moreover, $\beta_x^{-1/\theta} = \omega_x^{NE}$ and $\gamma_y^{-1/\theta} = \omega_y^{NE}$ solve the following implicit system of equations:*

$$\left(\frac{\rho}{1-\theta} - \omega_x^{NE}(\frac{1}{1-\theta} - n_x) + \delta_x\right)\left(\frac{\rho}{1-\theta} + \delta_y + n_y\omega_y^{NE}\right) + b_x b_y = 0, \tag{12}$$

$$\left(\frac{\rho}{1-\theta} - \omega_y^{NE}(\frac{1}{1-\theta} - n_y) + \delta_y\right)\left(\frac{\rho}{1-\theta} + \delta_x + n_x\omega_x^{NE}\right) + b_y b_x = 0, \tag{13}$$

*and are positively valued if and only if $n_x < \frac{1}{1-\theta}$ and $n_y < \frac{1}{1-\theta}$.*

**Remark 1.** *Setting $n_y = 0$ in Equation (12) and $n_x = 0$ in Equation (13) yields the corner solutions for only one active industry, as in Koulovatianos [16].*

The second important condition, as outlined in Dockner et al. [7], requires that we investigate whether the system dynamics (3) yield a unique and continuous solution when fishers choose to extract a fraction of the fish stock. Under linear harvesting strategies expressed as (6), the system dynamics (3) can be equivalently reformulated as follows:

$$\begin{aligned} \dot{x}(t) &= A_x x(t)^\theta - \delta_x x(t) - b_x y(t)^{1-\theta} x(t)^\theta - n_x \omega_x x(t), \quad x(0) > 0 \\ \dot{y}(t) &= A_y y(t)^\theta - \delta_y y(t) + b_y x(t)^{1-\theta} y(t)^\theta - n_y \omega_y y(t), \quad y(0) > 0 \end{aligned} \tag{14}$$

We make the following changes to the basis for our resource stock variables:

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} x(t)^{1-\theta} \\ y(t)^{1-\theta} \end{bmatrix}, \tag{15}$$

Taking the time derivative and replacing it with (14) yields the transformed system dynamics under linear strategies:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{(1-\theta)\begin{bmatrix} A_x \\ A_y \end{bmatrix}}_{\mathbf{C}} + \underbrace{(1-\theta)\begin{bmatrix} -\delta_x - n_x\omega_x & -b_x \\ b_y & -\delta_y - n_y\omega_y \end{bmatrix}}_{\mathbf{D}} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \tag{16}$$

The system dynamics become linear in states with constants $\mathbf{C}$ and $\mathbf{D}$. Given the initial conditions $X(0) = x(0)^{1-\theta} > 0$ and $Y(0) = y(0)^{1-\theta} > 0$, the solution is

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{t\mathbf{D}} \begin{bmatrix} X(0) - \bar{X} \\ Y(0) - \bar{Y} \end{bmatrix} + \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}, \tag{17}$$

where the steady-state stocks, $\bar{X}$ and $\bar{Y}$, are strictly positive under Assumption 1 and are given by

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} = -\mathbf{D}^{-1}\mathbf{C}, \tag{18}$$

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} = \frac{(1-\theta)^2}{Det(\mathbf{D})} \begin{bmatrix} \delta_y A_x - b_x A_y + \sum_{j=1}^{n_y} \omega_{y,j} A_x \\ b_y A_x + \delta_x A_y + \sum_{i=1}^{n_x} \omega_{x,i} A_y \end{bmatrix} >> 0. \tag{19}$$

The following lemma summarizes this discussion.

**Lemma 2.** *The linear dynamic system (16) admits a unique continuous function solution, as defined in Equation (17), which depends linearly on the initial conditions and converges to a globally stable positive steady state under Assumption 1.*

Finally, the last important condition is to verify the transversality conditions, which is crucial given our unbounded time horizon ($t \in [0, \infty)$) and unbounded payoffs ($\mathcal{J}^x, \mathcal{J}^y$). A consistent interpretation of the equilibrium requires analyzing the value function's convergence over an infinite timespan. Under the linear harvesting strategies, the equilibrium payoffs (as defined in (2)) exhibit linearity in the transformed state variables $X(t), Y(t)$. Consequently, the equilibrium payoff flows are driven only by stock trajectories, meaning that the convergence conditions predominantly rely on the matrix $\mathbf{D}$'s properties. Lemma 2 indicates that because of the trace and determinant criteria, the real parts of the eigenvalues of $\mathbf{D}$ are negative. However, convergence depends on the structure of $e^{t\mathbf{D}}$, which means we must consider: (*i*) two distinct real eigenvalues, (*ii*) two repeated real eigenvalues, and (*iii*) two complex eigenvalues, as none can be excluded. The subsequent lemma provides the results with detailed proofs in Appendix C.

**Lemma 3.** *Regardless of the properties of* $\mathbf{D}$*, the value functions* $V^x(x(t), y(t))$ *and* $V^x(x(t), y(t))$ *in (10) satisfy :*

$$\lim_{t\to\infty} e^{-\rho t} V^x(x(t), y(t)) = 0 \quad \textit{and} \quad \lim_{t\to\infty} e^{-\rho t} V^y(x(t), y(t)) = 0. \tag{20}$$

The main result of this section is now complete because all three key conditions specified in Dockner et al. [7] have been verified. Our feedback-Nash equilibrium is subgame perfect because the differential game in (4) and (5) occurs over an infinite time span and is autonomous. Consequently, any subgame of the entire game is of equal length and yields solutions independent of the initial conditions. Additionally, our equilibrium has the property that the value function in (10) satisfies the transversality conditions in (20), and serves as a unique functional form that solves (8) and (9) within the framework of linear species-specific strategies. This can be attributed to the linearity of payoffs with respect to the transformed state variables $X(t), Y(t)$ as well as its linearity with respect to the initial conditions in (17), providing a clear justification for the functional forms utilized in (10). The following proposition concludes this section:

**Proposition 1.** *Given the conditions established in Lemmas 1-3,* $(\omega_x^{NE}x(t), \ldots, \omega_x^{NE}x(t))$, *and* $(\omega_y^{NE}y(t), \ldots, \omega_y^{NE}y(t))$ *constitute a subgame perfect feedback-Nash equilibrium. The respective value functions for the prey and predator fishers are:*

$$V^x(x(0), y(0)) = \alpha_x + \frac{\beta_x}{1-\theta}x(0)^{1-\theta} + \frac{\gamma_x}{1-\theta}y(0)^{1-\theta} \tag{21}$$

*and*

$$V^y(x(0), y(0)) = \alpha_y + \frac{\beta_y}{1-\theta}x(0)^{1-\theta} + \frac{\gamma_y}{1-\theta}y(0)^{1-\theta} \tag{22}$$

*with the additional property that this is the unique functional form in which linear harvesting strategies occur.*

## 5. Nash versus sole-ownership

In the previous analysis, we explicitly characterized the private harvesting behavior in an unregulated two-species predator-prey ecosystem. This section aims to (*i*) characterize an optimal solution to the sole-owner problem, as in Definition 2, and (*ii*) compare fishing pressures under decentralized and centralized regimes. Having addressed the technical aspects earlier, we now focus on the implications of centralized management for ecosystem sustainability, contrasting it with outcomes under private, unregulated practices.

### *5.1. The sole-owner optimal allocation*

To manage the predator-prey ecosystem, a comprehensive exploitation model that accounts for all externalities is necessary. To do so, we aim to solve the sole-owner problem, as detailed in the maximization problem (7). Here, the sole owner acts as the manager and distributes the harvested quantities among the players. This approach assumes symmetric players within industries, akin to a decentralized equilibrium, with the goal of maximizing the combined industry payoffs. Given the concavity of payoffs with respect to harvesting (see (2)), the optimization hinges on identifying industry controls $h_x(t), h_y(t)$ that are symmetric within industries and maximize $(1-\theta)^{-1}(n_x h_x(t)^{1-\theta} + n_y h_y(t)^{1-\theta})$, considering the system dynamics[2]. The existence of an optimal solution is grounded in Theorem 3.4 of Dockner et al. [7], where the choice of the form of the value function, along with the optimality and transversality conditions, ensures a solution. Denoting the value function for a sole owner in state $(x, y)$ as $\hat{V}(x, y)$, the Hamilton-Jacobi-Bellman ($HJB$) equation is given by

$$\begin{aligned}\rho\hat{V}(x,y) = \max_{h_x,h_y\geq 0}\Bigg\{&(1-\theta)^{-1}\left(n_x h_x^{1-\theta}+n_y h_y^{1-\theta}\right)+\partial_x\hat{V}(x,y)\left(A_x x^\theta-\delta_x x-b_x y^{1-\theta}x^\theta-n_x h_x\right)\\&+\partial_y\hat{V}(x,y)\left(A_y y^\theta-\delta_y y+b_y x^{1-\theta}y^\theta-n_y h_y\right)\Bigg\}.\end{aligned} \tag{23}$$

This partial differential equation connects the optimal value function in states $(x, y)$ with the immediate rewards of optimal control choices along with the anticipated future gains from such optimal controls. Conjecturing the value function takes the following form:

$$\hat{V}(x,y)=\hat{\alpha}+\frac{\hat{\beta}}{1-\theta}x^{1-\theta}+\frac{\hat{\gamma}}{1-\theta}y^{1-\theta}. \tag{24}$$

The optimal conditions for $HJB$ are

$$\hat{h_x}=\hat{\beta}^{-1/\theta}x=\hat{\omega}_x x \quad\text{and}\quad \hat{h_y}=\hat{\gamma}^{-1/\theta}y=\hat{\omega}_y y. \tag{25}$$

These conditions imply that an optimal control strategy involves harvesting a fixed fraction of the stock for each species, denoted by $\hat{\beta}^{-1/\theta}=\hat{\omega}_x$ and $\hat{\gamma}^{-1/\theta}=\hat{\omega}_y$. By combining (24) and (25) into (23), we use the undetermined coefficient technique to obtain the values of $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$. Moreover, from Lemma 2, we derive stock solutions under linear harvesting, and from Lemma 3, we satisfy the transversality conditions. The following proposition summarizes the optimal harvesting by the sole owner.

