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Working Papers / Documents de travail

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WP 2023 - Nr 27

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Revised version: *November 2023*

Abstract: *When can exogenous changes in beliefs generate endogenous fluctuations in rational expectation models? We analyze this question in the canonical one-sector and two-sector models of the business cycle with increasing returns to scale. A key feature of our analysis is that we express the uniqueness/multiplicity condition of equilibirum paths in terms of restrictions on five critical and economically interpretable parameters: the Frisch elasticities of the labor supply curve with respect to the real wage and to the marginal utility of wealth, the intertemporal elasticity of substitution in consumption, the elasticity of substitution between capital and labor, and the degree of increasing returns to scale. We obtain two clear-cut conclusions: belief-driven fluctuations cannot exist in the one-sector version of the model for empirically consistent values for these five parameters. By contrast, belief-driven fluctuations are a robust property of the two-sector version of the model - with differentiated consumption and investment goods -, as they now emerge for a wide range of parameter values consistent with available empirical estimates. The key ingredients explaining these different outcomes are factor reallocation between sectors and the implied variations in the relative price of investment, affecting the expected return on capital accumulation.*

Keywords: *belief-driven business cycles, endogenous fluctuations, expectations, income effect.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

*The project leading to this publication has received funding from the french government under the "France 2030" investment plan managed by the NRA (reference: ANR-17-EURE-0020), and from the Excellence Initiative of Aix-Marseille University - A*MIDEX. We thank an anonymous referee and the Associate Editor Raouf Boucekkine for very valuable comments that allowed us to significantly improve our paper. We also thank J. Benhabib, R. Dos Santos Ferreira, C. Garcia-Penalosa, T. Lloyd-Braga, L. Modesto, X. Raurich and T. Seegmuller for useful comments and suggestions. This paper also benefited from presentations at the conference of the "Society of Nonlinear Dynamics and Econometrics", Tokyo, March 19-20, 2018, the International Conference on "Real and Financial Interdependencies: instability, international openness and regulation policies", AMSE, June 24-26, 2019, at the "2019 Africa Meeting of the Econometric Society", Rabat, July 11-13, 2019, and at the "International Conference on Public Economic Theory - PET 22", Marseille, June 8-10, 2022.

1 Introduction

Are exogenous changes in agents' expectations a significant cause of observed economic fluctuations? A positive answer to this question requires two necessary ingredients. First, belief-driven fluctuations must be a plausible outcome of realistic business cycle models, occurring for realistic values for all structural parameters. Second, to the extent that they exist, exogenous changes in expectations must generate endogenous dynamics for macroeconomic variables consistent with observed patterns of economic fluctuations, in particular in response to transitory shocks. The present paper addresses the first issue, while a thorough assessment of the second issue is proposed in an accompanying paper (Dufourt *et al.* [18]).

We analyze this question within two canonical models of the business cycle: the one-sector Real Business Cycle (RBC) model of Kydland and Prescott [37], stripped down to its essential ingredients by King, Plosser and Rebelo [36] and extended to consider positive externalities with increasing returns to scale by Benhabib and Farmer [6] ; and the two-sector version of the latter model proposed by Benhabib and Farmer [7]. The main factor distinguishing our analysis from these previous papers is that we do not restrict the specifications of individual preferences and of the production functions to have particular forms, imposing instead minimal sets of assumptions on these functions.

A key novelty of our approach is that we show that the local stability conditions of the steady state – influencing the possibility of emergence of belief-driven fluctuations – can then be characterized in terms of five critical and economically interpretable parameters, which we call *critical elasticities*. Two elasticities relate to the specification of the production function: the elasticity of substitution between capital and labor and the degree of increasing returns to scale. Three elasticities relate to the specification of individual preferences: the elasticity of intertemporal substitution in consumption (EIS), the Frisch elasticity of the labor supply curve with respect to the real wage and the Frisch elasticity of the labor supply curve with respect to the marginal utility of wealth.

Regarding the latter three elasticities, we show that the standard strict quasi-concavity and normality assumptions typically imposed on instantaneous utility functions naturally translate into *restrictions* on these elasticities, so that considering these elasticities enables to cover the whole set of admissible utility functions. Moreover, we argue that the Frisch elasticity with respect to the marginal utility of wealth provides a relevant measure of what is often called the degree of "wealth effect" on labor supply decisions, which has recently been shown to play a major role in the local stability properties of dynamic macroeconomic models (see Dufourt *et al.* [17, 19], Jaimovich [32]). We show that our general formulation based on these critical elasticities encompasses as special cases all the standard formulations for individual preferences proposed in the literature, such as the Greenwood *et al.* (GHH) [24] formulation with no income effect, Hansen's [28] formulation with separable consumption and labor, and the King *et al.* (KPR) [36] formulation with constant positive income effect. For each version of the model (one-sector and two-sector), we can then derive the range of parameter values consistent with belief-driven fluctuations and compare it with the range of available empirical estimates.

We derive two important conclusions. First, we prove that belief-driven fluctuations in one-sector RBC models can emerge with standard slopes for the labor demand and labor supply curves but are *ruled out* for any empirically plausible calibrations for the critical elasticities, regardless of the specifications for the individual utility function or the production function. Second, in sharp contrast, we prove that the existence of belief-driven fluctuations is a robust property of two-sector models, in the sense that they arise for a wide range of empirically credible values for the critical elasticities. For example, we show that sunspot fluctuations are compatible with any value for the wage elasticity of labor supply provided the other critical elasticities are in an appropriate range. Likewise, sunspot fluctuations can occur for an arbitrarily small value for the elasticity of intertemporal substitution (EIS) in consumption provided the degree of income effect is not too small.

We provide an in-depth analysis explaining how and why belief-driven fluctuations can emerge in each version of the model and we show that the main sources explaining the drastically different results are, in the two-sector model, factor reallocation between the consumption and the investment sector which, in the presence of externalities, affect the relative price of the investment good. This creates potential capital gains from allocating resources to the investment sector, which leaves the room for an investment boom triggered by optimistic self-fulfilling expectations about the return on capital accumulation.

These main results are derived from the canonical versions of the one-sector and two-sector RBC models, in which preferences are defined over current private consumption and leisure, and production functions are based on current capital and labor (with productive externalities). Clearly, if belief-driven fluctuations easily emerge in the stripped down version of the two-sector model, the scope for endogenous fluctuations is conceptually greater in richer models for which our benchmark model can be obtained as a special case. For example, models with variable capital utilization rate as in Wen [50], models with different production functions in the consumption and the investment sectors as in Benhabib and Nishimura [8] and Benhabib *et al.* [9], or models with static consumption spillovers in the utility function as in Chen and Hsu [15] and Alonso-Carrera *et al.* [2] should conceptually allow for an even greater scope for indeterminacy as long as our canonical model can be obtained as a limit case.[1],[2] We conclude that far from being an exotic feature or a theoretical curiosity arising under extreme parameter configurations,

[1]For example, in models with variable capital utilization, the limit case would be obtained as the one for which the elasticity of the depreciation rate to capital utilization variations is zero. In models with different sectoral production functions (e.g., in terms of their capital-labor intensity or their capital-labor elasticity of substitution), the limit case would be the one for which these elasticities are the same in both sectors. In models with static consumption spillovers, the limit case would the one for which the effect of aggregate consumption on individual preferences is nullified, etc.

[2]On the other hand, the case of models involving additional state variables is trickier, because the limit case enabling to recover our benchmark model may be associated with a change in the dimensionality of the dynamic system. When this happens, a discontinuity in the local stability properties of the model may (or may not) occur, so that a specific analysis for the larger model is required. This potentially concerns, for example, models with habit formation in the utility function (see e.g. Boldrin *et al.* [12] and Jaimovich and Rebelo [33]) for which past consumption is an additional state variable disappearing from the dynamic system when the influence of past consumption is nullified, or New-Keynesian models with sticky prices in which the past inflation rate is a state variable disappearing in the limit case of flexible prices.

belief-driven fluctuations should be considered as a central feature of standard dynamic (multisector) macroeconomic models.

The rest of this paper is organized as follows. In Section 2, we present a short review of the related literature. In Section 3, we analyze the aggregate (one-sector) model. We define the general technology and the general utility function considered throughout the paper. We present our new and innovative way of decomposing all the elasticities that characterize preferences, focusing in particular on income effect. We then study the existence and uniqueness of the steady state and we prove that belief-driven fluctuations are not a realistic outcome of standard aggregate models for all income effects. In Section 4, we consider the two-sector model under the same general specification of preferences and technologies as in Section 3. As a result the existence and uniqueness of the steady state is derived under the same basic conditions as in the aggregate case. We show that the existence of expectation-driven fluctuations is a generic property of two-sector models and fully compatible with empirically relevant values for all the structural parameters. Section 5 concludes. All the technical proofs are contained in the Appendix.

2 Literature review

The endogenous fluctuations and sunspot literature was initiated by the seminal contributions of Azariadis [3], Cass and Shell [14], Grandmont [22], and Woodford [51]. Yet Benhabib and Farmer (BF) [6] is the first paper to analyze these issues in the standard infinite-horizon one-sector model with endogenous labor supply, the workhorse model of the RBC literature. They show that in this model, indeterminacy occurs under the assumption of a large amount of externalities leading to an upward-sloping labor demand function. While this model subjected to sunspot shocks has been shown to account for the main "stylized facts" of the business cycles at least as well as standard RBC models (see Farmer and Guo [21]), the assumption of large aggregate IRS in production was found to be inconsistent with the data.[3] Since this seminal contribution, a major challenge in the literature has been to find extensions of this benchmark model capable of generating expectation-driven business-cycles under empirically realistic values for all structural parameters. Two strands of the literature address this challenge. The first seeks to determine how the local indeterminacy properties of the benchmark one-sector model evolve when some assumptions on preferences or the production function are relaxed. The second adds new ingredients to the benchmark model and reconsiders the issue of indeterminacy in the extended models. We briefly review some critical papers in these two strands of the literature.

The BF one-sector model is based on an additively separable utility function and a Cobb-Douglas technology. Many contributions have generalized this formulation to show that the existence of sunspot fluctuations can be compatible with a downward-sloping labor demand function. Pintus [43] introduced a general technology into the

[3]See e.g. Basu and Fernald [4] for empirical estimates of aggregate IRS in the US economy and Aiyagari [1] for a critique of macroeconomic sunspot models relying on a upward sloping aggregate labor demand curve.

Benhabib-Farmer framework, while Bennett and Farmer [11] and Hintermaier [31] considered non-separable preferences as defined by King *et al.* (KPR) [36], still assuming a Cobb-Douglas technology. Pintus [44] generalized their formulation to a general production function. Lloyd-Braga *et al.* [40] considered general homogenous preferences and a general technology. The overall message from this literature is that preference and technology parameters, like the elasticity of intertemporal substitution in consumption, the degree of income effect on labor supply, the degree of IRS in production, and the elasticity of substitution between capital and labor, all interact to influence the local stability properties of the model. Yet in all these models, the existence of expectation-driven business cycles requires at least one structural parameter value which appears to be outside the range of available empirical estimates.

In response to this critique of the sunspot literature, some authors have modified the production structure of the model. Wen [50] proposes a simple extension consisting in introducing a variable capital utilization rate into the Benhabib-Farmer setup, in the spirit of Greenwood *et al.* [24], and proves that this extension is sufficient to allow for the existence of sunspot fluctuations under empirically plausible values for the structural parameters.[4] However, unitary elasticities of intertemporal substitution in consumption and of capital-labor substitution are assumed, which restricts the possibilities to successfully match the data in a data confrontation perspective. Considering a more general utility function and a general production function, Dufourt *et al.* [20] show that the variable capital utilization model allows for the emergence of belief-driven fluctuations under a wider set of values for these elasticities if the elasticity of the labor supply curve is large enough. Yet, they also show that the model still faces difficulties in accounting for some crucial empirical facts associated with transitory shocks. Other authors have instead modified the specification for individual preferences. For example, Alonso-Carrera *et al.* [2] introduce consumption spillovers in the utility function and they show that this favors the occurrence of indeterminacy even when the labor supply curve is positively sloped. However, the extent to which such a model submitted to belief shocks can account for the data is left unexplored.

Two-sector models have also been considered, again following the seminal contribution of Benhabib and Farmer [7]. In this paper, Benhabib and Farmer extend their initial formulation to a two-sector economy producing differentiated consumption and investment goods but with the same Cobb-Douglas technology characterized by sector-specific output externalities leading to increasing returns. Building on the fact that capital and labor can be freely allocated between sectors, they prove that the existence of local indeterminacy becomes compatible with a downward-sloping labor demand function. Unlike their one-sector contribution, it clearly appears that when external effects in each sector depend on that sector's aggregate output, factor reallocations across sectors can have strong effects on marginal products and indeterminacy can occur with much smaller externalities. Harrison [29] builds on these results to show that indeterminacy occurs for a minimum value of the externality in the investment sector, even with no externality in the consumption sector. All these conclusions have been recently confirmed by Du-

[4] See also Benhabib and Wen [10] for a discrete-time version of the same model.

fourt *et al.* [17] considering GHH preferences instead of additively separable ones.[5] Guo and Harrison [26] also introduce a variable capital utilization rate into the Benhabib and Farmer model and confirm that lower externalities are required.

Worthy of mention too is all the literature departing from the contribution of Benhabib and Nishimura [8] and Benhabib *et al.* [9], in which sector-specific externalities are introduced in two-sector models with differentiated private technologies and constant social returns. In such a framework, the existence of local indeterminacy relies on a capital-intensity reversal between the private and social levels and, as shown in Nishimura and Venditti [41], requires extreme values for the elasticity of intertemporal substitution in consumption (EIS) which are not in line with empirical estimates.

All in all, although sunspot fluctuations have been shown to arise more easily in extended model formulations, there is still no general analysis in the literature.

3 A general aggregate model

We consider a closed economy framework in the spirit of Benhabib and Farmer [6] (BF). The economy is composed of a large number of identical infinitely-lived agents and a large number of identical producers. Agents consume, supply labor and accumulate capital. Firms produce the unique final good which can be used either for consumption or investment. All markets are perfectly competitive, but there are externalities in production.

3.1 The representative firm: a general technological structure

The production sector is composed of a large number of identical firms which operate under perfect competition. Output Y_t is produced by combining labor L_t and capital K_t. The technology of each firm exhibits constant returns to scale with respect to its own inputs and we consider that each of the many firms benefits from positive externalities due to the contribution of the average levels of labor $\bar{L}$ and capital $\bar{K}$. These external effects are exogenous and not traded in markets. The production function is

$$Y_t = f(K_t, L_t)e(\bar{K}_t, \bar{L}_t) \tag{1}$$

with $e(\bar{K}_t, \bar{L}_t)$ the externality variable. We follow BF by assuming that externalities affect the technology in a multiplicative way but we depart from them by not requiring the production function to be Cobb-Douglas. Rather, our production function is general and satisfies:

Assumption 1. *$f(K, L)$ is $\mathbf{C}^2$ over $\mathbb{R}^2_{++}$, increasing in (K, L), concave over $\mathbb{R}^2_{++}$ and homogeneous of degree one. $e(\bar{K}, \bar{L})$ is $\mathbf{C}^1$ over $\mathbb{R}_{++}$ and increasing in $(\bar{K}, \bar{L})$. Moreover, for any given $L > 0$,*

$$\lim_{K\to 0} f_1(K, L)e(K, L) = +\infty \text{ and } \lim_{K\to +\infty} f_1(K, L)e(K, L) = 0$$

and, for any given $K > 0$,

$$\lim_{L\to 0} f_2(K, L)e(K, L) = +\infty \text{ and } \lim_{L\to +\infty} f_2(K, L)e(K, L) = 0$$

[5] See also Guo and Harrison [27].