**Proposition 2.** *There exists a unique set of parameters $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ such that $\hat{V}(x,y)$ in (24) satisfies (23) for all $(x, y)$. Moreover, $\hat{\beta}^{-1/\theta}=\hat{\omega}_x$ and $\hat{\gamma}^{-1/\theta}=\hat{\omega}_y$ solve the following implicit system of equations:*

$$\hat{\omega}_x^{-\theta}\left(\frac{\rho}{1-\theta}+\delta_x-n_x\hat{\omega}_x\frac{\theta}{1-\theta}\right)-\hat{\omega}_y^{-\theta}b_y=0, \tag{26}$$

$$\hat{\omega}_y^{-\theta}\left(\frac{\rho}{1-\theta}+\delta_y-n_y\hat{\omega}_y\frac{\theta}{1-\theta}\right)+\hat{\omega}_x^{-\theta}b_x=0. \tag{27}$$

---

[2] Consider that symmetry is not the best choice and that there exist two asymmetric harvesting paths within industry $s=x,y$ such that $h_{s,l}\leq h_{s,l'}$, where $l\neq l'\in\{1,\ldots,n_s\}$. As the utility function is concave, we have $U_s(\lambda h_{s,l}+(1-\lambda)h_{s,l'})\geq\lambda U_s(h_{s,l})+(1-\lambda)U_s(h_{s,l'})\ \forall\lambda\in[0,1]$. Imposing $\lambda=1/2$ yields $U_s(h_{s,l}/2+h_{s,l'}/2)\geq 1/2U_s(h_{s,l})+1/2U_s(h_{s,l'})$, which means that the average of $U_s(h_{s,l})$ and $U_s(h_{s,l'})$ is less than the symmetric solution.

*The corresponding optimal value function is*

$$\hat{V}(x,y) = \hat{\alpha} + \frac{\hat{\beta}}{1-\theta}x(0)^{1-\theta} + \frac{\hat{\gamma}}{1-\theta}y(0)^{1-\theta}. \tag{28}$$

### *5.2. Comparing catch rates*

In renewable resource management, understanding optimal policies is crucial for understanding social efficiency and private-sector limitations. Many studies compare private and optimal exploitation in single-species scenarios; however, this becomes more complex with multiple species. This complexity stems from the technical challenges in solving dynamic models with several variables and various inefficiencies, including strategic and biological interactions, which affect the outcomes differently (refer to Bataille et al. [2]). Our model addresses these technicalities by comparing fishing pressures under decentralized and centralized regimes. We first analyze the effects of the number of prey and predators fishers on extraction rates and then compared fishing pressures in both regimes.

In our comparative statics analysis of fishing pressure relative to industry size, we introduce a new sufficient condition. Unlike Lemma 1 which bounds the fishery sizes separately, this condition limits the combined size of the industries, expressed as $n_x + n_y < \frac{1}{1-\theta}$. This condition, which is more restrictive than Lemma 1, enables a straightforward calculation of how the predator-prey industry size affects individual fishing pressure in the Nash equilibrium versus sole ownership. The findings, which are mainly based on the Implicit Function Theorem, are presented in Table 1, with detailed proofs in Section Appendix F.

| | $\omega_x^{NE}(n_x, n_y)$ | $\omega_y^{NE}(n_x, n_y)$ | $\hat{\omega}_x(n_x, n_y)$ | $\hat{\omega}_y(n_x, n_y)$ |
|---|---|---|---|---|
| **No. of prey fisheries** ($n_x$) | $+$ | $-$ | $-$ | $+$ |
| **No. of predator fisheries** ($n_y$) | $-$ | $+$ | $-$ | $-$ |

Table 1: Effects of varying the number of fisheries on the individual fishing pressure.

The first two columns reveal how the numbers of prey and predator fishermen influence individual catch pressure in a decentralized regime, whereas the last two columns show analogous effects under optimal management.

The within-industry effects show that increased competition in a decentralized regime leads to higher resource pressure, which aligns with the common-pool resource (CPR) literature. This is evident as $\forall s = x, y\ \partial_{n_s}\omega_s^{NE}(n_x, n_y) > 0$ implies $\partial_{n_s} n_s \omega_s^{NE}(n_x, n_y) > 0$. Conversely, under optimal management, a higher number of fishermen leads to reduced individual pressure, aligning with efficient resource allocation principles.

Regarding between-industry effects, fishers' strategic adaptations to changes in industry size are mainly influenced by biological interactions. In the decentralized regime, an increase in the number of prey fishermen reduces individual predator fishing pressure, owing to greater pressure on the prey and subsequent reduced predator regeneration. Conversely, under sole ownership, more prey fishermen lead to less pressure on prey, incentivizing increased pressure on predators to prevent prey stock depletion from predators. Similarly, more predator fishers decrease the predation intensity on prey in both regimes as competition among prey fishermen and natural predators decreased.

The comparative statistics reveal that in multispecies ecosystems with externalities, the standard common externality (i.e., the intensity of strategic interaction) significantly affects the entire system. This information is vital for managers to formulate fishery regulations. The subsequent step involved comparing fishing pressures under centralized and decentralized regimes to identify economic inefficiencies and underscore the complexity of managing these ecosystems.

Our first key finding is that the individual prey catch rates selected by fishers in a decentralized regime are invariably lower than those chosen by the sole owner in a centralized system. This outcome is primarily influenced by the fact that in the decentralized regime, an increase in predators consistently reduces the lifetime utility of each prey fisher; that is, $\partial_y V^x(x, y) < 0$ (refer to Appendix D). This dynamic leads to higher fishing pressure on the prey stock in the decentralized system compared to that of the sole owner, who benefits from the presence of more predators.

In contrast, the predator industry size mechanism differs, because more prey generally benefits predator fishers in both regimes. Thus, the comparison depended significantly on the relative numbers of prey and predator fishermen and the intensity of biological interactions between species. For instance, a sole owner considers that an increase in prey (i.e., less prey fishing pressure) might lead to a higher predator fishing intensity, but this advantage needs to be weighed against the negative impacts on prey fishers. However, in a decentralized system, predator fishermen always benefit from an increase in prey. Consequently, there could be scenarios where, given specific prey and predator industry sizes, decentralized fishing pressure is too low compared to a centralized system, because individual predator fishermen focus solely on their own harvest and utility.

To illustrate our findings, we provide a numerical example with fixed biological and economic parameters except for industry size. This approach enables us to compare decentralized and centralized predator fishing pressures across different predator-to-prey industry ratios within a consistent setting. The parameters for this example are listed in Table 2.

| $\theta$ | $\delta_x$ | $\delta_y$ | $A_x$ | $A_y$ | $b_x$ | $b_y$ | $\rho$ | $x(0)$,$y(0)$ |
|---|---|---|---|---|---|---|---|---|
| 0.9 | 0.5 | 0.5 | 0.5 | 0.5 | 0.2 | 1.5 | 0.5 | 0.1 |

Table 2: Parameter values

In this stylized example, we use Equations (12), (13), (26), and (27) to plot the implicit function that identifies all combinations of $n_x$ and $n_y$ satisfying $\omega_y^{NE}(n_x, n_y) = \hat{\omega}_y(n_x, n_y)$

for $n_x, n_y < \frac{1}{1-\theta}$. Based on the results in Table 1, this function increases in the $(n_x, n_y)$ plane. The corresponding graphical representation is presented in Figure 1.

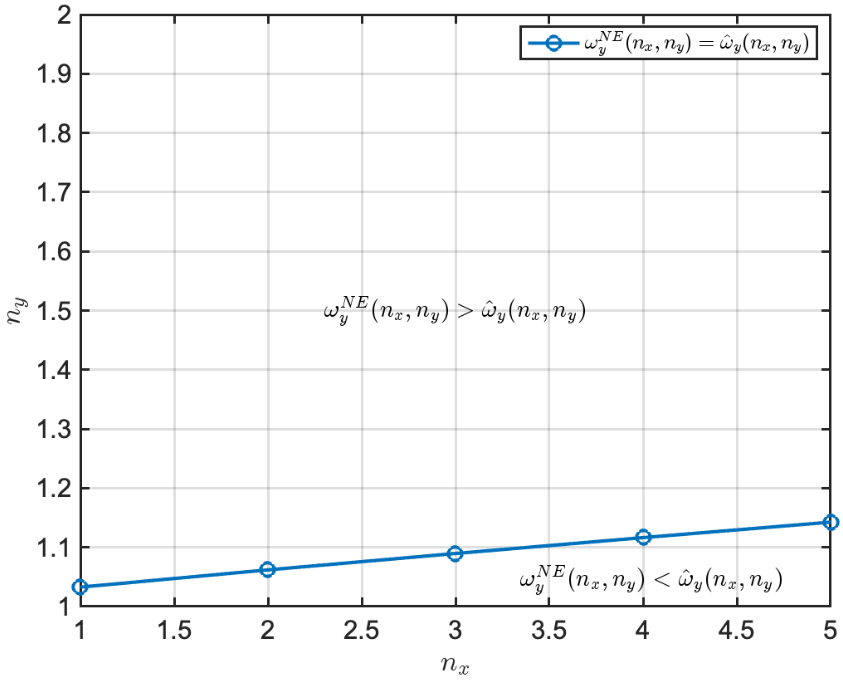

Figure 1: $\omega_y^{NE}(n_x, n_y)$ *vs* $\hat{\omega}_y(n_x, n_y)$

This figure divides the parameter space between prey and predator fisher numbers, allowing for a comparison of fishing pressures in the centralized and decentralized regimes. Given that these numbers are integers, there seems to be no exact industry size pairing in which the two fishing pressures equalize. However, in this example, if only one predator fisher is allowed, the fishing pressure under the Nash equilibrium is consistently lower than that under optimal management, regardless of the size of the prey industry. Conversely, with two predator fishers, the decentralized regime consistently led to excessive predator pressure, irrespective of the prey industry size. These results indicate that managing standard common externalities, specifically when exclusive rights are given to a single predator fisher, can lead to increased economic costs due to underfishing pressure, especially in the context of ecosystem-based spillovers. The following proposition summarizes catch rate comparisons under the two regimes.

**Proposition 3.** *A comparative analysis of fishing pressures under the Nash equilibrium and sole ownership reveal the following.*
*(i) Individual and aggregate prey fishing pressure are higher under a Nash equilibrium than under optimal management; that is,* $n_x \omega_x^{NE}(n_x, n_x) > n_x \hat{\omega}_x(n_x, n_y)$ *for all* $n_x, n_y < \frac{1}{1-\theta}$.
*(ii) A general ranking of individual and aggregate predator fishing pressure between the two regimes is impossible.*

A comparison between optimal and private fishing pressures revealed inefficiencies and underscores the impact of strategic and biological interactions. Although this is a significant first step, it leaves open the question of differences in species dynamics and harvesting quantities along the equilibrium path. This is an important question because $(i)$ predator-prey models have been studied extensively for their dynamic properties, making the analysis of solutions with endogenous harvesting interesting, and $(ii)$ understanding the harvesting dynamics in the multi-species context is necessary for managers to design time-consistent regulatory instruments such as quotas or harvesting taxes.

## 6. Industry size, stocks and harvesting flows

In the absence of species interactions, increased fishing pressure on a particular stock tends to lower its intrinsic growth rate or raise its mortality rate, both of which reduce stock size. However, in a multispecies scenario, the relationship between fishing pressure and harvest becomes complex and is influenced by ecosystem-wide spillover effects. To determine the effect of industry size on harvest quantities theoretically, it is important to consider its effects on predator-prey dynamics (as shown in Equation (17)) and fishing pressure (as in Table 1). Because of the mathematical complexity of the comparative statics of stock trajectory functions, we opted for a numerical simulation approach. Referring to the example in Table 2, we illustrate the stock and harvesting flows under both centralized and decentralized regimes for various industry sizes. In the subsequent figures, we set one industry (prey or predator) as a single harvester and vary the size of the other industry by $n_s = 1, 2, 3$. The legends for all graphs are as follows:

| **Regime** | $n_s = 1$ | $n_s = 2$ | $n_s = 3$ |
|---|---|---|---|
| Sole owner | - - - | - - - | - - - |
| Nash | — | — | — |

Table 3: Legend for stock and harvesting flows with $n_s = 1, 2, 3$

### *6.1. The effect on predator-prey dynamics*

Figure 2 shows how the prey and predator stocks change with more prey fishers (i.e., $n_x = 1, 2, 3$) in the decentralized and centralized regimes. More prey fishers lead to lower prey stocks and, consequently, fewer predators due to less prey. This demonstrates a strong form of the tragedy of the commons, where increased fishing depletes the ecosystem's biomass and negatively affects the interacting industries. This effect is observed not only in steady-state stocks, but also in the dynamic paths toward system convergence, highlighting the significant biological consequences of depleting lower trophic levels. Ecosystem depletion is more pronounced in the unregulated system than under sole ownership, where stock flows decrease but remain at relatively high levels.

Figure 3 shows the dynamics of the prey and predator populations with varying numbers of predator fishers (i.e., $n_y = 1, 2, 3$) under centralized and decentralized management. In an unregulated regime, increased predator fishing leads to predator scarcity and

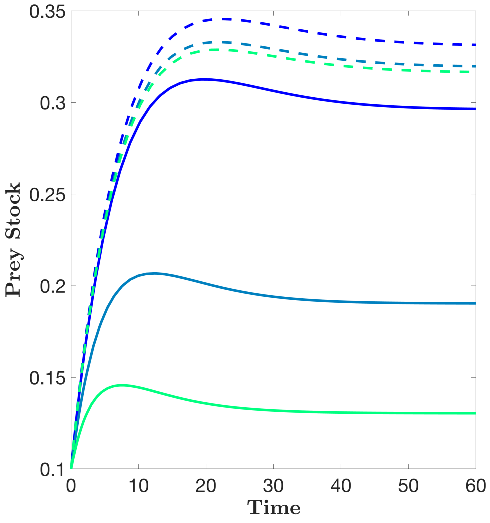

Figure 2: Predator-prey dynamics when $\bar{n}_y = 1$

a subsequent rise in prey owing to less predation. A comparison between the centralized and decentralized systems indicates potential long-term prey overpopulation. Specifically, in a decentralized system with fewer predator fishers, high predation initially suppressed prey numbers. However, as predator fishing increases, this effect diminishes, resulting in a higher prey stock than in the centralized system. These results highlight the complex trade-offs and potential counterintuitive outcomes in managing interdependent predator-prey species, in contrast to single-species models.

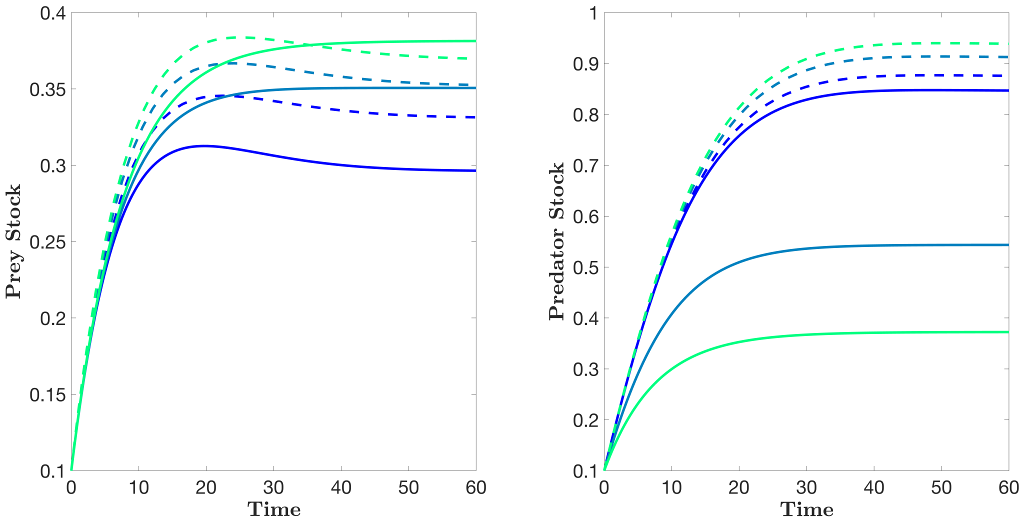

Figure 3: Predator-prey dynamics when $\bar{n}_x = 1$

### *6.2. The effects on predator-prey aggregate harvesting*

To bridge the gap between the natural resource management literature and the complexities of ecosystem management, we illustrate prey and predator harvesting dynamics in our case study. This approach mirrors our methodology for prey and predator flows in which we fix one industry size and vary it from 1 to 3.

Figure 4 demonstrates that increasing the number of prey fishers escalates fishing pressure (see Table 1), thereby depleting prey stocks and consequently reducing both harvesting quantities and payoffs. Conversely, diminished prey harvesting leads to excessive predator harvesting because predator fishers gain an advantage from species interactions under low prey harvest conditions.

Figure 5 illustrates that more predator fishers result in increased predator harvesting and substantial depletion under the Nash regime. This depletion subsequently facilitates increased prey harvesting. In a decentralized setting, a reduction in natural predation pressure prompts prey fishers to shift from underfishing (with a single predator fisher) to overfishing. Notably, temporal transition dynamics are critical, and are particularly evident in our case when $n_y = 2$, where we see a distinct switch from underfishing to overfishing at a specific time.