Firms rent capital units at the real rental rate r_t and hire labor at the unit real wage w_t. The profit maximization program of the representative firm,

$$\max_{\{Y_t,L_t,K_t\}} Y_t - w_tL_t - r_tK_t,$$

leads to the standard demand function for capital K_t and labor L_t:

$$r_t = f_1(K_t, L_t)e(\bar{K}_t, \bar{L}_t) \tag{2}$$

$$w_t = f_2(K_t, L_t)e(\bar{K}_t, \bar{L}_t) \tag{3}$$

As will become clear, the production function and the optimal decisions of firms influence the local dynamics of the model through four crucial elasticities: the elasticity of output with respect to capital stock $s(K, L)$ (which, at equilibrium, is also the share of capital in total income), the elasticity of capital-labor substitution $\sigma(K, L)$, and the elasticities of the externality variable with respect to labor, $\varepsilon_{eL}(\bar{K}, \bar{L})$, and capital, $\varepsilon_{eK}(\bar{K}, \bar{L})$:

$$s(K, L) = \frac{Kf_1(K,L)}{f(K,L)} \in (0, 1), \quad \sigma(K, L) = -\frac{(1-s(K,L))f_1(K,L)}{Kf_{11}(K,L)} > 0 \tag{4}$$

$$\varepsilon_{eK}(\bar{K}, \bar{L}) = \frac{e_1(\bar{K},\bar{L})\bar{K}}{e(\bar{K},\bar{L})}, \quad \varepsilon_{eL}(\bar{K}, \bar{L}) = \frac{e_2(\bar{K},\bar{L})\bar{L}}{e(\bar{K},\bar{L})} \tag{5}$$

Obviously, the choice of a Cobb-Douglas production function, as in BF, implies $\sigma(K, L) = 1$, whereas the use of a general production function entails $\sigma(K, L) \in (0, +\infty)$. To simplify notation, we will for now denote by s, σ, ε_{eK} and ε_{eL} the corresponding elasticities evaluated at the steady state. In order to allow for a direct comparison with BF, the externalities are also expressed as follows:

$$\varepsilon_{eK} = s\Theta_k \quad \varepsilon_{eL} = (1 - s)\Theta_l \tag{6}$$

where $\Theta_k, \Theta_l \geq 0$ are the degrees of increasing returns to scale in capital and labor. BF assume output externalities, implying $\Theta_k = \Theta_l = \Theta$. We allow for a more general formulation in which external effects can be factor-specific and independent of each-other.

Finally, we make a standard assumption requiring that the aggregate (i.e., taking external effects into account) labor demand and capital demand functions are decreasing in the real wage and in the rental rate of capital, respectively:

Assumption 2. $\Theta_k < (1 - s)/s\sigma \equiv \bar{\Theta}_k$ *and* $\Theta_l < s/(1 - s)\sigma \equiv \bar{\Theta}_l$

It is well known from BF analysis that when the slope of the aggregate labor demand curve is positive and greater than the slope of the aggregate labor supply curve, indeterminacy and sunspot fluctuations can occur in the one-sector infinite-horizon model. We rule out this possibility here because it entails extremely high degrees of increasing returns to scale that are at odds with the data. We will be more specific about realistic degrees of IRS later in the paper.

3.2 The representative household: a general utility function

The economy is composed of a continuum of mass 1 of identical households. In each period, the representative household is endowed with ℓ units of time. Given the real wage w_t and the rental rate of capital r_t, the household decides how much of its available time to allocate to leisure time $\mathcal{L}_t$ and hours worked l_t, and how much to consume c_t. It also rents its capital stock k_t to the representative firms, and accumulates capital according to the following capital accumulation constraint:

$$k_{t+1} = (1-\delta+r_t)k_t + w_t l_t + d_t - c_t \tag{7}$$

where $\delta \in (0,1)$ is the capital depreciation rate, and d_t are potential dividends redistributed ex-post by firms.

In each period, the utility that the household derives from consumption and leisure is described by a general instantaneous utility function $u(c, \mathcal{L})$. As in the case of the productive side of the economy, we want our analysis to be as general as possible. We thus make the following minimum standard assumptions on the utility function:

Assumption 3. *$u(c, \mathcal{L})$ is $\mathbf{C}^2$ over $\mathbb{R}^2_{++}$ increasing in each argument, strictly quasi-concave in $(c, \mathcal{L})$, and satisfies the Inada conditions*

$$\lim_{c\to 0} u_1(c,\mathcal{L}) = +\infty, \quad \lim_{c\to+\infty} u_1(c,\mathcal{L}) = 0, \quad \lim_{\mathcal{L}\to 0} u_2(c,\mathcal{L}) = +\infty \text{ and } \lim_{\mathcal{L}\to+\infty} u_2(c,\mathcal{L}) = 0.$$

The Inada conditions are introduced to ensure an interior optimum. Furthermore, to avoid basing our analysis of the local stability conditions and of the occurrence of sunspot fluctuations on exotic features regarding individual preferences, we introduce the following standard assumption on consumption and leisure:

Assumption 4. *Consumption c and leisure $\mathcal{L}$ are normal goods.*

Assuming that the intertemporal utility function is additively separable over time, the representative consumer solves the following lifetime utility maximization program (where $\beta \in (0,1)$ is the subjective discount factor):

$$\begin{array}{ll} \max\limits_{\{c_t,\, l_t,\, k_{t+1}\}_{t=0\ldots\infty}} & \sum\limits_{t=0}^{+\infty} \beta^t u(c_t, \ell - l_t) \\ s.t. & k_{t+1} = (1-\delta+r_t)k_t + w_t l_t + d_t - c_t, \quad t = 0\ldots\infty \\ & k_0 \text{ given} \end{array} \tag{8}$$

Denoting by λ_t the Lagrange multiplier on constraint (7) and $R_t = 1-\delta+r_t$ the net return factor on capital, the first-order conditions can be written as

$$u_1(c_t, \ell - l_t) = \lambda_t, \tag{9}$$

$$\frac{u_2(c_t, \ell - l_t)}{u_1(c_t, \ell - l_t)} = w_t \tag{10}$$

$$\lambda_t = \beta R_{t+1} \lambda_{t+1} \tag{11}$$

Equation (10) describes the optimal consumption-leisure trade-off, while equations (9) and (11) jointly describe the optimal arbitrage between consumption and saving (i.e., the Euler equation). An optimal path must also satisfy the transversality condition:

$$\lim_{t\to+\infty} \beta^t \lambda_t k_{t+1} = 0 \tag{12}$$

Following Rotemberg and Woodford [47], we can rewrite the optimality conditions (9-10) in terms of time-invariant Frisch consumption-demand and labor-supply curves involving the real wage w_t and the marginal utility of wealth λ_t :

$$c_t = c(w_t, \lambda_t), \quad l_t = l(w_t, \lambda_t) \tag{13}$$

As was the case for the productive size of the economy, we show later that the local dynamics of the model around the steady state is determined by a limited number of critical elasticities. Denote by

$$\epsilon_{cw} = \frac{c_1(w,\lambda)w}{c}, \ \epsilon_{c\lambda} = \frac{c_2(w,\lambda)\lambda}{c}, \ \epsilon_{lw} = \frac{l_1(w,\lambda)w}{l}, \ \epsilon_{l\lambda} = \frac{l_2(w,\lambda)\lambda}{l}, \tag{14}$$

the Frisch elasticities of the demand and supply functions (13), and by

$$\epsilon_{cc} = -\frac{u_1(c,\mathcal{L})}{u_{11}(c,\mathcal{L})c}, \tag{15}$$

the elasticity of intertemporal substitution in consumption. We can easily prove the following Lemma:

Lemma 1. *The three critical elasticities ϵ_{cc}, ϵ_{lw} and $\epsilon_{l\lambda}$ are related to the individual utility function by*

$$\epsilon_{cc} = -\frac{u_1}{u_{11}c}, \quad \epsilon_{lw} = \frac{1}{l}\left(\frac{-u_{11}u_2}{u_{11}u_{22}-u_{12}u_{21}}\right), \quad \epsilon_{l\lambda} = \frac{1}{l}\left(\frac{u_{21}u_1-u_{11}u_2}{u_{11}u_{22}-u_{12}u_{21}}\right) \tag{16}$$

Moreover, the remaining two elasticities ϵ_{cw} and $\epsilon_{c\lambda}$ are related to ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$ through the following equations

$$\epsilon_{cw} = \mathcal{C}\left(\epsilon_{lw} - \epsilon_{l\lambda}\right), \quad \epsilon_{c\lambda} = -\epsilon_{cc} + \mathcal{C}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\epsilon_{l\lambda} \tag{17}$$

where $\mathcal{C} = \theta(1-s)/(\theta - s\beta\delta) < 1$ is the steady state ratio of wage income over consumption (wl/c), which is independent of the specification of the individual utility function, and $\theta = 1-\beta(1-\delta)$.

Proof. See Appendix 6.1.

An implication of this Lemma is that, as far as the representative consumer's decision is concerned, the dynamic properties of the model are completely determined by the three elasticities ϵ_{lw}, $\epsilon_{l\lambda}$, and ϵ_{cc}, in addition to the parameter $\mathcal{C}$, which is independent of the specification of individual preferences.

Using this Lemma, we can immediately derive the following Proposition:

Proposition 1. *Under Assumption 3, $\epsilon_{cc} > 0$ and $\epsilon_{lw} > 0$. Moreover, under Assumption 4, $\epsilon_{l\lambda} \geq 0$, $\epsilon_{c\lambda} \leq 0$, and thus*

$$\epsilon_{cc} \geq \frac{\mathcal{C}\epsilon_{l\lambda}(\epsilon_{lw}-\epsilon_{l\lambda})}{\epsilon_{lw}} \equiv \epsilon_{cc}^N. \tag{18}$$

Proof. See Appendix 6.2.

The importance of this Proposition is that it shows how assumptions on preferences (namely Assumptions 3 and 4) naturally translate into *restrictions* on the critical elasticities ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$. Note that these restrictions ($\epsilon_{cc} > 0$, $\epsilon_{lw} > 0$, $\epsilon_{l\lambda} \geq 0$, and $\epsilon_{cc} \geq \epsilon_{cc}^N$) are actually much simpler than working directly with the standard concavity and normality assumptions based on the utility function.

Another nice feature of considering these elasticities, instead of considering the first-order and second-order derivatives of $u(\cdot)$, is that the former have a clear economic interpretation. The EIS in consumption ϵ_{cc} and the Frisch elasticity of labor supply ϵ_{lw} have a well-known interpretation that requires no further discussion. On the other hand, since λ_t is the marginal utility of wealth, the elasticity $\epsilon_{l\lambda}$ captures the extent to which a change in the household's expected wealth over its entire lifetime affects the current labor supply decision. Indeed, when, for any reason, lifetime income decreases, the intertemporal budget constraint obtained from aggregating (7) over time becomes more restrictive, and λ_t increases as the household's consumption choices are more constrained.

As implied by the normality assumption, it follows that consumption and leisure decrease while hours worked increase. The elasticity $\epsilon_{l\lambda}$ captures the extent to which such a change in lifetime income affects the current labor supply decision.

In short, $\epsilon_{l\lambda}$ is a properly defined measure of the *wealth effect on labor supply*. This elasticity is particularly important because the recent literature has shown that the intensity of this wealth effect plays a significant role in the local stability properties of many dynamic macroeconomic models (see Dufourt *et al.* [17, 19], Jaimovich [32]). However, this literature only addressed particular utility functions in which the wealth effect is known to be either "positive" (but not precisely defined) or zero. Moreover, these specific utility functions also require the introduction of other cross-restrictions on the three critical elasticities defined above. With our analysis, however, the intensity of the wealth effect on labor supply can be chosen independently of the values given to the other two elasticities (ϵ_{lw} and ϵ_{cc}).

To better illustrate these points, the following Proposition clarifies the restrictions implied by some of the most widely used classes of utility functions:

Proposition 2. *Under KPR preferences,*

$$u(c,\mathcal{L}) \;=\; \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}v(\mathcal{L}), & \textit{with } \gamma > 0,\ \gamma \neq 1 \\ \log(c) + \log(v(\mathcal{L})), & \textit{with } \gamma = 1 \end{cases}$$

with $\mathcal{L} = \ell - l$ *and* $v(\mathcal{L})$ *increasing and concave (if* $\gamma \leq 1$*) or decreasing and convex (if* $\gamma > 1$*), the critical elasticities satisfy:*

$$\epsilon_{cc} = \frac{1}{\gamma},\ \epsilon_{lw} = \frac{\mathcal{L}}{l}\frac{1}{(1-\epsilon_{cc})v'(\mathcal{L})\mathcal{L}/v(\mathcal{L}) - v''(\mathcal{L})\mathcal{L}/v'(\mathcal{L})} > 0,\ \epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}.$$

Under generalized GHH preferences,

$$u(c,l) = \frac{1}{1-\gamma}\left(c - \frac{l^{1+\chi}}{1+\chi}\right)^{1-\gamma},$$

with $\gamma > 0$ *and* $\chi \geq 0$*, the critical elasticities satisfy:*

$$\epsilon_{cc} = \frac{1}{\gamma}\left(1 - \frac{\mathcal{C}}{1+\chi}\right),\ \epsilon_{lw} = \frac{1}{\chi},\ \epsilon_{l\lambda} = 0.$$

Under Generalized Hansen [28] preferences,

$$u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\chi}}{1+\chi},$$

with $\gamma > 0$ *and* $\chi \geq 0$*, the critical elasticities satisfy:*

$$\epsilon_{cc} = \frac{1}{\gamma},\ \epsilon_{lw} = \epsilon_{l\lambda} = \frac{1}{\chi}. \tag{19}$$

According to Proposition 2, in the case of KPR preferences, only two of the three critical elasticities ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$ are independent since they are related through the equation $\epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}$. In the case of generalized GHH preferences, the restriction $\epsilon_{l\lambda} = 0$ is well-known, having been introduced on purpose to eliminate the wealth effect on labor supply. However, there is often far less awareness that, with this class of preferences, changing the calibration of the preference parameter χ to change the value of the wage elasticity of the labor supply, $\epsilon_{lw} = 1/\chi$, meanwhile generates a change in the EIS in consumption, ϵ_{cc}. In fact, Proposition 2 implies that changing the calibration of χ to $\chi' \neq \chi$ also requires adjusting the calibration of γ to $\gamma' = (1 - \mathcal{C}/(1+\chi'))\,\epsilon_{cc}$ if one wants to keep the initial value of the EIS unchanged. Finally, under generalized Hansen preferences, a strong restriction relating the two Frisch elasticities of the labor supply curve is introduced, since we have in this case: $\epsilon_{lw} = \epsilon_{l\lambda}$.