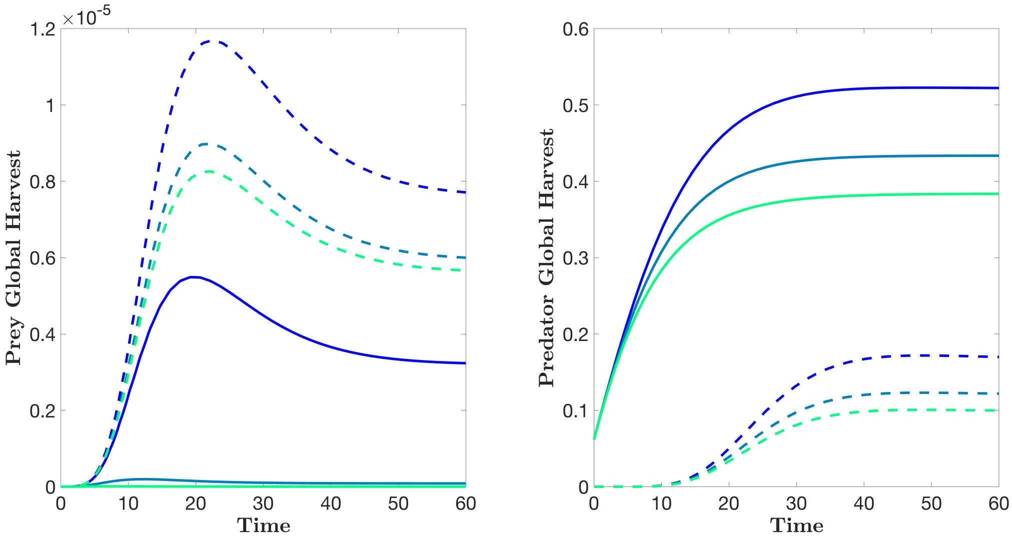

Figure 4: Prey and predator harvesting flows when $\bar{n}_y = 1$

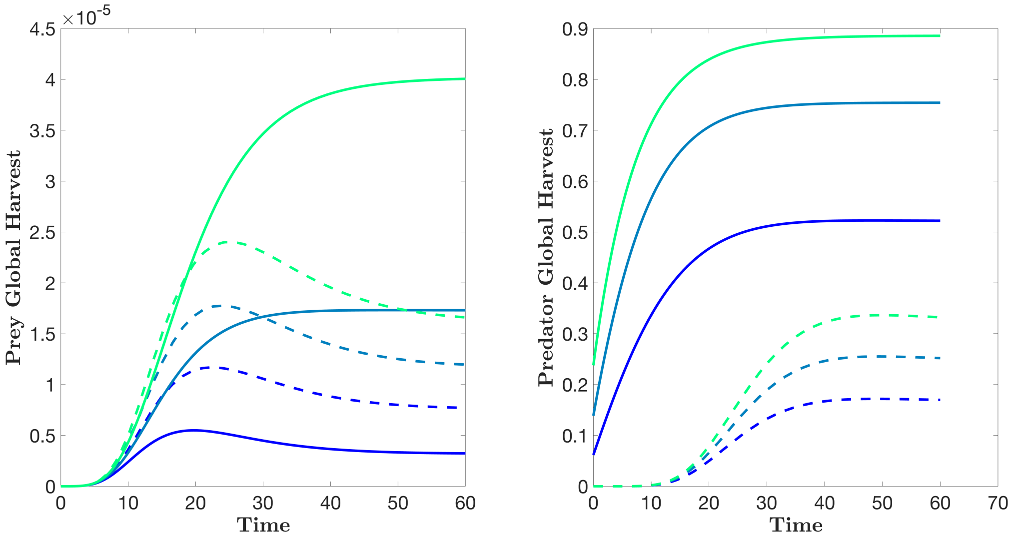

Figure 5: Prey and predator harvesting flows when $\bar{n}_x = 1$

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# Appendix A. Proof of Lemma 1

Essentially, we must prove that there exists a unique set of parameters, $\alpha_x$, $\beta_x$, $\gamma_x$, $\alpha_y$, $\beta_y$, and $\gamma_y$ that solve the Hamilton-Jacobi-Bellman equations. To do this, we rely on an undetermined coefficient technique to identify the values of the parameters that match the left- and right-hand sides of each equation after substituting (10) and (11) into (8) and (9).

We can assert that $\alpha_x$ and $\alpha_y$ are uniquely identifiable because of their linear dependence on the other parameters :

$$\rho\alpha_x = \beta_x A_x + \gamma_x A_y \qquad \text{(A.1)} \qquad\qquad \rho\alpha_y = \beta_y A_x + \gamma_y A_y \qquad \text{(A.2)}$$

Furthermore, $\gamma_x$ and $\beta_y$ are also directly identified through $\beta_x, \gamma_y$ which are parameters related to catch rates (see (11)).

$$\gamma_x = \frac{\beta_x}{b_y}\left(\frac{\rho}{1-\theta} - \beta_x^{-1/\theta}(\frac{1}{1-\theta} - n_x) + \delta_x\right) \tag{A.3}$$

$$\beta_y = -\frac{\gamma_y}{b_x}\left(\frac{\rho}{1-\theta} - \gamma_y^{-1/\theta}(\frac{1}{1-\theta} - n_y) + \delta_y\right) \tag{A.4}$$

Finally, the last two equations combined with the values of $\gamma_x$ and $\beta_y$ in (A.3) and (A.4) allow us to reduce the identification problem to only finding a solution to a system of two equations with two unknowns, $\beta_x, \gamma_y$ as follows:

$$\left(\frac{\rho}{1-\theta} - \beta_x^{-1/\theta}(\frac{1}{1-\theta} - n_x) + \delta_x\right)\left(\frac{\rho}{1-\theta} + \delta_y + n_y\gamma_y^{-1/\theta}\right) + b_x b_y = 0 \tag{A.5}$$

$$\left(\frac{\rho}{1-\theta} - \gamma_y^{-1/\theta}(\frac{1}{1-\theta} - n_y) + \delta_y\right)\left(\frac{\rho}{1-\theta} + \delta_x + n_x\beta_x^{-1/\theta}\right) + b_y b_x = 0. \tag{A.6}$$

It is immediate that at least one positive solution $\{\beta_x, \gamma_y\}$ emerges if and only if $n_x, n_y < \frac{1}{1-\theta}$ because all the parameters involved are strictly positive and this is a sum of positive terms. Using the first order conditions in (11); that is, $\omega_x = \beta_x^{-1/\theta}$ and $\omega_y = \gamma_y^{-1/\theta}$, and factorizing by $\omega_x$, $\omega_y$, and $\omega_x\omega_y$, the system simplifies to:

$$\underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_y)(\frac{\rho}{1-\theta} + \delta_x) - b_y b_x}{n_y(\frac{1}{1-\theta} - n_x)}}_{K} + \omega_x \underbrace{\frac{(\frac{\rho}{1-\theta} + \delta_y)}{n_y}}_{R} + \omega_y \underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_x)}{(\frac{1}{1-\theta} - n_x)}}_{S} + \omega_x\omega_y = 0 \tag{A.7}$$

$$\underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_y)(\frac{\rho}{1-\theta} + \delta_x) - b_y b_x}{n_x(\frac{1}{1-\theta} - n_y)}}_{U} + \omega_x \underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_y)}{(\frac{1}{1-\theta} - n_y)}}_{V} + \omega_y \underbrace{\frac{(\frac{\rho}{1-\theta} + \delta_x)}{n_x}}_{W} + \omega_x\omega_y = 0, \tag{A.8}$$

where the constants have the following signs: $K < 0, R > 0, S < 0, U < 0, V < 0, W > 0$ when $n_x < \frac{1}{1-\theta}$ and $n_y < \frac{1}{1-\theta}$. Taking the difference between (A.7) and (A.8) to eliminate the product of $\omega_x\omega_y$, Equation (A.7) becomes

$$(K - U) + (R - V)\omega_x + (S - W)\omega_y = 0 \tag{A.9}$$

Exploring the linearity of this equation leads to the following expression that links $\omega_x$ and $\omega_y$:

$$\omega_y(\omega_x) = \frac{1}{(S-W)}\bigg( - (K-U) - (R-V)\omega_x\bigg). \tag{A.10}$$

We can then substitute this into (A.8), resulting in the following second-order polynomial equation:

$$\Theta(\omega_x) = \bigg[\frac{SU-WK}{(S-W)}\bigg] + \omega_x \bigg[\frac{SV-WR-K+U}{(S-W)}\bigg] + \omega_x^2\bigg[-\frac{(R-V)}{(S-W)}\bigg] = 0 \tag{A.11}$$

Note that $\Theta(0) < 0$ and that the polynomial equation is of usual U-shape ($(R-V) > 0$ and $(S-W) < 0$), meaning it has two distinct roots: one positive and the other negative. Using (A.10), it remains to be verified whether the positive solution to (A.11) implies $\omega_y > 0$. To this end, let us observe that $\omega_y(\omega_x)$ is an increasing and affine function with the property $\omega_y(\omega_x^{min}) = 0$ where $\omega_x^{min}$ is the minimum prey catch rate, leading to a positive predator catch rate. To show that $\omega_x^{min}$ is indeed lower than the positive solution to (A.11), we must show that the polynomial equation evaluated in $\omega_x^{min}$ is negative.

$$\Theta(\omega_x^{min}) = \bigg[\frac{SU-WK}{(S-W)}\bigg] + \bigg(\frac{-K+U}{R-V}\bigg)\bigg[\frac{SV-WR-K+U}{(S-W)}\bigg] + \bigg(\frac{-K+U}{R-V}\bigg)^2\bigg[-\frac{(R-V)}{(S-W)}\bigg] \tag{A.12}$$

$\Leftrightarrow$

$$\Theta(\omega_x^{min}) = \underbrace{\frac{1}{(R-V)(S-W)}}_{<0}\bigg[\underbrace{SUR}_{>0} + \underbrace{WKV}_{>0}\underbrace{-KSV}_{>0}\underbrace{-WRU}_{>0}\bigg] < 0 \tag{A.13}$$

This concludes the identification process.