The particular case of the Jaimovich-Rebelo formulation

Jaimovich and Rebelo (JR) [33] were the first to discuss the importance of the income effect on the occurrence of indeterminacy and sunspot fluctuations. Their discussion is based on the following specification of the instantaneous utility function:

$$u(c_t, l_t, X_t) = \frac{\left[c_t - \frac{l_t^{1+\chi}}{1+\chi} X_t\right]^{1-\gamma} - 1}{1-\gamma} \tag{20}$$

with $X_t = c_t^\phi X_{t-1}^{1-\phi}$ and $\phi \in [0, 1]$. As they state, this specification nests as polar cases the GHH utility function (when $\phi = 0$) and the KPR utility function (when $\phi = 1$) formulations. The magnitude of the income effect is therefore controlled by varying the value of γ between these two extremes.

Assuming $\gamma = 1$ and $\Theta_k = \Theta_l = \Theta$, Jaimovich [32] shows that, for some values of Θ compatible with a negatively-slopped labor demand function, there exist two bounds $0 < \underline{\phi} < \overline{\phi} < 1$ such that when $\phi \in (\underline{\phi}, \overline{\phi})$ local indeterminacy and sunspot fluctuations occur under realistic values for all the structural parameters. The conclusion is that the income effect has a non-linear effect on the range of values consistent with indeterminacy, which appears to arise under intermediate amounts of income effect, and to be ruled out with either low or high amounts.

A difference between the JR specification and our specification is that, except for the two polar cases $\phi = 0$ and $\phi = 1$, the JR utility function assumes that an additional state variable X_t enters the utility function:

$$X_t = \prod_{s=0}^{t-1} c_{t-s}^{\phi(1-\phi)^s} X_0^{(1-\phi)^s}$$

It follows that the utility function at time t depends on the whole history of past consumption decisions. The result is that the consumption demand and labor supply decisions no longer write $c(w_t, \lambda_t)$ and $l(w_t, \lambda_t)$ but $c(w_t, \lambda_t, X_t)$ and $l(w_t, \lambda_t, X_t)$. In short, compared to our specification, such a formulation introduces two additional elasticities ϵ_{cX} and ϵ_{lX} associated with consumption habits which generate a dynamic link between current and future consumption and labor supply decisions – what Jaimovich refers to as a form of "dynamic" income effects.[6]

To avoid the complexities of introducing additional state variables, Nourry *et al.* [42] and Dufourt *et al.* [19] consider a modified JR utility function which only involves current-period variables, namely

$$u(c_t, l_t) = \frac{\left[c_t - \frac{l_t^{1+\chi}}{1+\chi} c_t^\phi\right]^{1-\gamma} - 1}{1-\gamma} \tag{21}$$

We recover the two polar cases of a GHH and a KPR utility function associated with $\phi = 0$ and $\phi = 1$, respectively. Moreover, it is now possible to vary the values of the three critical elasticities ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$ by considering alternative calibrations for the three parameters γ, χ, and ϕ. However, the critical elasticities are very cumbersome combinations of these parameters, and when one parameter is adjusted so as to change the value of one elasticity, the other parameters also have to be adjusted to maintain the

[6]As we will show later on, absent these additional ingredients, sunspot fluctuations cannot occur under empirically relevant calibrations for any size of income effect.

value of the other elasticities constant. It is therefore much better to work with a general utility function and calibrate the three critical elasticities directly, as we do in this paper.

3.3 Intertemporal equilibrium and steady state

At a symmetric general equilibrium of the economy, prices $\{w_t, r_t, \lambda_t\}$ adjust so that all markets clear at any date t, with the externality variables satisfying $(\bar{k}_t, \bar{l}_t) = (k_t, l_t)$ for any t. Imposing the latter equalities in the set of physical constraints and optimality conditions (1)-(3), (11), and (13), we obtain that a symmetric general equilibrium satisfies in any t,

$$\begin{array}{rcl} \lambda_t & = & \beta(1-\delta+r_{t+1})\lambda_{t+1} \\ k_{t+1} & = & (1-\delta)k_t + y_t - c_t \\ r_t & = & f_1(k_t, l_t)e(k_t, l_t) \\ w_t & = & f_2(k_t, l_t)e(k_t, l_t) \\ c_t & = & c(w_t, \lambda_t) \\ l_t & = & l(w_t, \lambda_t) \\ y_t & = & f(k_t, l_t)e(k_t, l_t) \end{array} \tag{22}$$

together with the initial condition k_0 given and the transversality condition (12).

From these dynamic equations, we immediately derive that if a steady state exists, the rental rate of capital at the steady state is

$$r^* = \frac{1-\beta(1-\delta)}{\beta} \equiv \frac{\theta}{\beta}$$

In order to study the existence and uniqueness of a steady state, we analyze the existence of a 6-uple $(k^*, y^*, l^*, c^*, w^*, \lambda^*)$ solution to the set of equations

$$\begin{array}{c} f_1(k^*, l^*)e(k^*, l^*) = \frac{\theta}{\beta}, \quad f_2(k^*, l^*)e(k^*, l^*) = \frac{u_2(c^*, \ell - l^*)}{u_1(c^*, \ell - l^*)} \\ c^* = f(k^*, l^*)e(k^*, l^*) - \delta k^*, \quad w^* = f_2(k^*, l^*)e(k^*, l^*) \\ y^* = f(k^*, l^*)e(k^*, l^*), \quad c^* = c(w^*, \lambda^*) \end{array} \tag{23}$$

Note that, for analytical convenience, instead of considering the Frisch labor supply equation $l^* = l(w^*, \lambda^*)$, we reintroduce the initial optimality condition involving the marginal rate of substitution between consumption and labor.

We first prove the following Lemma:

Lemma 2. *At the steady state, the ratios y^*/k^*, c^*/k^* and w^*l^*/c^* satisfy*

$$\frac{y^*}{k^*} = \frac{\theta}{s\beta}, \quad \frac{c^*}{k^*} = \frac{\theta - s\beta\delta}{s\beta}, \quad \text{and} \quad \frac{w^*l^*}{c^*} = \frac{\theta(1-s)}{\theta - s\beta\delta} \equiv \mathcal{C}$$

Proof. See Appendix 6.3.

Using this Lemma, we derive the following Proposition:

Proposition 3. *Under Assumptions 1-4, a unique steady state generically exists. Moreover, for any given calibration of structural parameters, there always exists a value $\ell^* > 0$ such that when $\ell = \ell^*$, the steady state is constant across calibrations with $l^* = \bar{l}^* < \ell^*$.*

Proof. See Appendix 6.4.

An implication of Proposition 3 is that, when analyzing how alternative calibrations for the structural parameters affect the stability properties of the model, it is possible to maintain the steady state $(k^*, y^*, l^*, c^*, w^*, \lambda^*)$ unchanged by adjusting the value for ℓ^* accordingly. In other words, we can follow the usual practice of "calibrating" the level of hours worked at the steady state without difficulty.

3.4 Local stability analysis

We now carry out a thorough analysis of the local stability properties of the steady state when the dynamics is defined by (22). In order to do so, we log-linearize the set of equations in (22) around the unique steady state. Using Lemmata 1 and 2, we obtain (where hatted variables denote percentage deviations from the steady state):

$$\begin{aligned}
\widehat{\lambda}_t &= \widehat{\lambda}_{t+1} + \theta \widehat{r}_{t+1} \\
\widehat{k}_{t+1} &= (1-\delta)\widehat{k}_t + \tfrac{\theta}{s\beta}\widehat{y}_t - \left(\tfrac{\theta - s\beta\delta}{s\beta}\right)\widehat{c}_t \\
\widehat{r}_t &= \left(-\tfrac{1-s}{\sigma} + s\Theta_k\right)\widehat{k}_t + \left(\tfrac{1-s}{\sigma} + (1-s)\Theta_l\right)\widehat{l}_t \\
\widehat{w}_t &= \left(\tfrac{s}{\sigma} + s\Theta_k\right)\widehat{k}_t + \left(-\tfrac{s}{\sigma} + (1-s)\Theta_l\right)\widehat{l}_t \\
\widehat{c}_t &= \mathcal{C}\left(\epsilon_{lw} - \epsilon_{l\lambda}\right)\widehat{w}_t + \left(\mathcal{C}\left(1 - \tfrac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\epsilon_{l\lambda} - \epsilon_{cc}\right)\widehat{\lambda}_t \\
\widehat{l}_t &= \epsilon_{lw}\widehat{w}_t + \epsilon_{l\lambda}\widehat{\lambda}_t \\
\widehat{y}_t &= s(1+\Theta_k)\widehat{k}_t + (1-s)\left(1+\Theta_l\right)\widehat{l}_t
\end{aligned}$$

This is a system of seven equations in seven variables, only two of these equations being dynamic. To analyze the local stability properties of the model, we first reduce the system by using the five static equations to eliminate five variables, $\widehat{y}_t$, $\widehat{c}_t$, $\widehat{l}_t$, $\widehat{w}_t$, and $\widehat{r}_t$, from the dynamic equations. The obtained system of *minimal dimension* – two dynamic equations in two variables, $\widehat{k}_t$ and $\widehat{\lambda}_t$ – can be expressed as:

$$\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix} \equiv J \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix}$$

where J is the Jacobian matrix of the underlying non-linear 2-dimensional system evaluated at the steady state, which is given in the following Proposition:

Proposition 4. *The elements of the Jacobian matrix J are:*

$$J_{11} = \tfrac{1}{\beta}\left\{1 + \theta\Theta_k + \frac{\theta(1-s)\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\}, \quad J_{12} = \tfrac{1}{s\beta}\left\{\frac{\theta(1-s)\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} + (\theta - s\beta\delta)\epsilon_{cc}\right\}$$

$$J_{21} = \theta\frac{\frac{1-s}{\sigma}-s\Theta_k-\frac{\epsilon_{lw}}{\sigma}\left[\Theta_l(1-s)+s\Theta_k\right]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}J_{11}, \quad J_{22} = \frac{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\theta\left\{\frac{1-s}{\sigma}-s\Theta_k-\frac{\epsilon_{lw}}{\sigma}\left[\Theta_l(1-s)+s\Theta_k\right]\right\}J_{12}}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}$$

Proof. See Appendix 6.5.

The local dynamics of the model is thus determined by the nine structural parameters constituting the matrix J: four of them concern the productive side of the economy, namely s, σ, Θ_k, and Θ_l, four of them concern individual preferences, namely β, ϵ_{cc}, ϵ_{lw}, $\epsilon_{l\lambda}$, and finally there is the depreciation rate of capital δ.

Using the geometrical methodology of Grandmont *et al.* [23] as presented in Appendix 6.6, we prove that there exist two bifurcation loci in the parameter space such that, when ϵ_{cc} is increased from 0 to $+\infty$, a change in the stability properties of the steady state occurs when ϵ_{cc} crosses any of the two loci. These results are formally summarized in the following Lemma:

Lemma 3. *Under Assumptions 1-4, let $\Omega = (\beta, \delta, s, \sigma, \Theta_k, \Theta_l)$ be the set of structural parameters. For any $\omega \in \Omega$ such that $\sigma \leq \bar{\sigma} \equiv \theta/(1-\beta)$, there exist two* bifurcation curves *crossing the 3-dimensional plane $(\epsilon_{lw}, \epsilon_{l\lambda}, \epsilon_{cc})$ and generating a change in the local stability properties of the steady state:*

– *a* flip bifurcation curve $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ *associated with one real eigenvalue of J crossing -1,*

– *a (degenerate)* transcritical bifurcation curve $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ *associated with one real eigenvalue of J crossing 1.*

These bifurcation curves appear for any $\epsilon_{lw} > \underline{\epsilon}_{lw}$, with

$$\underline{\epsilon}_{lw} \equiv \frac{1-s-\sigma s \Theta_k}{(1-s)\Theta_l + s\Theta_k}.$$

There also exists one critical bound $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ *such that $\mathcal{D} = 1$ when $\epsilon_{l\lambda} = \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$. This critical bound exists for any (Θ_k, Θ_l) such that $\Theta_k \in [0, \underline{\Theta}_k)$ and $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$, with*

$$\underline{\Theta}_k \equiv \frac{s\beta}{(1-s)\sigma - (1-\beta)}, \qquad \underline{\Theta}_l \equiv \frac{1-\beta+\Theta_k}{\beta}.$$

The formal expressions of the bifurcation curves and the critical bound are given in Appendix 6.6.

Proof. See Appendix 6.6.

The following Theorem now provides a complete picture of the local stability properties of the aggregate model.

Theorem 1. *Under Assumptions 1-4, let $\sigma \leq \bar{\sigma} \equiv \theta/(1-\beta)$ and consider the bifurcation curves, critical bound, and thresholds defined in Lemma 3. Then the following results hold:*

Case 1 - Low wage elasticity of labor supply: $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$.

The steady state is a saddle-point.

Case 2 - High wage elasticity of labor supply: $\epsilon_{lw} > \underline{\epsilon}_{lw}$.

– *Under* low capital externalities $\Theta_k \in (0, \underline{\Theta}_k)$, *the steady state is*

i) for $\Theta_l \in [0, \underline{\Theta}_l)$,

- *a saddle-point if $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$,*
- *a source if $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.*

ii) for $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$,

- *a saddle-point if $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$,*
- *a source if $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$ and $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$,*
- *a sink if $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$ and $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.*

– *Under* large capital externalities $\Theta_k \in (\underline{\Theta}_k, \bar{\Theta}_k)$, *the steady state is*

- *a saddle-point if* $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$,
- *a source if* $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.

Proof. See Appendix 6.7.

3.5 Discussion

Theorem 1 – characterizing the local stability properties of the one-sector model for any specification of individual preferences, any specification for the production function, and any degrees of IRS in capital and labor consistent with downward-sloping labor and capital demand curves – considerably generalizes existing results surveyed in the literature review.[7] Three main results can be drawn from this Theorem. First, from a theoretical standpoint, it is possible to identify an area in the parameter space such that indeterminacy and sunspot fluctuations exist in the one-sector model, even though externalities are mild enough to ensure downward-sloping capital demand and labor demand Yet, this area of indeterminacy occurs for empirically implausible configurations of parameter values (Result 1). Second, under various minor additional assumptions on parameters frequently encountered in the literature, local indeterminacy is excluded (Result 2). Third, for parameter values in the range of available empirical estimates, the steady-state is a saddle path (Result 3). Before elaborating further on these conclusions, we first provide an explanation as for why indeterminacy can occur in this standard model.

Interpretation for indeterminacy:

Case 2(ii) of Theorem 1 shows that indeterminacy and sunspot fluctuations can occur in the one-sector model (with standard slopes on the labor supply and labor demand curves) when the following necessary conditions are met: $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$, $\epsilon_{lw} > \underline{\epsilon}_{lw}$, $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ and $\epsilon_{cc} > \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$. To understand why these conditions are needed, it is simpler to start with a situation of fixed labor supply ($\epsilon_{lw} = \epsilon_{l\lambda} = 0$) and analyze why indeterminacy and sunspot equilibria *cannot occur* in this case. Consider the Euler equation $\lambda_t = \beta(1 - \delta + r_{t+1})\lambda_{t+1}$ and assume that agents expect an increase in the future return on capital accumulation, r_{t+1}. Can such an expectation be self-fulfilling? According to the Euler equation, an increase in r_{t+1} is associated with an increase in the marginal utility of wealth λ_t and an incentive to substitute investment for consumption (since $\epsilon_{c\lambda} < 0$), leading to a lower consumption c_t, a higher investment i_t and a higher capital stock k_{t+1} at $t + 1$. With a fixed labor supply, the larger capital stock k_{t+1} generates a decrease in the marginal productivity of capital at $t + 1$ and thus a decrease in $r_{t+1} = f_1(k_{t+1}, l)e(k_{t+1}, l)$. The initial expectation cannot be self-fulfilling.

To make this expectation self-fulfilling, the negative influence of the larger capital stock on the real return on capital must be more than compensated by a proportionately

[7] The literature shows that sunspot fluctuations require a sufficiently large (possibly larger than one) elasticity of capital-labor substitution and/or a sufficiently large elasticity of intertemporal substitution in consumption. Moreover, local indeterminacy is ruled out under GHH preferences with no-income effect and KPR preferences (see Bennett and Farmer [11], Hintermaier [31], Lloyd-Braga *et al.* [40], and Pintus [43, 44]).

larger positive influence associated with an *increase in labor* at $t+1$. This is only possible if both the labor demand curve and the labor supply curve are *sufficiently elastic*. Positives externalities in labor $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ make a step in that direction by rendering the aggregate labor demand curve more elastic with respect to the real wage. Consider now the log-linearized versions of the Frisch consumption-demand and labor-supply curves:

$$\begin{aligned} \widehat{c}_t &= \underbrace{\mathcal{C}\left(\epsilon_{lw} - \epsilon_{l\lambda}\right)}_{\epsilon_{cw}} \widehat{w}_t + \underbrace{\left(\mathcal{C}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\epsilon_{l\lambda} - \epsilon_{cc}\right)}_{\epsilon_{c\lambda}<0} \widehat{\lambda}_t \\ \widehat{l}_t &= \epsilon_{lw}\widehat{w}_t + \epsilon_{l\lambda}\widehat{\lambda}_t \end{aligned}$$

At period t, an increase in λ_t generates an increase in the labor supply curve due to the positive income effect, $\epsilon_{l\lambda} > 0$. Since the capital stock is predetermined at t, labor demand is unchanged and the increase in labor supply generates a decrease in the equilibrium real wage w_t. At $t+1$, the change in labor supply must be sufficiently large to compensate the negative effect of capital accumulation on the real return on capital, which requires $\epsilon_{l\lambda}$ to be very large, i.e. $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw}) > \epsilon_{lw}$, as shown in Theorem 1. However, for the story to remain fully consistent, one must also ensure that consumption still *decreases* at t and investment still increases at t when λ_t increases. In the log-linearized version of the Frisch consumption demand above, one easily observe that $\epsilon_{cw} = \mathcal{C}\left(\epsilon_{lw} - \epsilon_{l\lambda}\right) < 0$ when $\epsilon_{l\lambda} > \epsilon_{lw}$, so that the first component is positive ($\widehat{w}_t < 0$ since w_t decreases at t). A decrease in consumption c_t then requires $\epsilon_{c\lambda}$ to be *sufficiently negative*. This in turn requires that the intertemporal substitution effect be sufficiently large, i.e. that $\epsilon_{cc} > \epsilon_{cc}^T\left(\epsilon_{lw}, \epsilon_{l\lambda}\right)$ as shown in Theorem 1.

Result 1: Implausibility of indeterminacy in the one-sector model

Indeterminacy requires *at the same time* a sufficiently large degree of IRS in labor, a sufficiently large wage elasticity of the labor supply curve, a sufficiently large degree of income effect on labor supply, and a sufficiently large EIS in consumption: $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$, $\epsilon_{lw} > \underline{\epsilon}_{lw}$, , $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ and $\epsilon_{cc} > \epsilon_{cc}^T\left(\epsilon_{lw}, \epsilon_{l\lambda}\right)$. However, it is easy to verify that these four conditions can only be simultaneously satisfied for extremely high (thus highly unrealistic) values for several critical elasticities. To see this, consider that the wage elasticity of the labor supply curve is close to the lower bound $\underline{\epsilon}_{lw}$ required for indeterminacy. From Appendix 6.6, we know that ϵ_{cc}^T tends to $+\infty$ when ϵ_{lw} tends to $\underline{\epsilon}_{lw}$, making the condition $\epsilon_{cc} > \epsilon_{cc}^T\left(\epsilon_{lw}, \epsilon_{l\lambda}\right)$ impossible to satisfy for plausible EIS values. Conversely, consider now that the aggregate labor supply curve is very elastic (due, for example, to labor indivisibility at the individual level combined with perfect unemployment insurance, as in Hansen [37] and Rogerson [88]). We know that $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ is an increasing function of ϵ_{lw} so the condition $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ required for indeterminacy now imposes very large degrees of income effect on the labor supply. Moreover, $\epsilon_{cc}^T\left(\epsilon_{lw}, \epsilon_{l\lambda}\right)$ is increasing in $\epsilon_{l\lambda}$, so the large degree of income effect has a retroactively large effect on the value for the EIS in consumption required for indeterminacy ($\epsilon_{cc} > \epsilon_{cc}^T\left(\epsilon_{lw}, \epsilon_{l\lambda}\right)$). No empirically realistic calibration ensures this outcome.

To fix ideas, consider a simple calibration with $\sigma = 1$, $\Theta_k = 0.25$, Θ_l set to its upper bound $\overline{\Theta}_l$ (the most favorable case for indeterminacy), and $\epsilon_{lw} = 3$ (the value advocated by Rogerson and Wallenius [70] and Prescott and Wallenius [68] to calibrate the wage elasticity of the aggregate labor supply curve in standard RBC/DSGE models). Indeterminacy requires in this case $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda} \simeq 55$ and $\epsilon_{cc} > \epsilon_{cc}^T \simeq 1169$. If, at the other extreme, the wage elasticity of the labor supply curve is increased to 1000 for the same other parameter values (approximating Hansen's [37] type of preferences with an infinitely elastic labor supply curve), indeterminacy now requires $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda} \simeq 1259$ and $\epsilon_{cc} > \epsilon_{cc}^T \simeq 254$. Clearly, no configuration is empirically realistic.[8]

Result 2: Impossibility of indeterminacy under additional assumptions frequently made in the literature

Consider some various additional restrictions on structural parameters frequently imposed in the literature. We derive from Theorem 1 the following Proposition:

Proposition 5. *Under Assumptions 1-4, for any $\sigma > 0$, local indeterminacy is ruled out in the following cases: i) $\Theta_k = \Theta_l$, ii) $\epsilon_{lw} = 0$, iii) $\epsilon_{l\lambda} = 0$, iv) $\Theta_l = 0$.*

Proof. See Appendix 6.8.

Case i) of Proposition 5 corresponds to the case of output externalities considered in, e.g., Benhabib and Farmer [6], and was initially proved by Hintermaier [31] in the case of a Cobb-Douglas technology. Here, we extend this result to any production function. Cases ii), iii), and iv) correspond, respectively, to the case of an inelastic labor supply, to the case of GHH preferences with no-income effect on labor supply, and to the case in which externalities occur solely through the aggregate capital stock. In all these cases, indeterminacy is ruled out under otherwise standard assumptions regarding the utility and production functions defined in Assumptions 1-4.

Result 3: Saddle-path stability for realistic calibrations

Finally, we can introduce some restrictions on the nine structural parameters (s, σ, Θ_k, Θ_l, β, ϵ_{cc}, ϵ_{lw}, $\epsilon_{l\lambda}$ and δ) influencing the local stability properties of the steady state based on empirical considerations. We take advantage of the fact that a narrow range of empirical estimates exist for several of these parameters. In particular, it is widely accepted in the literature that, at a quarterly frequency, the subjective discount factor is close to $\beta = 0.99$, consistent with a long-run annual return on capital of around 4%. Likewise, empirical estimates for the annual depreciation rate of capital are typically around 10%, implying $\delta = 0.025$. In the US, the share of capital income in total income is typically estimated around 30%, implying a capital elasticity in the production function close to $s = 0.3$. Estimates for other critical elasticities are often more variables across empirical studies, but a range including most available empirical estimates can nonetheless be defined. For example, based on the recent empirical literature (see e.g. León-Ledesma *et al.* [39], Klump *et al.* [38], Duffy and Papageorgiou [16] and Karagiannis *et al.* [34]), a plausible range for the capital-labor elasticity of substitution is $\sigma \in (0, 2)$. Likewise, using the various empirical estimates provided by Campbell [13], Vissing-Jorgensen [49],

[8] Similar unrealistic values appear for the whole range of potential calibrations regarding σ, Θ_k and Θ_l.

and Gruber [25], a plausible range for the EIS in consumption is $\epsilon_{cc} \in (0,2)$. Finally, estimates of increasing returns to scale by Basu and Fernald [4] for US manufacturing industry provide a value of around 10% with standard deviation 0.33, which enables us to define a range of empirically credible values for the aggregate degree of IRS in the model, $\Theta = (1-s)\Theta_l + s\Theta_k$, of $\Theta \in (0, 0.43)$. Regarding the Frisch wage-elasticity of the labor supply curve, it is well known from the literature that for various reasons this elasticity can be large at the aggregate level even though it is small at the individual level (see e.g. Rogerson and Wallenius [46], and Prescott and Wallenius [45] for a discussion). Our choice here is to not restrict this elasticity *a priori* in order to include Hansen's [28] specification of individual preferences – associated with an infinitely elastic aggregate labor supply curve – into the analysis, since these preferences are widely used in the DSGE literature. This leads us to introduce the following Assumption:

Assumption 5. ***Realistic structural parameters**: $\beta = 0.99$, $\delta = 0.025$, $s = 0.3$, $\sigma \in (0,2)$, $\epsilon_{cc} \in (0,2)$ and $\Theta \in (0, 0.43)$ with $\Theta = (1-s)\Theta_l + s\Theta_k$.*

We then obtain the following Proposition which follows directly from Theorem 1:

Proposition 6. *Under Assumptions 1-5, the steady state is a saddle-point.*

Proof. See Appendix 6.9.

Proposition 6 generalizes a well-known property of the RBC model with constant returns to scale to the case of increasing returns to scale associated with positive externalities: for any empirically plausible calibration of structural parameters, the steady-state is a saddle path, regardless of the specification of individual preferences and the specification of the production function.

4 A general two-sector model

As emphasized by Jaimovich and Rebelo [33], aggregate and sectoral comovement are central features of business-cycles. We now assess whether indeterminacy and sunspot fluctuations are a more likely outcome of multisector infinite horizon models. There are of course many possible ways of constructing multisector economies. To facilitate comparison with the existing literature, we choose to focus our analysis on a two-sector model similar to the one analyzed by Benhabib and Farmer [7], except that we do not restrict the specifications of the utility and the production functions.[9]

Thus, we consider a two-sector economy in which firms produce differentiated consumption and investment goods using capital and labor. As in Benhabib and Farmer [7], we assume that capital and labor are perfectly mobile across sectors, and that both sectors produce their goods with the same technology at the private level. However, we assume, as in Dufourt *et al.* [17], that only the firms in the investment good sector are

[9] In Benhabib and Farmer [7], consumers have Hansen's type of individual preferences and the production functions are Cobb-Douglas.

affected by productive externalities. This choice is based on the fact that empirical estimates for the degree of IRS in the consumption sector are close to zero, while they are positive and significant in the investment sector (see e.g. Harrison [30]).

Given these assumptions, firms in the consumption sector produce output Y_{ct} according to the production function:

$$Y_{ct} = f(K_{ct}, L_{ct}) \tag{24}$$

where K_{ct} and L_{ct} are capital and labor allocated to the consumption sector.

In the investment sector, output Y_{It} is also produced according to the same production function, but is affected by a productive externality

$$Y_{It} = f(K_{It}, L_{It})e(\bar{K}_{It}, \bar{L}_{It}) \tag{25}$$

where K_{It} and L_{It} are the numbers of capital and labor units used in the production of the investment good, and $e(\bar{K}_{It}, \bar{L}_{It})$ is the externality variable. The functions $f(.,.)$ and $e(.,.)$ of course satisfy Assumption 1. Following Benhabib and Farmer [7], we also restrict the specifications of externalities to consider *output externalities,* satisfying $\Theta_k = \Theta_l = \Theta \geq 0$.[10] Recall from Proposition 5 that under such a restriction, local indeterminacy is completely ruled out in the aggregate model.

Assuming that factor markets are perfectly competitive and that capital and labor inputs are perfectly mobile across the two-sectors, the first-order conditions for profit maximization of the representative firm in each sector are:

$$r_t = f_1(K_{ct}, L_{ct}) = p_t f_1(K_{It}, L_{It})e(\bar{K}_{It}, \bar{L}_{It}), \tag{26}$$

$$w_t = f_2(K_{ct}, L_{ct}) = p_t f_2(K_{It}, L_{It})e(\bar{K}_{It}, \bar{L}_{It}) \tag{27}$$

where r_t, p_t, and w_t are respectively the rental rate of capital, the price of the investment good, and the real wage rate at time t, all in terms of the price of the consumption good, which is chosen here as the numeraire.

As in the previous section, we restrict the degree of IRS to ensure that the capital and labor demand functions are *negatively sloped.* Under output externalities, this is ensured by the following Assumption, replacing Assumption 2 above:

Assumption 6. $\Theta < s/(1-s)\sigma$

Denoting by i_t the investment, the budget constraint faced by the representative household is

$$c_t + p_t i_t = r_t k_t + w_t l_t + d_t, \tag{28}$$

where again dividends d_t are zero at equilibrium. The law of motion of the capital stock is:

$$k_{t+1} = (1-\delta)k_t + i_t \tag{29}$$

The household then maximizes its present discounted lifetime utility

$$\max_{\{k_{t+1}, c_t, l_t, i_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t, \ell - l_t) \tag{30}$$

subject to (28), (29), and k_0 given. The first-order conditions and the transversality condition are the same as (9)-(12), with the return factor now defined as:

$$R_{t+1} = \frac{(1-\delta)p_{t+1} + r_{t+1}}{p_t} \tag{31}$$

[10]This assumption allows us to avoid having to consider a much larger number of cases.