## Appendix B. Proof of Lemma 2

From the theory of linear differential equations (e.g., Hirsch et al. [15]) it is immediate that there exist a unique solution $X(t), Y(t)$ to the initial value problem that satisfies the initial conditions because of the linearity of $X(0), Y(0)$ in (17). By considering the system dynamics in (16), we obtain

$$Det(\mathbf{D}) = (1-\theta)^2\bigg((\delta_x + n_x\omega_x)(\delta_y + n_y\omega_y) + b_x b_y\bigg) > 0 \tag{B.1}$$

and

$$Tr(\mathbf{D}) = -(\delta_x + n_x\omega_x) - (\delta_y + n_y\omega_y) < 0 \tag{B.2}$$

to ensure that the solution is globally stable.

## Appendix C. Proof of Lemma 3

We essentially need to demonstrate that the payoffs along the equilibrium path converge as time approaches infinity in both industries; that is,

$$\lim_{t\to\infty} e^{-\rho t}V^x(x(t), y(t)) = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t}V^y(x(t), y(t)) = 0, \tag{C.1}$$

or, written differently,

$$\lim_{t\to\infty} e^{-\rho t}\mathcal{V}^x(X(t), Y(t)) = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t}\mathcal{V}^y(X(t), Y(t)) = 0, \tag{C.2}$$

depending on the change in the variable in (15). To avoid confusion during the proof, if the system dynamics begin at time $\tau \in [0,\infty)$, we establish the relations $x(\tau)^{1-\theta} = X(\tau)$ and $y(\tau)^{1-\theta} = Y(\tau)$, which connect the value function in the initial states, denoted as $V(x(t), y(t))$, with the value function in the transformed state, denoted as $\mathcal{V}(X(t), Y(t))$. We examine three distinct cases based on the properties of matrix $\mathbf{D}$ to confirm the convergence of $\mathcal{V}^x(X(t), Y(t))$, and consequently, the convergence of $V^x(x(t), y(t))$. It is important to note that the same methodology applies to the predator fisher value function, which is not addressed here.

*(i)* $\mathbf{D}$ *has two real eigenvalues*

Let $\lambda_1$ and $\lambda_2$ be the two distinct negative real eigenvalues of the matrix $\mathbf{D}$, which are calculated as $\lambda_1, \lambda_2 = (Tr(\mathbf{D}) \pm \sqrt{Tr(\mathbf{D})^2 - 4Det(\mathbf{D})})/2$. It follows that

$$e^{t\mathbf{D}} = P \begin{bmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{bmatrix} P^{-1}, \tag{C.3}$$

where $P$ represents the eigenvectors associated with the eigenvalues of $\mathbf{D}$. Initializing the system at $X(\tau), Y(\tau)$ implies the existence of constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ where after conducting algebraic manipulations, the solutions for the stock variables in (17) can be expressed as follows:

$$X(t) = \Big(c_1(X(\tau) - \bar{X}) + c_2(Y(\tau) - \bar{Y})\Big)e^{t\lambda_1} + \Big(c_3(X(\tau) - \bar{X}) + c_4(Y(\tau) - \bar{Y})\Big)e^{t\lambda_2} + \bar{X} \tag{C.4}$$

and

$$Y(t) = \Big(c_5(X(\tau) - \bar{X}) + c_6(Y(\tau) - \bar{Y})\Big)e^{t\lambda_1} + \Big(c_7(X(\tau) - \bar{X}) + c_8(Y(\tau) - \bar{Y})\Big)e^{t\lambda_2} + \bar{Y}. \tag{C.5}$$

The value function of an individual prey fisher, that is, the flow of payoffs along the equilibrium path when the system starts at $t = \tau$, is by definition equal to

$$\mathcal{V}^x(X(\tau), Y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta} \int_\tau^\infty X(t, X(\tau), Y(\tau)) e^{-\rho(t-\tau)} dt. \tag{C.6}$$

Because $X(t, X(\tau), Y(\tau))$ is as in (C.4), we can write the value function as

$$\mathcal{V}^x(X(\tau), Y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta} \Big(C_1(X(\tau), Y(\tau)) \int_\tau^\infty e^{(\lambda_1-\rho)(t-\tau)} dt + C_2(X(\tau), Y(\tau)) \int_\tau^\infty e^{(\lambda_2-\rho)(t-\tau)} dt \tag{C.7}$$
$$+ \bar{X} \int_\tau^\infty e^{-\rho(t-\tau)} dt\Big),$$

which is identical to

$$V^x(x(\tau), y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta} \Big(C_1(x(\tau), y(\tau)) \int_\tau^\infty e^{(\lambda_1-\rho)(t-\tau)} dt + C_2(x(\tau), y(\tau)) \int_\tau^\infty e^{(\lambda_2-\rho)(t-\tau)} dt \tag{C.8}$$
$$+ \bar{X} \int_\tau^\infty e^{-\rho(t-\tau)} dt\Big),$$

where $C_1$ and $C_2$ are constants that depend on the initial conditions and other model parameters. The integrals are computed as follows:

$$\int_{\tau}^{\infty} e^{(\lambda_1-\rho)(t-\tau)}dt = \frac{e^{\tau(\lambda_1-\rho)}}{\rho-\lambda_1}; \quad \int_{\tau}^{\infty} e^{(\lambda_2-\rho)(t-\tau)}dt = \frac{e^{\tau(\lambda_2-\rho)}}{\rho-\lambda_2}; \quad \int_{\tau}^{\infty} e^{-\rho(t-\tau)}dt = \frac{e^{-\rho\tau}}{\rho} \tag{C.9}$$

and converge to finite positive values.

*(ii)* **D** *has two complex eigenvalues*

To address this case, we decompose the real and imaginary components of each eigenvalue as follows: $\lambda_1 = h + vi$ and $\lambda_2 = h - vi$, where $h = Tr(\mathbf{D})/2 < 0$ represents the real part and $v = (Tr(\mathbf{D})^2 - 4Det(\mathbf{D}))/2 < 0$ the imaginary part. Consequently, $e^{t\mathbf{D}}$ takes the form

$$e^{t\mathbf{D}} = P \begin{bmatrix} e^{t(h+vi)} & 0 \\ 0 & e^{t(h-vi)} \end{bmatrix} P^{-1}, \tag{C.10}$$

where matrix $P$ represents the (complex) eigenvectors associated with the complex eigenvalues. We use Euler's formula; that is, the continuous-time equivalent of DeMoivre's theorem, which states that $e^{t(h\pm vi)} = e^{ht}(\cos(vt) \pm i\sin(vt))$. Subsequently, (C.10) is transformed into

$$e^{t\mathbf{D}} = Pe^{ht} \begin{bmatrix} \cos(vt) + i\sin(vt) & 0 \\ 0 & \cos(vt) - i\sin(vt) \end{bmatrix} P^{-1}. \tag{C.11}$$

Initializing the system at $t = \tau$ and combining (C.11) with (17), we can say that there exists constants, denoted as $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$, where after a series of algebraic computations, and leveraging the fundamental identity $i^2 = -1$, the solutions for $X(t)$ and $Y(t)$ are

$$X(t) = e^{ht}\left[\left(k_1(X(\tau)-\bar{X})+k_2(Y(\tau)-\bar{Y})\right)\cos(vt)+\left(k_3(X(\tau)-\bar{X})+k_4(Y(\tau)-\bar{Y})\right)\sin(vt)\right]+\bar{X} \tag{C.12}$$

and

$$Y(t) = e^{ht}\left[\left(k_5(X(\tau)-\bar{X})+k_6(Y(\tau)-\bar{Y})\right)\cos(vt)+\left(k_7(X(\tau)-\bar{X})+k_8(Y(\tau)-\bar{Y})\right)\sin(vt)\right]+\bar{Y}. \tag{C.13}$$

The value function in the transformed state of an individual prey fisher is

$$\begin{aligned} \mathcal{V}^x(X(\tau), Y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta} \Bigg( & K_1(X(\tau), Y(\tau)) \int_{\tau}^{\infty} \cos(vt) e^{(h-\rho)(t-\tau)}dt \\ & + K_2(X(\tau), Y(\tau)) \int_{\tau}^{\infty} \sin(vt) e^{(h-\rho)(t-\tau)}dt \\ & + \bar{X} \int_{\tau}^{\infty} e^{-\rho(t-\tau)}dt \Bigg) \end{aligned} \tag{C.14}$$

and equivalently, after changing $X(\tau) = x(\tau)^{1-\theta}$,

$$
\begin{aligned}
V^x(x(\tau), y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta} \Bigg( & K_1(x(\tau), y(\tau)) \int_\tau^\infty \cos(vt) e^{(h-\rho)(t-\tau)} dt \\
& + K_2(x(\tau), y(\tau)) \int_\tau^\infty \sin(vt) e^{(h-\rho)(t-\tau)} dt \\
& + \bar{X} \int_\tau^\infty e^{-\rho(t-\tau)} dt \Bigg),
\end{aligned} \tag{C.15}
$$

where $K_1$ and $K_2$ are constants that depend on the initial conditions and parameters. The integrals are computed as follows:

$$
\int_\tau^\infty \cos(vt) e^{(h-\rho)(t-\tau)} dt = \left[ \frac{e^{(h-\rho)(t-\tau)}((h-\rho)\cos(vt) + v\sin(vt))}{h^2 - 2h\rho + v^2 + \rho^2} \right]_\tau^\infty \tag{C.16}
$$

$$
\int_\tau^\infty \sin(vt) e^{(h-\rho)(t-\tau)} dt = \left[ \frac{e^{(h-\rho)(t-\tau)}((h-\rho)\sin(vt) - v\cos(vt))}{h^2 - 2h\rho + v^2 + \rho^2} \right]_\tau^\infty \tag{C.17}
$$

and again

$$
\int_\tau^\infty e^{-\rho(t-\tau)} dt = \frac{e^{-\rho\tau}}{\rho}. \tag{C.18}
$$

Given that $h < 0$ and $v < 0$ and that co-sinus and sinus functions are bounded between $-1$ and 1, we can say that all three integrals converge to finite-positive values.