4.1 Intertemporal equilibrium and steady state

We consider symmetric rational expectation equilibria which consist of prices $\{r_t, p_t, w_t\}_{t\geq 0}$ and quantities $\{c_t, l_t, i_t, k_t, Y_{ct}, Y_{It}, K_{ct}, K_{It}, L_{ct}, L_{It}\}_{t\geq 0}$, with the externality variables satisfying $(\overline{K}_{It}, \overline{L}_{It}) = (K_{It}, L_{It})$ for any t, thereby satisfying the household's and the firms' first-order conditions as given by (9)-(11) and (26)-(27), the technological and budget constraints (24)-(25) and (28)-(29), the market equilibrium conditions for the consumption and investment goods

$$c_t = Y_{ct}, \quad i_t = Y_{It}, \tag{32}$$

with GDP defined as $y_t = c_t + p_t i_t$, the market equilibrium conditions for capital and labor

$$K_{ct} + K_{It} = k_t, \quad L_{ct} + L_{It} = l_t, \tag{33}$$

and the transversality condition (12).

Combining (24)-(25) and firms' first-order conditions (26)-(27), we derive $p_t e(K_{It}, L_{It}) = 1$ and that the equilibrium capital-labor ratios in the consumption and the investment sectors are identical and equal to $k_t/l_t = K_{ct}/L_{ct} = K_{It}/L_{It} = sw_t/((1-s)r_t)$. Combining this with Assumption 1, aggregate output y_t can be rewritten as $y_t = f(k_t, l_t)$, and the first-order conditions with respect to capital and labor in the consumption and investment sectors can be expressed as $r_t = f_1(k_t, l_t)$ and $w_t = f_2(k_t, l_t)$. It follows that a symmetric general equilibrium satisfies in any t,

$$\begin{aligned}
\lambda_t &= \beta\lambda_{t+1}\left[\frac{(1-\delta)p_{t+1} + r_{t+1}}{p_t}\right] && (34)\\
k_{t+1} &= (1-\delta)k_t + \frac{y_t - c_t}{p_t} && (35)\\
r_t &= f_1(k_t, l_t) && (36)\\
w_t &= f_2(k_t, l_t) && (37)\\
p_t &= \frac{1}{e(K_{It}, L_{It})} && (38)\\
c_t &= c(w_t, \lambda_t) && (39)\\
l_t &= l(w_t, \lambda_t) && (40)\\
y_t &= f(k_t, l_t) && (41)\\
K_{ct} &= \frac{sc_t}{r_t} && (42)\\
L_{ct} &= \frac{(1-s)r_t K_{ct}}{sw_t} && (43)\\
k_t &= K_{It} + K_{ct} && (44)\\
l_t &= L_{It} + L_{ct} && (45)
\end{aligned}$$

together with the initial condition k_0 given and the transversality condition (12).

It is easy to show that the same conclusion as in Proposition 3 applies here: under Assumptions 1, 3, 4, and 6, there exists a unique steady state. Moreover, this steady state can be maintained constant across calibrations by adjusting the value of ℓ accordingly.[11]

[11] A proof of this statement can be provided upon request.

4.2 Local stability analysis

As in the former Section, we log-linearize the system of equations (34)-(45) around the steady state. Once again, this system contains only two dynamic equations, so that it can be reduced to a system of *minimal dimension,* i.e. a system involving two dynamic equations in two variables $\widehat{k}_t$ and $\widehat{\lambda}_t$. This reduced system can be expressed as:

$$\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix} \equiv J \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix}$$

where the Jacobian matrix J is given in the following Proposition:

Proposition 7. *The elements of the Jacobian matrix J are:*

$$J_{11} = \frac{A_{22}B_{11}+B_{21}}{A_{21}}, \quad J_{12} = \frac{A_{22}B_{12}-B_{22}}{A_{21}}, \quad J_{21} = B_{11}, \quad J_{22} = -B_{22}$$

with

$$A_{21} = 1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} - \frac{(1-\delta)\Theta}{s\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right], \; A_{22} = \frac{\frac{\theta(1-s)}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \frac{\theta(1-\delta)\Theta}{\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}$$

$$B_{11} = \frac{1+\Theta}{s\beta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right], \; B_{12} = \frac{1}{\beta}\left[1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \theta\Theta\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}\right]$$

$$B_{21} = 1 - \frac{\Theta}{s\beta\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right], \; B_{22} = \frac{\theta\Theta}{\beta\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}$$

Proof. See Appendix 6.10.

We can thus carry out the same kind of analysis as in Section 2 and provide a detailed local stability analysis of the steady state, considering a family of economies parameterized by the three elasticities (ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$) that govern the EIS in consumption, the wage elasticity, and the income effect, and by the technological parameters σ and Θ governing the elasticity of capital-labor substitution and the degree of increasing returns to scale (IRS) in the investment sector.

Similar to the previous Section, we prove in Appendix 6.11 that there exist three bifurcation loci in the parameter space such that, when ϵ_{cc} is increased from 0 to $+\infty$, a change in the stability properties of the steady state occurs when ϵ_{cc} crosses any of the three loci. We establish the following Lemma:

Lemma 4. *Under Assumptions 1, 3, 4 and 6, let $\Omega = (\beta, \delta, s, \sigma, \Theta)$ be the set of structural parameters. For any $\omega \in \Omega$, there exist three* bifurcation curves *crossing the 3-dimensional plane $(\epsilon_{lw}, \epsilon_{l\lambda}, \epsilon_{cc})$ and generating a change in the local stability properties of the steady state:*

- *a* flip bifurcation curve $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ *combined with one real eigenvalue of J crossing -1,*

- *a* Hopf bifurcation curve $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$ *combined with two complex conjugate eigenvalues of J crossing the unit circle,*

- *a (degenerate)* transcritical bifurcation curve $\epsilon_{cc}^{T}(\epsilon_{lw}, \epsilon_{l\lambda})$ *combined with one real eigenvalue of J crossing 1.*

There also exist four critical bounds $\bar{\epsilon}_{l\lambda}(\epsilon_{lw})$, $\tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$, $\hat{\epsilon}_{l\lambda}(\epsilon_{lw})$ *and* $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ *such that:*

- $\epsilon_{cc}^{H}(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$ *when* $\epsilon_{l\lambda} = \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$,

- $\epsilon_{cc}^{F}(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$ *when* $\epsilon_{l\lambda} = \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$,

- $\epsilon_{cc}^{H}(\epsilon_{lw}, \epsilon_{l\lambda}) = \epsilon_{cc}^{T}(\epsilon_{lw}, \epsilon_{l\lambda})$ *when* $\epsilon_{l\lambda} = \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$,

- $\epsilon_{cc}^{F}(\epsilon_{lw}, \epsilon_{l\lambda}) = \epsilon_{cc}^{T}(\epsilon_{lw}, \epsilon_{l\lambda})$ *when* $\epsilon_{l\lambda} = \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

The formal expressions of these bifurcation curves and critical bounds are given in Appendix 6.11.

Proof. See Appendix 6.11.

The critical bounds help us to define areas in the 3-dimensional plane where the bifurcation curves exist (or not) when ϵ_{cc} is gradually increased from 0 to $+\infty$. For example, the flip bifurcation exists whenever $\epsilon_{l\lambda} \in (0, \tilde{\epsilon}_{l\lambda})$.[12] The transcritical bifurcation has a vertical asymptote at $\underline{\epsilon}_{lw} = (\sigma_{sup} - \sigma)/s$, so that the transcritical bifurcation exists whenever $\epsilon_{lw} > \underline{\epsilon}_{lw}$. Finally, if $\epsilon_{lw} \leq \underline{\epsilon}_{lw}$, the Hopf bifurcation exists whenever $\epsilon_{l\lambda} \in (0, \bar{\epsilon}_{l\lambda})$, whereas if $\epsilon_{lw} > \underline{\epsilon}_{lw}$, the Hopf bifurcation exists whenever $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$.

It is also easy to show that whenever both curves exist, the flip and Hopf bifurcations satisfy $0 < \epsilon_{cc}^{F} < \epsilon_{cc}^{H} < \infty$. Likewise, whenever both curves exist, the flip and transcritical bifurcations satisfy $0 < \epsilon_{cc}^{F} < \epsilon_{cc}^{T} < \infty$ if $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}$, and $0 < \epsilon_{cc}^{T} < \epsilon_{cc}^{F} < \infty$ if $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}$.

Following a similar discussion as in Section 3.4, we introduce the following Assumption on the empirically relevant values of the structural parameters, excluding here the elasticity of intertemporal substitution in consumption which is used as a bifurcation parameter:

Assumption 7. *$\beta = 0.99$, $\delta = 0.025$, $s = 0.3$, $\sigma \in (0, 2)$ and $\Theta \in (0, 0.43)$.*

We can now establish the following Theorem, providing a complete picture of the stability properties of the 2-sector model.

Theorem 2. *Let Assumptions 1, 3, 4, 6 and 7 hold. Consider the bifurcation curves and critical curves defined by Lemma 4, and define by $\underline{\epsilon}_{lw} = (\sigma_{sup} - \sigma)/s$, with $\sigma_{sup} = (1-s)(1+\Theta)/\Theta$, and by $\bar{\epsilon}_{lw}$ the unique solution of $\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}) = \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$. Then, if $\sigma > \sigma_{inf}$ with*

$$\sigma_{inf} \equiv \frac{\delta\theta(1-s)(1+\Theta)}{\Theta[4\beta(1-\delta)+\delta\theta]} \tag{46}$$

we have:

Case 1 - Low wage elasticity of labor supply: $\epsilon_{lw} \in [0, \underline{\epsilon}_{lw})$.

i) when $\epsilon_{l\lambda} \in [0, \tilde{\epsilon}_{l\lambda})$, the steady state is

- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^{F})$,

[12]To simplify notations, from now on we no longer explicitly mention the dependence of the critical bounds on ϵ_{lw} and the dependence of the bifurcation curves on ϵ_{lw} and $\epsilon_{l\lambda}$.

- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$,
- a source for any $\epsilon_{cc} > \epsilon_{cc}^H$.

ii) when $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$*, the steady state is*
- a sink for any $\epsilon_{cc} \in [0, \epsilon_{cc}^H)$,
- a source for any $\epsilon_{cc} > \epsilon_{cc}^H$.

iii) when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}$*, the steady state is a source for any* $\epsilon_{cc} \geq 0$.

Case 2 - Intermediate wage elasticity of labor supply: $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$.

i) when $\epsilon_{l\lambda} \in [0, \underline{\epsilon}_{l\lambda})$*, the steady state is*
- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$,
- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

ii) when $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}, \hat{\epsilon}_{l\lambda})$*, the steady state is*
- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$,
- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

iii) when $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}, \tilde{\epsilon}_{l\lambda})$*, the steady state is*
- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$,
- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$,
- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^H, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

iv) when $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$*, the steady state is*
- a sink for any $\epsilon_{cc} \in [0, \epsilon_{cc}^H)$,
- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^H, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

v) when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}$*, the steady state is*
- a source for any $\epsilon_{cc} \in [0, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

Case 3 - High wage elasticity of labor supply: $\epsilon_{lw} > \bar{\epsilon}_{lw}$.

i) when $\epsilon_{l\lambda} \in [0, \underline{\epsilon}_{l\lambda})$*, the steady state is*
- a saddle for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$,
- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

ii) when $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}, \tilde{\epsilon}_{l\lambda})$*, the steady state is*
- a saddle for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$,
- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

iii) when $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}, \hat{\epsilon}_{l\lambda})$*, the steady state is*
- a sink for any $\epsilon_{cc} \in [0, \epsilon_{cc}^T)$,
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

iv) when $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$*, the steady state is*
- a sink for any $\epsilon_{cc} \in [0, \epsilon_{cc}^H)$*,*
- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^H, \epsilon_{cc}^T)$*,*
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$*.*

v) when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}$*, the steady state is*
- a source for any $\epsilon_{cc} \in [0, \epsilon_{cc}^T)$*,*
- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$*.*

Proof. See Appendix 6.12.

It is worth noticing that the lower bound σ_{inf} on the capital-labor elasticity of substitution as given by (46) implies that local indeterminacy requires a strictly positive amount of externalities. However, the lower bounds is extremely close to zero even for very low values of Θ, i.e. for instance $\sigma_{inf} = 0.016$ when $\Theta = 0.01$.

4.3 Discussion

Theorem 2 enables us to derive the fourth and most central result of our paper:

Result 4: plausibility of indeterminacy in the two-sector model

Indeterminacy and belief-driven fluctuations are robust properties of the standard two-sector model of the business cycle, emerging for a large range of empirically-consistent values for all the critical elasticities considered: $\epsilon_{l\lambda}$, ϵ_{lw}, ϵ_{cc}, Θ and σ.

Before providing numerical examples supporting this claim, we first provide a conceptual explanation as for why indeterminacy easily emerges in the two-sector model.

Interpretation for indeterminacy

What makes indeterminacy much easier to obtain in the two-sector model compared to it's one-sector counterpart? The key answer lies on the Euler equation, which is now expressed as $\lambda_t = \beta\lambda_{t+1}R_{t+1}$, with $R_{t+1} = ((1-\delta)p_{t+1} + r_{t+1})/p_t$ the real return on capital accumulation. The latter no longer exclusively depends on the capital rental rate at $t+1$, r_{t+1}, as in the one-sector model, but also on the evolution of the relative price of the investment good between t and $t+1$. For example, an increase in p_{t+1}/p_t generates a capital gain which must be taken into account in the decision of how much to consume and invest. Consider now as before that agents expect an increase in this real return on capital accumulation, with R_{t+1} replacing r_{t+1} in the 2-sector model. As in the one-sector model, this leads to an increase in the marginal utility of wealth λ_t which, since $\epsilon_{c\lambda} < 0$, creates an incentive to substitute investment for consumption, leading to a higher investment i_t and a higher capital stock k_{t+1} at $t+1$. When $\epsilon_{lw} \in [0, \underline{\epsilon}_{lw})$ and the labor supply is almost fixed with respect to the wage rate, the larger capital stock generates a decrease in the capital rental rate r_{t+1}. However, at t, the increase in the demand for investment also creates an incentive to reallocate capital and labor from the consumption to the investment sector: Both K_{It} and L_{It} increase, while K_{ct}

and L_{ct} decrease. With positive externalities $e(K_{It}, L_{It}) > 0$ in the investment sector, this reallocation of inputs increases the supply of the investment good which more than compensates the increase in demand. As a result, the equilibrium price $p_t = 1/e(K_{It}, L_{It})$ *decreases*, which generates an expected capital gain from investing in the investment good (with $p_{t+1} > p_t$). When this expected capital gain is large enough, the expected rate of return on capital accumulation $R_{t+1} = ((1-\delta)p_{t+1}+r_{t+1})/p_t$ can increase even though the expected rental rate of capital r_{t+1} decreases. This makes the initial expectation that R_{t+1} will increase self-fulfilling. This situation is depicted in case 1(i) of Theorem 2, covering the almost fixed labor supply case with $\epsilon_{lw} \in [0, \underline{\epsilon}_{lw})$, where indeterminacy emerges under positive externalities $\Theta > 0$ in the investment sector for any positive $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$ which guarantees a large enough substitution between investment and consumption. In this case indeed, $\epsilon_{l\lambda} \in [0, \tilde{\epsilon}_{l\lambda})$ is also small, the labor supply is also almost fixed with respect to the marginal utility of wealth, so that $\epsilon_{c\lambda}$ can be sufficiently negative if ϵ_{cc} is large enough. In case 1(ii) however, since $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$ is larger and the labor supply is more affected by the marginal utility of wealth, $\epsilon_{c\lambda}$ can be sufficiently negative even for arbitrarily small EIS $\epsilon_{cc} \in [0, \epsilon_{cc}^H)$.