*(iii)* <u>**D** *has two repeated real eigenvalues*</u>

When **D** possesses a symmetric real eigenvalue, the eigenspace's dimension is smaller than that of matrix **D**. In this scenario, **D** is not diagonalizable, necessitating an alternative approach. The repeated real eigenvalues are denoted by $\lambda = Tr(\mathbf{D})/2 < 0$. To address this issue, we define the Jordan canonical form of $\mathbf{P^{-1}DP}$ as

$$
\mathbf{P^{-1}DP} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \mathbf{J}, \tag{C.19}
$$

where **P** represents the Jordan vectors and **J** is the Jordan canonical form. In this case, the expression for the matrix exponential $e^{t\mathbf{D}}$ takes the form

$$
e^{t\mathbf{D}} = \mathbf{P} e^{t\mathbf{J}} \mathbf{P^{-1}} = \mathbf{P} e^{t\lambda} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{P^{-1}}. \tag{C.20}
$$

Given that the system dynamics starts at $t = \tau$ with initial conditions $X(\tau)$ and $Y(\tau)$, and based on the form of (C.20), this leads to the presence of constants, denoted as $r_1, r_2, r_3, r_4, r_5, r_6, r_7$, and $r_8$, which determine the solutions for $X(t)$ and $Y(t)$, respectively, as follows:

$$
X(t) = \Big( r_1(X(\tau) - \bar{X}) + r_2(Y(\tau) - \bar{Y}) \Big) e^{t\lambda} + \Big( r_3(X(\tau) - \bar{X}) + r_4(Y(\tau) - \bar{Y}) \Big) t e^{\lambda} \Big] + \bar{X} \tag{C.21}
$$

and

$$
Y(t) = \Big( r_5(X(\tau) - \bar{X}) + r_6(Y(\tau) - \bar{Y}) \Big) e^{t\lambda} + \Big( r_7(X(\tau) - \bar{X}) + r_8(Y(\tau) - \bar{Y}) \Big) t e^{\lambda} \Big] + \bar{Y} \tag{C.22}
$$

By computing the value function of a specific prey fisher $\mathcal{V}^x(X(t), Y(t))$, we obtain

$$\mathcal{V}^x(X(\tau),Y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta}\Bigg(R_1(X(\tau),Y(\tau))\int_\tau^\infty e^{(\lambda-\rho)(t-\tau)}dt + R_2(X(\tau),Y(\tau))\int_\tau^\infty te^{(\lambda-\rho)(t-\tau)}dt + \bar{X}\int_\tau^\infty e^{-\rho(t-\tau)}dt\Bigg). \tag{C.23}$$

Equivalently, upon reverting the change in the variable for the prey fish stock to its original form:

$$V^x(x(\tau),y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta}\Bigg(R_1(x(\tau),y(\tau))\int_\tau^\infty e^{(\lambda-\rho)(t-\tau)}dt + R_2(x(\tau),y(\tau))\int_\tau^\infty te^{(\lambda-\rho)(t-\tau)}dt + \bar{X}\int_\tau^\infty e^{-\rho(t-\tau)}dt\Bigg), \tag{C.24}$$

where $R_1$ and $R_2$ are constants that depend on the initial conditions. By computing the integrals, we obtain:

$$\int_\tau^\infty e^{(\lambda-\rho)(t-\tau)}dt = \frac{e^{\tau(\lambda-\rho)}}{\rho-\lambda}, \quad \int_\tau^\infty te^{(\lambda-\rho)(t-\tau)}dt = \frac{-\lambda\tau+\rho\tau+1}{(\lambda-\rho)^2}, \quad \int_\tau^\infty e^{-\rho(t-\tau)}dt = \frac{e^{-\rho\tau}}{\rho}. \tag{C.25}$$

Recall that $\lambda < 0$ and $\rho > 0$; all integrals converge to finite positive values, thus concluding the proof.

## Appendix D. Proof of Proposition 1

We now need to demonstrate that along the equilibrium path, the payoffs follow the pattern described in (10). To establish this, we draw from the findings of Lemma 3 and its corresponding proof in Appendix C. In addition to the convergence results in Equations (C.7), (C.14), and (C.23), we must establish that this quantity is both separable and linear under the initial conditions. Similar to the approach used in Appendix C, we elucidate the structure of the equilibrium value function for a prey fisher based on the characteristics of matrix $\mathbf{D}$, keeping in mind that the same reasoning applies to the value function of an individual predator fisher.

*(i)* **D** *has two real eigenvalues*

Considering Equations (C.7) and (C.4), it is clear that $C_1(X(\tau),Y(\tau))$ and $C_2(X(\tau),Y(\tau))$ have an affine relationship with respect to initial conditions. Specifically, there exist coefficients $c_1', c_2', c_3', c_4', c_5', c_6'$ such that

$$\mathcal{V}^x(X(\tau),Y(\tau)) = \frac{\omega_x^{1-\theta}}{1-\theta}\Bigg((c_1'+c_2'X(\tau)+c_3'Y(\tau))\int_\tau^\infty e^{(\lambda_1-\rho)(t-\tau)}dt + (c_4'+c_5'X(\tau)+c_6'Y(\tau))\int_\tau^\infty e^{(\lambda_2-\rho)(t-\tau)}dt + \bar{X}\int_\tau^\infty e^{-\rho(t-\tau)}dt\Bigg). \tag{D.1}$$

By applying the results of Lemma 3 regarding the convergence of integrals and reorganizing the terms, the value function is expressed as follows:

$$\mathcal{V}^x(X(\tau),Y(\tau)) = \alpha_x + \frac{\beta_x}{1-\theta}X(\tau) + \frac{\gamma_x}{1-\theta}Y(\tau), \tag{D.2}$$

or alternatively,

$$V^x(x(\tau), y(\tau)) = \alpha_x + \frac{\beta_x}{1-\theta} x(\tau)^{1-\theta} + \frac{\gamma_x}{1-\theta} y(\tau)^{1-\theta}, \tag{D.3}$$

where the coefficients $\alpha_x$, $\beta_x$, and $\gamma_x$ can be determined directly from (D.1) as follows:

$$\alpha_x = \frac{\omega_x^{1-\theta}}{1-\theta}\left(c_1' \int_\tau^\infty e^{(\lambda_1-\rho)(t-\tau)}dt + c_4' \int_\tau^\infty e^{(\lambda_2-\rho)(t-\tau)}dt + \bar{X}\int_\tau^\infty e^{-\rho(t-\tau)}dt\right) \tag{D.4}$$

$$\beta_x = \omega_x^{1-\theta}\left(c_2' \int_\tau^\infty e^{(\lambda_1-\rho)(t-\tau)}dt + c_5' \int_\tau^\infty e^{(\lambda_2-\rho)(t-\tau)}dt\right) \tag{D.5}$$

$$\gamma_x = \omega_x^{1-\theta}\left(c_3' \int_\tau^\infty e^{(\lambda_1-\rho)(t-\tau)}dt + c_6' \int_\tau^\infty e^{(\lambda_2-\rho)(t-\tau)}dt\right). \tag{D.6}$$

*(i)* **D** *has two complex eigenvalues*

Once more, by leveraging (C.14) and (C.12), we can represent $K_1$ and $K_2$ as affine functions in terms of initial conditions with $k_1', k_2', k_3', k_4', k_5', k_6'$, yielding updated values for $\alpha_x, \beta_x$, and $\gamma_x$:

$$\alpha_x = \frac{\omega_x^{1-\theta}}{1-\theta}\left(k_1' \int_\tau^\infty \cos(vt)e^{(h-\rho)(t-\tau)}dt + k_4' \int_\tau^\infty \sin(vt)e^{(h-\rho)(t-\tau)}dt + \bar{X}\int_\tau^\infty e^{-\rho(t-\tau)}dt\right) \tag{D.7}$$

$$\beta_x = \omega_x^{1-\theta}\left(k_2' \int_\tau^\infty \cos(vt)e^{(h-\rho)(t-\tau)}dt + k_5' \int_\tau^\infty \sin(vt)e^{(h-\rho)(t-\tau)}dt\right) \tag{D.8}$$

$$\gamma_x = \omega_x^{1-\theta}\left(k_3' \int_\tau^\infty \cos(vt)e^{(h-\rho)(t-\tau)}dt + k_6' \int_\tau^\infty \sin(vt)e^{(h-\rho)(t-\tau)}dt\right). \tag{D.9}$$

*(iii)* **D** *has two repeated real eigenvalues*

By applying the same techniques and defining the coefficients by $r_1', r_2', r_3', r_4', r_5', r_6'$, we obtain

$$\alpha_x = \frac{\omega_x^{1-\theta}}{1-\theta}\left(r_1' \int_\tau^\infty e^{(\lambda-\rho)(t-\tau)}dt + r_4' \int_\tau^\infty te^{(\lambda-\rho)(t-\tau)}dt + \bar{X}\int_\tau^\infty e^{-\rho(t-\tau)}dt\right) \tag{D.10}$$

$$\beta_x = \omega_x^{1-\theta}\left(r_2' \int_\tau^\infty e^{(\lambda-\rho)(t-\tau)}dt + r_5' \int_\tau^\infty te^{(\lambda-\rho)(t-\tau)}dt\right) \tag{D.11}$$

$$\gamma_x = \omega_x^{1-\theta}\left(r_3' \int_\tau^\infty e^{(\lambda-\rho)(t-\tau)}dt + r_6' \int_\tau^\infty te^{(\lambda-\rho)(t-\tau)}dt\right). \tag{D.12}$$

This concludes the proof according to the unique functional form of the value function in which linear harvesting strategies occur.