This complex mechanism associated with factor reallocation and variations in the relative price of the investment good in the presence of positive externalities is what makes the two-sector model much more prone to endogenous fluctuations triggered by exogenous changes in beliefs. Obviously, if indeterminacy is possible in the version with fixed labor supply, the scope for indeterminacy is even larger when the labor supply curve is allowed to vary. Indeed, in this case, the increase in equilibrium labor reduces the extent to which the capital rental rate r_{t+1} decreases when capital accumulates, which in turn reduces the extent to which the relative price of the investment good p_t has to decrease to generate the requested increase in the expected return factor R_{t+1}. Clearly, it is no longer *required* to have a sufficiently large variations in equilibrium labor for indeterminacy to emerge (i.e.,. ϵ_{lw}, $\epsilon_{l\lambda}$ and ϵ_{cc} no longer have to be very large, as in the one sector model).

Graphical illustrations

To illustrate our claim in Result 4, we represent in the 3-dimensional plane of Figure 1 the local stability properties of the two-sector model for economies displaying different degrees of Frisch labor supply elasticities and different EIS in consumption ($\epsilon_{l\lambda}$, ϵ_{lw} and ϵ_{cc}), while the other parameters are fixed and calibrated according to the benchmark calibration $\beta = 0.99$, $\delta = 0.025$, $s = 0.3$, $\sigma = 1$ and $\Theta = 0.3$.[13] The value $\sigma = 1$ corresponds to a Cobb-Douglas production function while the value $\Theta = 0.3$ for the

[13] As implied by Proposition 1, there always exist admissible utility functions $u(c, \ell - l)$ such that $\epsilon_{l\lambda}$, ϵ_{lw} and ϵ_{cc} reach their targeted value at the steady-state. It should then be emphasized that Figure 1 (and related figures below) do *not* represent what happens when one structural parameter of a specific utility function changes, but rather represents how economies differing in terms of their critical elasticities $\epsilon_{l\lambda}$, ϵ_{lw} and ϵ_{cc} have similar or different local stability properties, independently of the implicit utility function considered. This is, in our view, the relevant approach to take since deep preference parameters of specific utility functions are never observed while estimates for the critical elasticities can be obtained from the data.

degree of IRS in the investment sector is close to the point estimate obtained by Harrison [30] for the US economy (4-digit data).

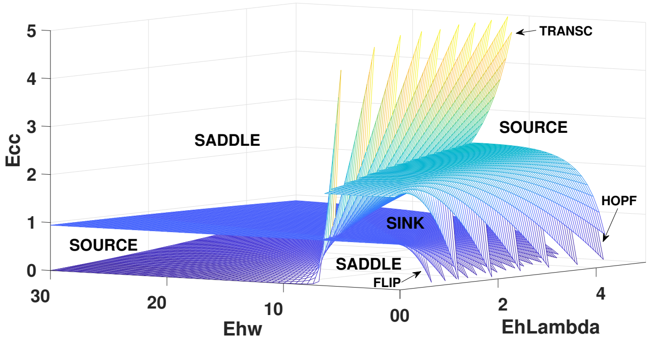


Figure 1: *Local stability properties of the two sector model. Benchmark calibration with* $\sigma = 1$ *and* $\Theta = 0.3$.

We see that, unlike with the one-sector model, there now exists a wide range of values for $(\epsilon_{lw}, \epsilon_{l\lambda}, \epsilon_{cc})$ such that the steady state is locally indeterminate and belief-driven fluctuations emerge. Indeterminacy occurs for a large set of values of the EIS in consumption in the range $\epsilon_{cc} \in (0, 2)$ and for various configurations for the wage and wealth elasticities of the labor supply curve $(\epsilon_{lw}, \epsilon_{l\lambda})$. This includes very large values for the latter two elasticities (as in Hansen's type of preferences with infinitely elastic labor supply, $\epsilon_{lw} = \epsilon_{l\lambda} = +\infty$) or very small values for the wage-elasticity ϵ_{lw} consistent with micro-level estimates (also covering the fixed labor supply case).[14] Clearly, the range of values associated with indeterminacy is entirely consistent with the range of credible empirical estimates defined in Assumption 5.

For better clarity, Figure 2 displays the stability property areas when the range of values considered for ϵ_{lw} and $\epsilon_{l\lambda}$ is restricted to (0,5), close to the value of $\epsilon_{lw} = 3$ advocated by Rogerson and Wallenius [46] and Prescott and Wallenius [45] to calibrate dynamic macroeconomic models. As can be seen, indeterminacy arises in this configuration for a very wide range of values for ϵ_{cc} in the realistic interval (0,2), including values that are arbitrarily close to 0.

[14]As is well kown, a lengthy discussion exists in the literature about how to calibrate the Frisch wage-elasticity of the aggregate labor supply curve. Both theoretical considerations and empirical evidence point toward small values at the individual level but greater values at the aggregate level (see for example Rogerson and Wallenius [46] for a discussion). Meanwhile, it is well known that standard RBC-DSGE models do not perform well when the wage elasticity of the aggregate labor supply is too low, which explains the popularity of the class of preferences suggested by Hansen [28].

On the other hand, there is very little empirical evidence on the wealth-elasticity $\epsilon_{l\lambda}$. The main difficulty is that this elasticity captures the effect of a *marginal* increase in intertemporal wealth on labor supply, and that exogenous variations enabling this elasticity to be identified are very difficult to find in the data (see the lengthy discussion and relevant references to the literature in Kimball and Shapiro, [35]). Analyzing data from a thought-experiment survey conducted by the Health and Retirement Study (HRS), Kimball and Shapiro [35] tend toward the conclusion that the elasticities ϵ_{lw} and $\epsilon_{l\lambda}$ are rather small. Yet, their estimations assume an equality between both elasticities (as imposed by Hansen's type of preferences), while from a theoretical standpoint there is no reason to assume that they are equal.

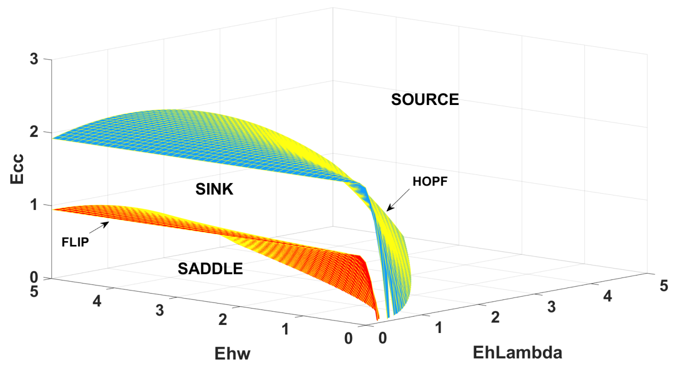


Figure 2: *Local stability properties for low labor suply elasticities (benchmark calibration with* $\sigma = 1$ *and* $\Theta = 0.3$*).*

4.4 Robustness

Since we emphasized that five critical elasticities have a strong influence on the local stability properties of the 2-sector model , it is worthwhile to analyze how the elasticities related to the productive side of the economy influence the local stability properties of the two-sector model displayed in Figures 1 and 2 in the case of a Cobb-Douglas production function ($\sigma = 1$) and $\Theta = 0.3$. In Figure 3, we display how these stability properties vary when we consider four different values for Θ, namely $\Theta \in (0.1, 0.2, 0.3, 0.4)$. We see that increasing the degree of increasing returns to scale has the effect of shifting down the flip of Hopf bifurcation curves, so that indeterminacy is easier to obtain for larger degrees of IRS. Note however that in all cases there exist a range of values $\epsilon_{cc} \in (0, 2)$ such that indeterminacy prevails, including values arbitrarily close to 0

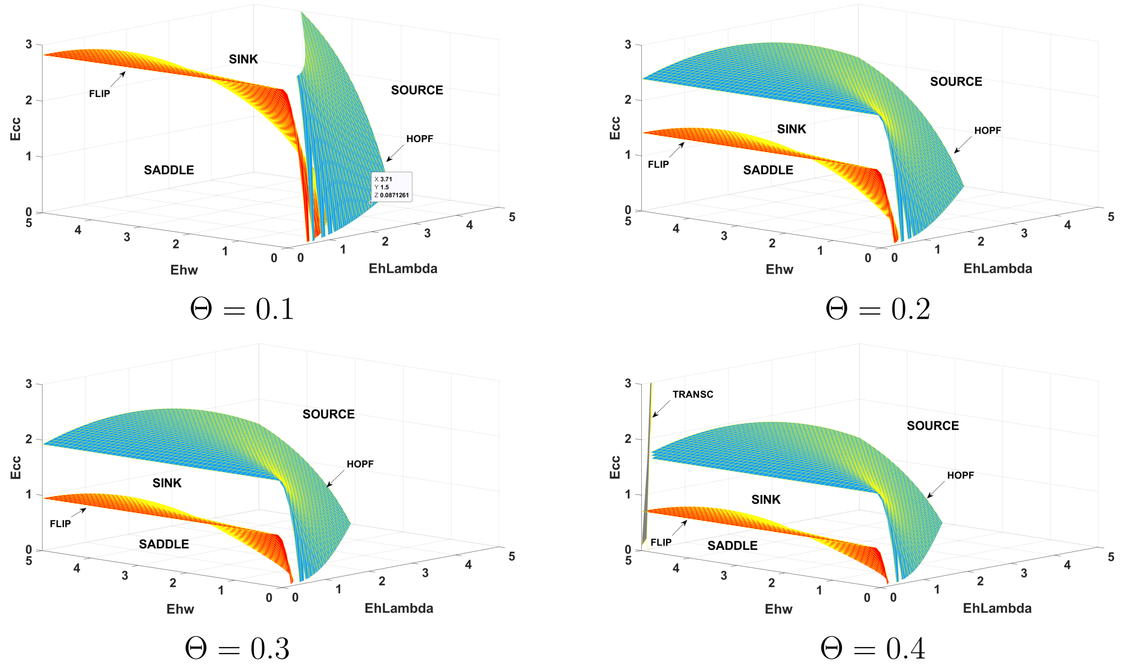
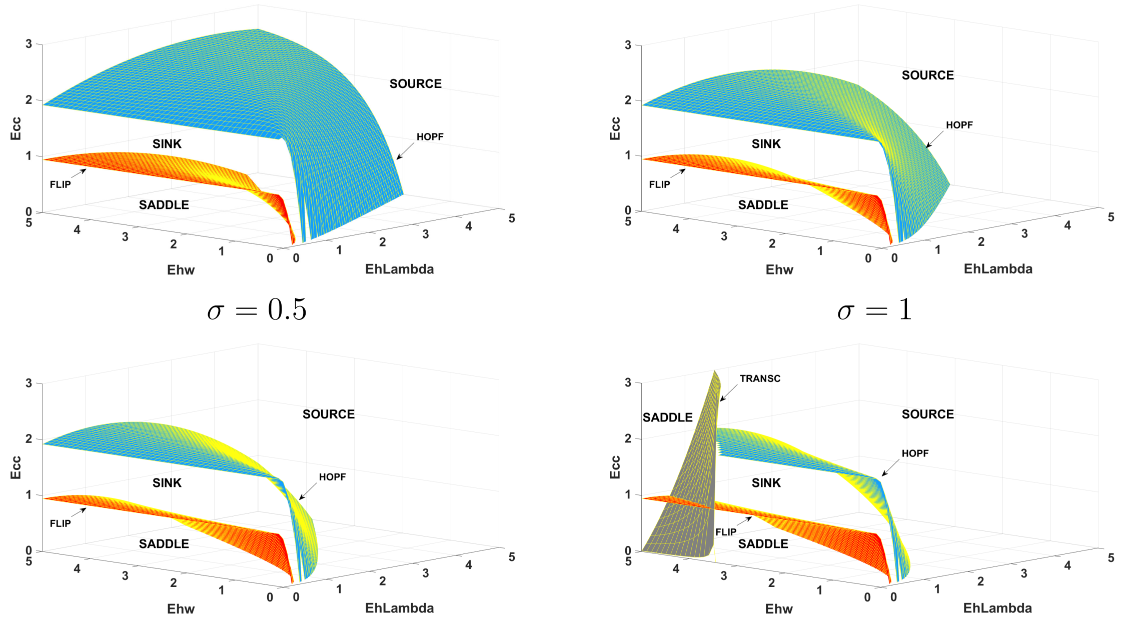


Figure 3: *Effects of varying* Θ *on the indeterminacy area*

In Figure 4, we display the local stability properties for four different values for the elasticity of substitution between capital and labor σ, namely $\sigma \in (0.5, 1, 1.5, 2)$.



$\sigma = 0.5$ $\sigma = 1$

$\sigma = 1.5$ $\sigma = 2$

Figure 4: *Effects of varying* σ *on the indeterminacy area*

We see that the area for which the steady-state is a sink tends to shrink when larger values for σ are considered, especially for larger intensities of the wealth effect $\epsilon_{l\lambda}$. Moreover, the transcritical bifurcation curve shifts to the right when σ increases, implying that indeterminacy is also harder to obtain for large wage elasticities of the labor supply curve ϵ_{lw}. However, in all cases, a significant range of parameter configurations in the set of empirically plausible values (as defined in Assumption 7) are consistent with an indeterminate steady-state, in particular for the EIS in consumption $\epsilon_{cc} \in (0, 2)$. This reinforces our claim that indeterminacy and the emergence of expectation-driven fluctuations are robust properties of the standard two-sector model.

4.5 Specific utility functions

Since Theorem 2 applies to any form of instantaneous utility functions satisfying the usual assumptions, results regarding any particular utility function can be derived as a direct application of this Theorem. We consider here the three most popular utility functions used in the macroeconomic literature, namely the KPR, generalized Hansen and GHH utility functions, with cross restrictions on the three critical elasticities ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$ summarized in Proposition 2. Using these cross restrictions enables us to obtain a slightly modified lower bound $\tilde{\sigma}_{inf}$ compared to the one, σ_{inf}, introduced in the general case (see Theorem 2), which is slightly larger:

Assumption 8. $\sigma > \tilde{\sigma}_{inf} \equiv \frac{(1-\beta+\Theta\theta)\delta(1-s)(1+\Theta)}{\Theta[4(1-\delta)\Theta\beta+\delta(1-\beta+\Theta\theta)]}$

We then obtain for the KPR case:

Corollary 1. *Under Assumptions 1, 3, 4, 6, 7 and 8, consider a KPR utility function such that* $\epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}$. *Then there exist bifurcation curves* $0 < \epsilon_{cc}^F < \epsilon_{cc}^H$ *such that the steady state is:*

- *a saddle-point for any* $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$*,*
- *a sink for any* $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$*,*
- *a source for any* $\epsilon_{cc} > \epsilon_{cc}^H$*,*

The formal expressions for the flip and Hopf bifurcation curves (ϵ_{cc}^F *and* ϵ_{cc}^H) *are given in Appendix 6.13.*

Proof. See Appendix 6.13.