## Appendix E. Proof of Proposition 2

Let $\hat{\alpha}, \hat{\beta}$, and $\hat{\gamma}$ be the parameters that solve the Hamilton-Jacobi-Bellman equation for the sole-owner problem. Utilizing a methodology analogous to that used in the proof of Lemma 1 (referenced in Appendix A), we employ an undetermined coefficient technique. Substituting the first-order conditions from (25) and the functional form of the value function from (24) into the Hamilton-Jacobi-Bellman equation in (23), we obtain the following system of equations:

$$\rho\hat{\alpha} = \hat{\beta}A_x + \hat{\gamma}A_y \tag{E.1}$$

$$\rho\frac{\hat{\beta}}{1-\theta} = n_x\frac{\hat{\beta}^{-(1-\theta)/\theta}}{1-\theta} - \hat{\beta}(\delta_x + n_x\hat{\beta}^{-1/\theta}) + \hat{\gamma}b_y \tag{E.2}$$

$$\rho\frac{\hat{\gamma}}{1-\theta} = n_y\frac{\hat{\gamma}^{-(1-\theta)/\theta}}{1-\theta} - \hat{\gamma}(\delta_y + n_y\hat{\gamma}^{-1/\theta}) - \hat{\beta}b_x. \tag{E.3}$$

Equation (E.1) can be identified directly using $\hat{\beta}$ and $\hat{\gamma}$. However, (E.2) and (E.3) become:

$$\hat{\beta}\left(\frac{\rho}{1-\theta} + \delta_x - n_x\hat{\beta}^{-1/\theta}\frac{\theta}{1-\theta}\right) - \hat{\gamma}b_y = 0 \tag{E.4}$$

$$\hat{\gamma}\left(\frac{\rho}{1-\theta} + \delta_y - n_y\hat{\gamma}^{-1/\theta}\frac{\theta}{1-\theta}\right) + \hat{\beta}b_x = 0. \tag{E.5}$$

Equations (E.4) and (E.5) are equivalent to Equations (26) and (27), respectively. Examining the linearity of Equation (E.4) in $\hat{\gamma}$, we derive the implicit function $\hat{\gamma}(\hat{\beta})$ with the following characteristics: $\hat{\gamma}^{'} > 0$, $\hat{\gamma}(\hat{\beta}) = 0 \Leftrightarrow \hat{\beta} = (n_x(\frac{\rho}{1-\theta} + \delta_x)\frac{1-\theta}{\theta})^{-\theta}$, and $\lim_{\hat{\beta}\to 0}\hat{\gamma} = -\infty$. A similar approach applies to Equation (E.5), yielding the implicit function $\hat{\beta}(\hat{\gamma})$ with the characteristics $\hat{\beta}^{'} < 0$, $\hat{\beta}(\hat{\gamma}) = 0 \Leftrightarrow \hat{\gamma} = (n_y(\frac{\rho}{1-\theta} + \delta_y)\frac{1-\theta}{\theta})^{-\theta}$, and $\lim_{\hat{\gamma}\to 0}\hat{\beta} = \infty$. These properties guarantee the existence of a unique pair $\{\hat{\beta}, \hat{\gamma}\}$, and consequently, a unique pair $\{\hat{\omega}_x, \hat{\omega}_y\}$.

## Appendix F. The effect of industry size on equilibrium catch rates (Table 1)

The proofs of the comparative statics rely on the application of the implicit function theorem.

### *Appendix F.1. The effect of $n_x$ and $n_y$ on $\omega_x^{NE}(n_x, n_y)$ and $\omega_y^{NE}(n_x, n_y)$*

Recall that the system of equations that provides the solutions $\{\omega_x^{NE}, \omega_y^{NE}\}$ is given by (A.7) and (A.8):

$$\varphi(\omega_x, \omega_y) = \left[\begin{array}{c} \varphi_1(\omega_x, \omega_y) \\ \varphi_2(\omega_x, \omega_y) \end{array}\right] = \left[\begin{array}{c} \underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_y)(\frac{\rho}{1-\theta} + \delta_x) - b_y b_x}{n_y(\frac{1}{1-\theta} - n_x)}}_{K} + \omega_x \underbrace{\frac{(\frac{\rho}{1-\theta} + \delta_y)}{n_y}}_{R} + \omega_y \underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_x)}{(\frac{1}{1-\theta} - n_x)}}_{S} + \omega_x\omega_y \\ \underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_y)(\frac{\rho}{1-\theta} + \delta_x) - b_y b_x}{n_x(\frac{1}{1-\theta} - n_y)}}_{U} + \omega_x \underbrace{\frac{-(\frac{\rho}{1-\theta} + \delta_y)}{(\frac{1}{1-\theta} - n_y)}}_{V} + \omega_y \underbrace{\frac{(\frac{\rho}{1-\theta} + \delta_x)}{n_x}}_{W} + \omega_x\omega_y. \end{array}\right] \tag{F.1}$$

The Jacobian matrix is :

$$\partial\varphi = \left[\begin{array}{cc} R + \omega_y & S + \omega_x \\ V + \omega_y & W + \omega_x \end{array}\right] = \left[\begin{array}{cc} + & + \\ + & +, \end{array}\right] \tag{F.2}$$

where $R + \omega_y$ and $W + \omega_x$ are both positive. In contrast, $S + \omega_x$ and $V + \omega_y$ can be rewritten as

$$S + \omega_x = \frac{-\rho/(1-\theta) - \delta_x + \omega_x/(1-\theta) - \omega_x n_x}{1/(1-\theta) - n_x} \tag{F.3}$$

and

$$V + \omega_y = \frac{-\rho/(1-\theta) - \delta_y + \omega_y/(1-\theta) - \omega_y n_y}{1/(1-\theta) - n_y}. \tag{F.4}$$

Using the fact that, at the equilibrium, (12) and (13) are satisfied, we obtain

$$b_x b_y = \Big( -\rho/(1-\theta) - \delta_x + \omega_x/(1-\theta) - \omega_x n_x \Big)\Big( \rho/(1-\theta) + \delta_y + n_y \omega_y \Big) > 0 \tag{F.5}$$

and

$$b_x b_y = \Big( -\rho/(1-\theta) - \delta_y + \omega_y/(1-\theta) - \omega_y n_y \Big)\Big( \rho/(1-\theta) + \delta_x + n_x \omega_x \Big) > 0, \tag{F.6}$$

which allows us to conclude that $S + \omega_x > 0$ and $V + \omega_y > 0$. The determinant of the Jacobian matrix is

$$\det\left( \partial\varphi\big|_{\varphi(\omega_x,\omega_y)=0} \right) = \begin{vmatrix} R+\omega_y & S+\omega_x \\ V+\omega_y & W+\omega_x \end{vmatrix} = RW - SV + (R-V)\omega_x + (W-S)\omega_y, \tag{F.7}$$

where $R - V > 0$ and $W - S > 0$. To ensure the positivity of the determinant, we must verify that $RW-SV$ is positive. Using (F.1) we can verify that $RW-SV > 0$ if $(1-n_x(1-\theta))(1-n_y)(1-\theta)-n_x n_y > 0$ which is true under the following additional sufficient condition: $n_x + n_y \leq \frac{1}{1-\theta}$. By applying the implicit function theorem to (F.1), we obtain

$$\begin{bmatrix} \frac{\partial \omega_x}{\partial p} \\ \frac{\partial \omega_y}{\partial p} \end{bmatrix}_{p\in\{n_x,n_y\}} = -\left( \partial_{(\omega_x,\omega_y)}\varphi\big|_{\varphi(\omega_x,\omega_y)=0} \right)^{-1} \left( \partial_{n_x,n_y}\varphi\big|_{\varphi(\omega_x,\omega_y)=0} \right), \tag{F.8}$$

where

$$-\left( \partial_{(\omega_x,\omega_y)}\varphi\big|_{\varphi(\omega_x,\omega_y)=0} \right)^{-1} = -\left( \det\left( \partial\varphi\big|_{\varphi(\omega_x,\omega_y)=0} \right) \right)^{-1} \begin{bmatrix} W+\omega_x & -(S+\omega_x) \\ -(V+\omega_y) & R+\omega_y \end{bmatrix} = \begin{bmatrix} - & + \\ + & -. \end{bmatrix}$$