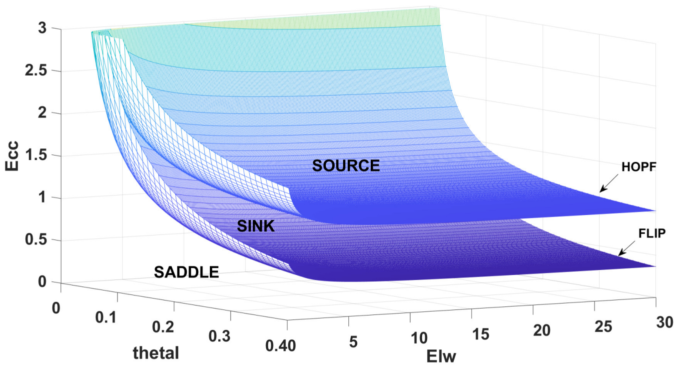


Figure 5: *Local stability properties with KPR utility function* ($\sigma = 1$).

Notice that with a KPR utility function there is no transcritical bifurcation. Moreover, the existence of local indeterminacy is compatible with an arbitrarily large wage-elasticity of the labor supply curve. As an illustration, Figure 5 displays the local stability properties of the steady state in the KPR case as a function of ϵ_{lw}, ϵ_{cc} and Θ when the production function is Cobb-Douglas ($\sigma = 1$). We see that with a KPR utility function, indeterminacy can easily be obtained for moderate values for the EIS in consumption $\epsilon_{cc} \in (0, 2)$ and an arbitrarily large elasticity of the labor supply curve, provided that the degree of IRS is large enough.

With GHH preferences, we obtain:

Corollary 2. *Under Assumptions 1, 3, 4, 6, 7 and 8, consider a GHH utility function such that* $\epsilon_{l\lambda} = 0$*. Then there exist a* critical bound $\underline{\epsilon}_{lw} \equiv (\sigma_{sup} - \sigma)/s$*, with* $\sigma_{sup} = (1-s)(1+\Theta)/\Theta$*, and bifurcation curves* $0 < \epsilon_{cc}^F < \epsilon_{cc}^H$ *such that the following cases hold:*

i) when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$*, then the steady state is:*

- *a saddle-point for any* $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$*,*
- *a sink for any* $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$*,*
- *a source for any* $\epsilon_{cc} > \epsilon_{cc}^H$*.*

ii) when $\epsilon_{lw} > \underline{\epsilon}_{lw}$*, the steady state is unstable for any* $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$ *and saddle-point stable for any* $\epsilon_{cc} > \epsilon_{cc}^F$*.*

The formal expressions for the flip and Hopf bifurcation curves (ϵ_{cc}^F *and* ϵ_{cc}^H) *are given in Appendix 6.14.*

Proof. See Appendix 6.14.

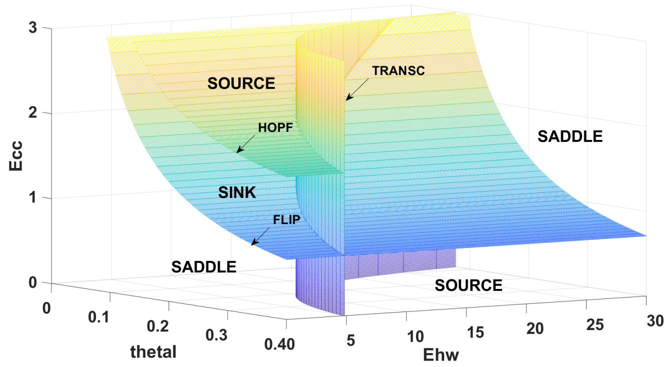


Figure 6: *Local stability properties with GHH utility function* ($\sigma = 1$).

A graphical illustration of this case in the 3-dimensional plane defined by $(\epsilon_{lw}, \Theta, \epsilon_{cc})$ is displayed in Figure 6, again in the case $\sigma = 1$. Notice that with a GHH utility function, the critical bound $\underline{\epsilon}_{lw}$ actually corresponds to a transcritical bifurcation curve when ϵ_{lw} is taken as bifurcation parameter. However, this critical bound is independent of the elasticity of intertemporal substitution in consumption ϵ_{cc}, implying that the locus is vertical with respect to ϵ_{cc}. As can be seen, compared to the KPR case, the existence of local indeterminacy for realistic $\epsilon_{cc} \in (0, 2)$ is somewhat more difficult to obtain, as the range of parameter values consistent with a sink equilibrium is shrunk by this transcritical bifurcation curve, implying that indeterminacy is ruled out for large elasticities of the labor supply curve.

With generalized Hansen preferences, we obtain:

Corollary 3. *Under Assumptions 1, 3, 4, 6, 7 and 8, consider a generalized Hansen utility function such that* $\epsilon_{l\lambda} = \epsilon_{lw}$. *Then there exist critical bounds* $\epsilon_{lw}^F < \underline{\epsilon}_{lw} = (\sigma_{sup} - \sigma)/s < \epsilon_{lw}^H$, $\tilde{\sigma}_{inf} < \sigma^F < \sigma^H < \sigma_{sup}$ *and bifurcation curves* $0 \leq \epsilon_{cc}^F \leq \epsilon_{cc}^H \leq \epsilon_{cc}^T \leq +\infty$ *such that the steady state is:*

- *a saddle-point for any* $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$,
- *a sink for any* $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$,
- *a source for any* $\epsilon_{cc} \in (\epsilon_{cc}^H, \epsilon_{cc}^T)$,
- *a saddle-point for any* $\epsilon_{cc} > \epsilon_{cc}^T$.

. *Moreover:*

- $\epsilon_{cc}^F = 0$ *if* $\sigma > \sigma^F$ *and* $\epsilon_{lw} > \epsilon_{lw}^F$,
- $\epsilon_{cc}^H = 0$ *if* $\sigma > \sigma^H$ *and* $\epsilon_{lw} > \epsilon_{lw}^H$,
- $\epsilon_{cc}^T = +\infty$ *if* $\epsilon_{lw} < \underline{\epsilon}_{lw}$.

The formal expressions for the flip, Hopf and transcritical bifurcation curves (ϵ_{cc}^F, ϵ_{cc}^H *and* ϵ_{cc}^T), *as well as those of all critical bounds* ($\epsilon_{lw}^F, \epsilon_{lw}^H, \sigma^F$ *and* σ^H), *are given in Appendix 6.15.*

Proof. See Appendix 6.15.

Figure 7 displays again the corresponding stability properties associated with a Cobb-Douglas technology ($\sigma = 1$). We see that with generalized Hansen preferences, indeterminacy is very easy to obtain since a large range of values $\epsilon_{cc} \in (0, 2)$ exists such that

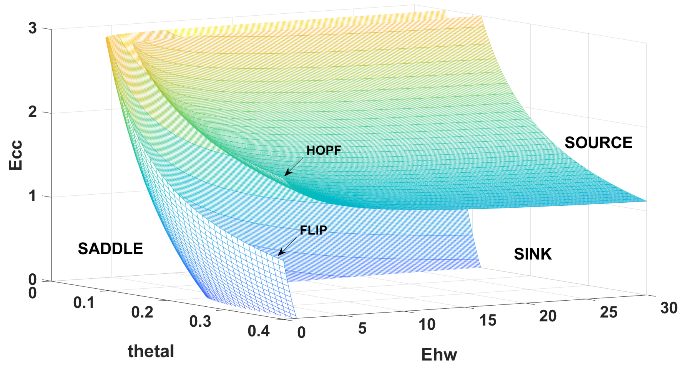


Figure 7: *Local stability properties with generalized Hansen utility function* ($\sigma = 1$).

the steady-state is a sink, including values arbitrarily close to 0. This is true for a large set of values for the wage labor supply elasticity, including values that are either small or arbitrarily large.

5 Concluding comments

This paper established two main results: First, the existence of belief-driven fluctuations in the standard one-sector RBC model with increasing returns to scale is ruled out for any empirically plausible calibration regarding the critical elasticities defining the utility function and the production function. Second, on the contrary, the existence of such fluctuations is a very likely outcome of the standard two-sector version of the model, in the sense that they now arise for a large set of empirically plausible values for these elasticities. While this latter result is true in general, it also holds when some of the most frequently used utility function in the macroeconomic literature are considered (KPR, generalized Hansen and GHH utility functions) even though the model is restricted in this case by cross-restrictions on the critical elasticities. Hence, far from being an exotic feature emerging under extreme assumptions, the possibility of expectation-driven fluctuations is a likely outcome of workhorse models in the macroeconomic literature, provided they are at least bisectoral (arguably a reasonable assumption).

At this stage, a critical issue remains. Are business cycles triggered by exogenous changes in expectations important quantitatively? So far, the literature has been largely negative. In a well-known paper, Schmitt-Grohé [48] showed that a standard two-sector model submitted to beliefs shocks is unable to account for several defining features of observed business cycle, in particular in response to transitory shocks. However, the model and its extensions are evaluated under the maintained assumptions of KPR preferences. Whether the same negative outcome prevails for a larger set of preferences is an open question. In Dufourt *et al.* [18], we address this issue and we show that the belief-driven two-sector model can actually account for all the dimensions of observed business cycles emphasized by Schmitt-Grohé [48], as well as other dimensions of the data underlined in the recent literature (see e.g. Beaudry *et al.,* [5]). We refer the reader to this paper for a thorough discussion.

6 Appendix

6.1 Proof of Lemma 1

From the definition of ϵ_{lw} and $\epsilon_{l\lambda}$ as given by (15)-(16), a total differentiation of the optimality conditions (9)-(10) gives

$$\begin{aligned} u_{11}dc_t - u_{12}dl_t &= d\lambda_t \\ u_{21}dc_t - u_{22}dl_t &= d\lambda_t w_t + \lambda_t dw_t \end{aligned}$$

Solving this system with respect to dl_t yields to the expressions (19) and (20). We also derive

$$\begin{aligned} \epsilon_{cw} &= -\frac{u_{12}u_1}{u_{11}u_{22}-u_{12}u_{21}}\frac{w}{c} = \frac{wl}{c}(\epsilon_{lw} - \epsilon_{l\lambda}) \\ \epsilon_{c\lambda} &= \frac{u_1}{u_{11}c} - \frac{u_{12}}{u_{11}c}\frac{u_{11}u_2 - u_{21}u_1}{u_{11}u_{22}-u_{12}u_{21}} = -\epsilon_{cc} + \frac{wl}{c}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\epsilon_{l\lambda} \end{aligned}$$

We show in Appendix 6.3 below that at the steady state

$$\frac{wl}{c} = \frac{\theta(1-s)}{\theta - s\beta\delta} \equiv \mathcal{C} < 1 \tag{47}$$

The result follows. □

6.2 Proof of Proposition 1

By Assumption 3, u is an increasing function over $\mathbb{R}^2_{++}$, implying $(c, l, u_1, u_2) > 0$. Moreover, the strict quasi-concavity of u implies $u_{11} < 0$ and $u_{11}u_{22} - u_{12}u_{21} > 0$. Using Lemma 1, we straightforwardly obtain $\epsilon_{cc} > 0$ and $\epsilon_{lw} > 0$. By Assumption 4, c and $\mathcal{L}$ are normal goods. The normality of $\mathcal{L}$ requires $u_{21}u_1 - u_{11}u_2 \geq 0$. Combined with the strict quasi-concavity of u, we straightforwardly obtain $\epsilon_{l\lambda} \geq 0$. Using a similar reasoning, we obtain that the normality of c requires $\epsilon_{c\lambda} \leq 0$ and therefore, using Lemma 1, $\epsilon_{cc} \geq \mathcal{C}\epsilon_{l\lambda}(\epsilon_{lw} - \epsilon_{l\lambda})/\epsilon_{lw} \equiv \epsilon_{cc}^N$. □

6.3 Proof of Lemma 2

We know from constant-returns-scale of the technology at the private level that

$$\frac{rk}{y} = s \text{ and } \frac{wl}{y} = 1 - s$$

Considering that at the steady state we have $R^* = 1/\beta$ with $R^* = r^* + 1 - \delta = sy^*/k^* + 1 - \delta$ we get

$$\frac{y^*}{k^*} = \frac{\theta}{s\beta}$$

with $\theta = 1 - \beta(1-\delta)$. It follows from the capital accumulation equation evaluated at the steady state that $c = y - k$ and thus

$$\frac{c^*}{k^*} = \frac{\theta - s\beta\delta}{s\beta}$$

We conclude from this

$$\frac{w^*l^*}{c^*} = \frac{\theta(1-s)}{\theta - s\beta\delta} \equiv \mathcal{C} < 1 \tag{48}$$

□

6.4 Proof of Proposition 3

Considering again that $R^* = r^* + 1 - \delta = 1/\beta$, we get

$$f_1(k^*, l^*)e(k^*, l^*) \equiv g(k^*, l^*) = \frac{\theta}{\beta} \tag{49}$$

It is then easy to compute under Assumption 2

$$\frac{g_1(k^*,l^*)k^*}{g(k^*,l^*)} = s\Theta_k - \frac{1-s}{\sigma} < 0$$

Therefore, applying the implicit function theorem, we conclude that there exists a unique function $k(.)$ such that $k^* = k(l^*)$. Considering that

$$\frac{g_2(k^*,l^*)l^*}{g(k^*,l^*)} = (1-s)\Theta_l + \frac{1-s}{\sigma}$$

we conclude that

$$\frac{k'(l^*)l^*}{k(l^*)} = -\frac{(1-s)\Theta_l + \frac{1-s}{\sigma}}{s\Theta_k - \frac{1-s}{\sigma}} > 0$$

Recalling now that

$$\frac{y^*}{k^*} = \frac{\theta}{s\beta} \text{ and } c^* = \frac{\theta - s\beta\delta}{s\beta}k^* \tag{50}$$

we derive

$$c^* = c(l^*) = \frac{\theta - s(k(l^*),l^*)\beta\delta}{s(k(l^*),l^*)\beta}k(l^*) \equiv h(l^*)k(l^*)$$

Straightforward computations give

$$\frac{h'(l^*)l^*}{h(l^*)} = \frac{\theta(1-s)}{\theta - s\beta\delta}\left(1 - \frac{1}{\sigma}\right)\frac{(1-s)\Theta_l + s\Theta_k}{s\Theta_k - \frac{1-s}{\sigma}}$$

and we easily conclude under Assumption 2

$$\frac{c'(l^*)l^*}{c(l^*)} = -\frac{(1-s)\left\{\Theta_l\left[s(1-\beta) + \frac{\theta(1-s)}{\sigma}\right] + \frac{\theta s\Theta_k}{\sigma} + \frac{\theta - s\beta\delta}{\sigma} - \theta s\Theta_k\right\}}{(\theta - s\beta\delta)\left(s\Theta_k - \frac{1-s}{\sigma}\right)} > 0$$

Moreover we also get from (25)

$$w^* = w(l^*) = f_2(k(l^*), l^*)e(k(l^*), l^*)$$

and thus

$$\frac{w'(l^*)l^*}{w(l^*)} = -\frac{\frac{1-s}{\sigma}\Theta_l + \frac{s}{\sigma}\Theta_k}{s\Theta_k - \frac{1-s}{\sigma}} > 0$$

Consider then the third equation of (26) which becomes

$$\frac{u_2(c(l^*), \ell - l^*)}{u_1(c(l^*), \ell - l^*)} \equiv \psi(l^*) = w(l^*) \tag{51}$$

Under Assumptions 2, 3 and 4, we get

$$\frac{\psi'^*)l^*}{\psi(l^*)} = \frac{c'(l^*)l^*}{c(l^*)}\frac{\epsilon_{l\lambda}}{\epsilon_{cc}\epsilon_{lw}} + \frac{1}{\epsilon_{cc}\epsilon_{lw}}\left[\epsilon_{cc} - \mathcal{C}\epsilon_{l\lambda}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\right] \geq 0 \tag{52}$$

It follows that the existence of a unique steady state value l^* is obtained if $g'(l^*) \neq w'(l^*)$. Straightforward computations show that this condition is satisfied if

$$\frac{c'(l^*)l^*}{c(l^*)}\frac{\epsilon_{l\lambda}}{\epsilon_{cc}\epsilon_{lw}} + \frac{1}{\epsilon_{cc}\epsilon_{lw}}\left[\epsilon_{cc} - \mathcal{C}\epsilon_{l\lambda}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\right] - \frac{w'(l^*)l^*}{w(l^*)} \neq 0 \tag{53}$$

Such a condition is generically satisfied so that the existence and uniqueness of a steady state is generically ensured.