It remains to differentiate (F.1) with respect to industry size

$$\left( \partial_{(n_x,n_y)}\varphi\big|_{\varphi(\omega_x,\omega_y)=0} \right) = \begin{bmatrix} \partial_{n_x}\phi_1 & \partial_{n_y}\phi_1 \\ \partial_{n_x}\phi_2 & \partial_{n_y}\phi_2 \end{bmatrix} = \begin{bmatrix} - & + \\ + & - \end{bmatrix}$$

$$\partial_{n_x}\phi_1 = \frac{-n_y\left( (\frac{\rho}{1-\theta}+\delta_y)(\frac{\rho}{1-\theta}+\delta_x) + B_y B_x \right)}{n_y^2(\frac{1}{1-\theta}-n_x)^2} - \omega_y \frac{(\frac{\rho}{1-\theta}+\delta_x)}{(\frac{1}{1-\theta}-n_x)^2} < 0 \tag{F.9}$$

$$\partial_{n_y}\phi_2 = \frac{-n_x\left( (\frac{\rho}{1-\theta}+\delta_y)(\frac{\rho}{1-\theta}+\delta_x) + B_y B_x \right)}{n_x^2(\frac{1}{1-\theta}-n_y)^2} - \omega_x \frac{(\frac{\rho}{1-\theta}+\delta_y)}{(\frac{1}{1-\theta}-n_y)^2} < 0 \tag{F.10}$$

$$\partial_{n_x}\phi_2 = \frac{1}{n_x^2(1/(1-\theta)-n_y)}\left( -\frac{\rho}{(1-\theta)} - \delta_y + \frac{\omega_y}{(1-\theta)} - \omega_y n_y \right) n_x \omega_x > 0 \tag{F.11}$$

$$\partial_{n_y}\phi_1 = \frac{1}{n_y^2(1/(1-\theta)-n_x)}\left( -\frac{\rho}{(1-\theta)} - \delta_x + \frac{\omega_x}{(1-\theta)} - \omega_x n_x \right) n_y \omega_y > 0, \tag{F.12}$$

where $(-\frac{\rho}{(1-\theta)} - \delta_y + \frac{\omega_y}{(1-\theta)} - \omega_y n_y)$ and $(-\frac{\rho}{(1-\theta)} - \delta_x + \frac{\omega_x}{(1-\theta)} - \omega_x n_x)$ are positives using the optimality conditions in (12) and (13). Thus, we deduce the following directly:

$$\frac{\partial \omega_x^{NE}}{\partial n_x} > 0, \quad \frac{\partial \omega_x^{NE}}{\partial n_y} < 0, \quad \frac{\partial \omega_y^{NE}}{\partial n_x} < 0, \quad \frac{\partial \omega_y^{NE}}{\partial n_y} > 0. \tag{F.13}$$

### *Appendix F.2. The effect of* $n_x$ *and* $n_y$ *on* $\hat{\omega}_x(n_x, n_y)$ *and* $\hat{\omega}_y(n_x, n_y)$

In an efficient allocation, the system that provides the solution $\{\hat{\omega_x}, \hat{\omega_y}\}$ is given by Equations (26) and (27):

$$\hat{\varphi}(\hat{\omega}_x, \hat{\omega}_y) = \begin{bmatrix} \hat{\varphi}_1(\hat{\omega_x}, \hat{\omega_y}) \\ \hat{\varphi}_2(\hat{\omega_x}, \hat{\omega_y}) \end{bmatrix} = \begin{bmatrix} \hat{\omega}_x^{-\theta}\left(\frac{\rho}{1-\theta} + \delta_x - n_x\hat{\omega}_x\frac{\theta}{1-\theta}\right) - \hat{\omega}_y^{-\theta}b_y \\ \hat{\omega}_y^{-\theta}\left(\frac{\rho}{1-\theta} + \delta_y - n_y\hat{\omega}_y\frac{\theta}{1-\theta}\right) + \hat{\omega}_x^{-\theta}b_x. \end{bmatrix} \tag{F.14}$$

The Jacobian matrix is

$$\partial\hat{\varphi} = \begin{bmatrix} -\theta\hat{\omega}_x^{-\theta}[(\frac{\rho}{1-\theta} + \delta_x)\hat{\omega}_x^{-1} + n_x] & \theta\hat{\omega}_y^{-\theta-1}b_y \\ -\theta\hat{\omega}_x^{-\theta-1}b_x & -\theta\hat{\omega}_y^{-\theta}[(\frac{\rho}{1-\theta} + \delta_y)\hat{\omega}_y^{-1} + n_y] \end{bmatrix} = \begin{bmatrix} - & + \\ - & -. \end{bmatrix} \tag{F.15}$$

Through computations, we obtain det $\left(\partial\hat{\varphi}|_{\hat{\varphi}(\hat{\omega}_x,\hat{\omega}_y)=0}\right) > 0$*directly*. By applying the Implicit Function Theorem, we know that

$$\begin{bmatrix} \frac{\partial\hat{\omega}_x}{\partial p} \\ \frac{\partial\hat{\omega}_y}{\partial p} \end{bmatrix}_{p\in\{n_x,n_y\}} = -\left(\partial_{(\hat{\omega}_x,\hat{\omega}_y)}\hat{\varphi}\big|_{\hat{\varphi}(\hat{\omega}_x,\hat{\omega}_y)=0}\right)^{-1}\left(\partial_{n_x,n_y}\hat{\varphi}\big|_{\hat{\varphi}(\hat{\omega}_x,\hat{\omega}_y)=0}\right), \tag{F.16}$$

where

$$-\left(\partial_{(\hat{\omega}_x,\hat{\omega}_y)}\hat{\varphi}\big|_{\hat{\varphi}(\hat{\omega}_x,\hat{\omega}_y)=0}\right)^{-1} = -\left(\det\left(\partial\hat{\varphi}|_{\hat{\varphi}(\hat{\omega}_x,\hat{\omega}_y)=0}\right)\right)^{-1}\begin{bmatrix} - & - \\ + & - \end{bmatrix} = \begin{bmatrix} + & + \\ - & +. \end{bmatrix}$$

Taking the derivative of (F.14) with respect to industry size, $n_x$ and $n_y$, we obtain

$$\left(\partial_{(n_x,n_y)}\hat{\varphi}\big|_{\hat{\varphi}(\hat{\omega}_x,\hat{\omega}_y)=0}\right) = \begin{bmatrix} -\frac{\theta\hat{\omega}_x^{1-\theta}}{1-\theta} & 0 \\ 0 & -\frac{\theta\hat{\omega}_y^{1-\theta}}{1-\theta} \end{bmatrix} = \begin{bmatrix} - & 0 \\ 0 & -. \end{bmatrix} \tag{F.17}$$

From (F.16), we directly deduce the following:

$$\frac{\partial\hat{\omega}_x}{\partial n_x} < 0, \quad \frac{\partial\hat{\omega}_x}{\partial n_y} < 0, \quad \frac{\partial\hat{\omega}_y}{\partial n_x} > 0, \quad \frac{\partial\hat{\omega}_y}{\partial n_y} < 0. \tag{F.18}$$

## Appendix G. Proof of proposition 3

To prove the first point of this proposition, we essentially need to compare the systems of Equations (A.3), (A.4), (A.5), and (A.6) characterizing the Nash solution with those describing the optimal allocation in Equations (E.4) and (E.5) for individual prey catch rate. The second point in the proposition arises directly from the counterexample in Figure 1.

Using Equations (A.3) and (A.5), we obtain $\gamma_x < 0$. Using (E.4) and (E.5), we find that $\hat{\gamma} > 0$. Recall that $\gamma_x$ and $\hat{\gamma}$ are identified by the following equations:

$$\gamma_x = f_1(\beta_x, n_x) = \frac{\beta_x}{b_y}\left(\frac{\rho}{1-\theta} + \delta_x - \beta_x^{-1/\theta}\left(\frac{1}{1-\theta} - n_x\right)\right) < 0 \tag{G.1}$$

$$\hat{\gamma} = f_2(\hat{\beta}, n_x) = \frac{\hat{\beta}}{b_y}\left(\frac{\rho}{1-\theta} + \delta_x - n_x\hat{\beta}^{-1/\theta}\frac{\theta}{1-\theta}\right) > 0. \tag{G.2}$$

For all $n_x$ satisfying our assumptions, we can assert that

$$f_1(\beta_x, n_x) < f_2(\hat{\beta}, n_x). \tag{G.3}$$

This holds true when $n_x = 1$, which implies that

$$f_1(\beta_x, 1) < f_1(\beta_x, n_x) < f_2(\hat{\beta}, n_x) < f_2(\hat{\beta}, 1) = f_1(\hat{\beta}, 1). \tag{G.4}$$

Because $\partial_{n_x} f_1 > 0$, $\partial_{n_x} f_2 < 0$ and $f_1$ are increasing in $\beta$, we conclude that $\beta_x < \hat{\beta}$. Therefore, using the first-order conditions, we can infer that $\omega_x^{NE}(n_x, n_y) > \hat{\omega}_x(n_x, n_y)$, but also that $n_x \omega_x^{NE}(n_x, n_y) > n_x \hat{\omega}_x(n_x, n_y)$.