Now let us normalize the steady state considering the value $\bar{l}^*$ corresponding to the average amount of working hours relative to the total amount of time ℓ. Substituting $l^* = \bar{l}^*$ into equation (51), we get

$$\frac{u_2(c(\bar{l}^*),\ell-\bar{l}^*)}{u_1(c(\bar{l}^*),\ell-\bar{l}^*)} \equiv \phi(\ell) = w(\bar{l}^*) \tag{54}$$

Straightforward computations give

$$\frac{\phi'(\ell)\ell}{\phi(\ell)} = -\frac{\ell}{\bar{l}^*}\frac{1}{\epsilon_{cc}\epsilon_{lw}}\left[\epsilon_{cc} - \mathcal{C}\epsilon_{l\lambda}\left(1-\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\right]$$

It follows under Assumptions 3 and 4 that $\phi'(\ell)\ell/\phi(\ell) \leq 0$ and there exists a unique value $\ell^* > \bar{l}^*$ solution of equation (54). We conclude finally that if $\ell = \ell^*$, then the unique steady state (k^*, l^*, c^*) is such that $l^* = \bar{l}^*$.

□

6.5 Proof of Proposition 4

From the optimality conditions (9)-(10) and Lemma 1, we derive

$$\widehat{l}_t = \epsilon_{lw}\widehat{w}_t + \epsilon_{l\lambda}\widehat{\lambda}_t \tag{55}$$

$$\widehat{c}_t = \mathcal{C}\left(1-\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\widehat{l}_t - \epsilon_{cc}\widehat{\lambda}_t \tag{56}$$

and (25) implies

$$\widehat{w}_t = \left(s\Theta_k + \frac{s}{\sigma}\right)\widehat{k}_t + \left[(1-s)\Theta_l - \frac{s}{\sigma}\right]\widehat{l}_t$$

Using this expression in (55) yields

$$\widehat{l}_t = \frac{\epsilon_{lw}s\left(\frac{1}{\sigma}+\Theta_k\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\widehat{k}_t + \frac{\epsilon_{l\lambda}}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\widehat{\lambda}_t \tag{57}$$

Using (57) into (56) gives

$$\widehat{c}_t = \frac{\mathcal{C}s\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}-\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\widehat{k}_t - \left(\epsilon_{cc} - \frac{\epsilon_{l\lambda}\mathcal{C}\left(1-\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right)\widehat{\lambda}_t \tag{58}$$

Using (25), the system of difference equations describing the intertemporal equilibrium can be stated as follows

$$\begin{aligned} f(k_t, l(k_t,\lambda_t))e(k_t, l(k_t,\lambda_t)) + (1-\delta_t)k_t - c(k_t,\lambda_t) - k_{t+1} &= 0 \\ \beta\left[1-\delta+f_1(k_{t+1}, l(k_{t+1},\lambda_{t+1}))e(k_{t+1}, l(k_{t+1},\lambda_{t+1}))\right]\lambda_{t+1} - \lambda_t &= 0 \end{aligned} \tag{59}$$

Linearizing the first equation around the steady state using (50), (57) and (58) gives after simplifications

$$\begin{aligned} \widehat{k}_{t+1} = {} & \widehat{k}_t\frac{1}{\beta}\left\{1+\theta\Theta_k + \frac{\theta(1-s)\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\} \\ & + \widehat{\lambda}_t\frac{1}{s\beta}\left\{\frac{\epsilon_{l\lambda}\theta(1-s)\left(\Theta_l+\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} + (\theta - s\beta\delta)\epsilon_{cc}\right\} \end{aligned} \tag{60}$$

Linearizing the second equation of (59) around the steady state gives

$$\widehat{\lambda}_{t+1} = \widehat{\lambda}_t + \widehat{k}_{t+1}\left[s\Theta_k - \frac{1-s}{\sigma}\right]\theta - \widehat{l}_{t+1}\left[\Theta_l + \frac{1}{\sigma}\right]\theta(1-s) \tag{61}$$

Using (57) finally gives

$$\widehat{\lambda}_{t+1}\left\{1+\frac{\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\}-\widehat{k}_{t+1}\theta\frac{\frac{1-s}{\sigma}-s\Theta_k-\frac{\epsilon_{lw}}{\sigma}[s(\Theta_k-\Theta_l)+\Theta_l]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}=\widehat{\lambda}_t \tag{62}$$

Equations (60) and (62) can be expressed as follows

$$\begin{pmatrix} 1 & 0 \\ -A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix}=\begin{pmatrix} B_{11} & B_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \widehat{k}_{t} \\ \widehat{\lambda}_{t} \end{pmatrix}$$

with

$$\begin{aligned}
A_{21} &= \theta\frac{\frac{1-s}{\sigma}-s\Theta_k-\frac{\epsilon_{lw}}{\sigma}[s(\Theta_k-\Theta_l)+\Theta_l]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} \\
A_{22} &= 1+\frac{\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} \\
B_{11} &= \frac{1}{\beta}\left\{1+\theta\Theta_k+\frac{\theta(1-s)\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\} \\
B_{12} &= \frac{1}{\beta s}\left\{\frac{\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\theta(1-s)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}+(\theta-s\beta\delta)\epsilon_{cc}\right\}
\end{aligned}$$

The Jacobian matrix J follows after straightforward computations and simplifications.□

6.6 Proof of Lemma 3

We easily derive from Proposition 4 the following characteristic polynomial

$$\mathcal{P}(\lambda)=\lambda^2-\lambda\mathcal{T}(\epsilon_{cc})+\mathcal{D} \tag{63}$$

with

$$\mathcal{D}=\frac{1}{\beta}\left\{1+\theta\frac{\Theta_k\left[1+s\frac{\epsilon_{lw}}{\sigma}+(1-s)\epsilon_{l\lambda}\right]+(1-s)\Theta_l\left(\frac{\epsilon_{lw}}{\sigma}-\epsilon_{l\lambda}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}\right\}$$

$$\mathcal{T}(\epsilon_{cc})=1+\mathcal{D}+\frac{\theta(\theta-s\beta\delta)(1-s)}{\beta s}\frac{\epsilon_{cc}\left[\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}-\frac{\epsilon_{lw}}{\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)\right]+\epsilon_{l\lambda}\left[\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}s\Theta_k}{1-s}\right)+\Theta_l\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}}{\sigma}\right)+\mathcal{C}\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)\right]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}$$

The analysis of the local stability properties of the model is based on the geometrical methodology of Grandmont *et al.* [30]. In Figure 8, we draw a graph in the trace-determinant $(\mathcal{T},\mathcal{D})$ space where three relevant lines are considered: line AC ($\mathcal{D}=\mathcal{T}-1$) along which one eigenvalue of $\mathcal{D}$ is equal to 1, line AB ($\mathcal{D}=-\mathcal{T}-1$) along which one eigenvalue of $\mathcal{D}$ is equal to -1 and segment BC ($\mathcal{D}=1$, $|\mathcal{T}|<2$) along which the two eigenvalues of $\mathcal{D}$ are complex conjugates with modulus equal to 1. These three lines divide the space $(\mathcal{T},\mathcal{D})$ into three different types of regions according to the number of eigenvalues with modulus smaller than, equal to, and greater than 1. This determines whether the steady state is a sink (locally indeterminate), a source (locally unstable) or a saddle-point (see the corresponding areas in Figure 8).

Then, for any particular calibration of structural parameters, we can compute the trace and determinant using the expression for the Jacobian matrix obtained in Proposition 4 and assess in which area the model is located. We can also assess how these local stability properties change when the calibration of any particular parameter is varied over its admissible range.

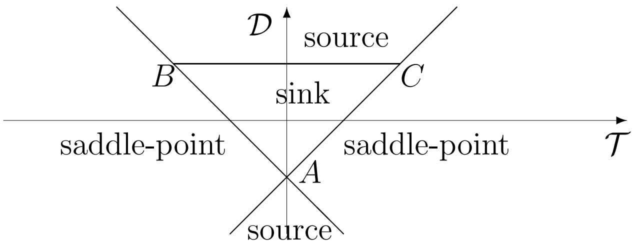


Figure 8: Area of local indeterminacy.

In the one-sector stochastic growth model considered so far, the analysis is greatly simplified by observing that $\mathcal{D}$ does not depend on ϵ_{cc}, implying that the pair $(\mathcal{T}(\epsilon_{cc}), \mathcal{D})$ describes an horizontal line in the $(\mathcal{T}, \mathcal{D})$ space when ϵ_{cc} increases from 0 to $+\infty$. As a result, any Hopf bifurcation related to a Determinant equal to 1 is generically ruled out.

To prove the possible existence of local indeterminacy we need to show that there exist some parameters' configurations such that $\mathcal{D} < 1$ and $1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} > 0$. It is easy to show from the expression of $\mathcal{T}(\epsilon_{cc})$ that a necessary condition to get $1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} > 0$ is

$$\epsilon_{lw} > \frac{1-\frac{\sigma s\Theta_k}{1-s}}{\Theta_l+\frac{s\Theta_k}{1-s}} \equiv \underline{\epsilon}_{lw}$$

Let us now write the determinant as

$$\mathcal{D} = \frac{1}{\beta}\frac{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\left(1-\frac{\theta}{\sigma}\right)\right]+\epsilon_{l\lambda}\frac{\theta(1-s)}{\sigma}+\theta\Theta_k\left[1+s\frac{\epsilon_{lw}}{\sigma}+(1-s)\epsilon_{l\lambda}\right]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}$$

Since under Assumption 2 the expression $\frac{s}{\sigma} - \Theta_l(1-s)\left(1-\frac{\theta}{\sigma}\right)$ is necessarily positive for any $\sigma > 0$, we get $\mathcal{D} > 0$ for any $\sigma > 0$. Moreover, $\mathcal{D} \leq 1$ if and only if $\Theta_l > \underline{\Theta}_l$ and $\epsilon_{l\lambda} \geq \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ with

$$\underline{\Theta}_l \equiv \frac{1-\beta}{\beta} + \frac{\Theta_k}{\beta}, \ \underline{\epsilon}_{l\lambda}(\epsilon_{lw}) \equiv \frac{1-\beta+\theta\Theta_k+\epsilon_{lw}\left[\frac{s(1-\beta+\theta\Theta_k)}{\sigma}-\Theta_l(1-s)\left(1-\beta-\frac{\theta}{\sigma}\right)\right]}{\theta(1-s)(1-\beta+\Theta_k)(\Theta_l-\underline{\Theta}_l)} \tag{64}$$

Under $\sigma \leq \bar{\sigma} \equiv \theta/(1-\beta)$ we get $1 - \beta - \frac{\theta}{\sigma} \leq 0$ so that $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) > 0$ for any $\Theta_l > \underline{\Theta}_l$ and $\epsilon_{lw} \geq 0$. We need therefore to show that $\underline{\Theta}_l < \bar{\Theta}_l$ which is obtained if and only if

$$\Theta_k < \underline{\Theta}_k \equiv \frac{s\beta}{(1-s)\sigma} - (1-\beta)$$

with $\underline{\Theta}_k \in (0, \bar{\Theta}_k)$ under Assumption 5 and $\sigma \leq \bar{\sigma}$.

Obviously, we conclude that $\mathcal{D} > 1$ when $\Theta_l < \underline{\Theta}_l$ for any $\epsilon_{l\lambda} \geq 0$, or when $\Theta_k \in [0, \underline{\Theta}_k)$, $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ and $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

Let us compute the critical values $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ and $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ respectively associated with flip and transcritical bifurcations. The first one is obtained as the solution of $1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} = 0$, namely

$$\epsilon_{cc}^T \equiv \epsilon_{l\lambda}\frac{\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}s\Theta_k}{1-s}\right)+\Theta_l\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}}{\sigma}\right)+\mathcal{C}\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)}{\frac{\epsilon_{lw}}{\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)-\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)} \tag{65}$$

while the second one is obtained as the solution of $1 + \mathcal{T}(\epsilon_{cc}) + \mathcal{D} = 0$, namely

$$\epsilon_{cc}^F \equiv \frac{2\left\{1+\beta+\theta\Theta_k+\epsilon_{lw}\left[(1+\beta)\left(\frac{s}{\sigma}-\Theta_l(1-s)\right)+\frac{\theta\Theta_l(1-s)}{\sigma}\right]+\epsilon_{l\lambda}\theta(1-s)\left[\frac{1+\beta}{\sigma}+\beta\Theta_l+\Theta_k\right]\right\}}{\frac{\theta(\theta-s\beta\delta)(1-s)}{s\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)(\epsilon_{lw}-\underline{\epsilon}_{lw})}$$

$$+ \frac{\epsilon_{l\lambda}\frac{\theta(1-s)(\theta-s\beta\delta)}{s}\left[\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}s\Theta_k}{1-s}\right)+\Theta_l\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}}{\sigma}\right)+\mathcal{C}\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)\right]}{\frac{\theta(\theta-s\beta\delta)(1-s)}{s\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)(\epsilon_{lw}-\underline{\epsilon}_{lw})} \tag{66}$$

□

6.7 Proof of Theorem 1

We immediately derive for any Θ_l:

$$1-\mathcal{T}(0)+\mathcal{D}<0 \text{ and } \lim_{\epsilon_{cc}\to+\infty}\mathcal{T}(\epsilon_{cc})=\pm\infty \text{ when } \epsilon_{lw}\lesseqgtr\underline{\epsilon}_{lw}$$

Case 1 - Let us consider first the case with a low wage elasticity for the labor supply, i.e. $\epsilon_{lw}<\underline{\epsilon}_{lw}$. We get the following two graphical configurations depending on the values of Θ_l, Θ_k and $\epsilon_{l\lambda}$:

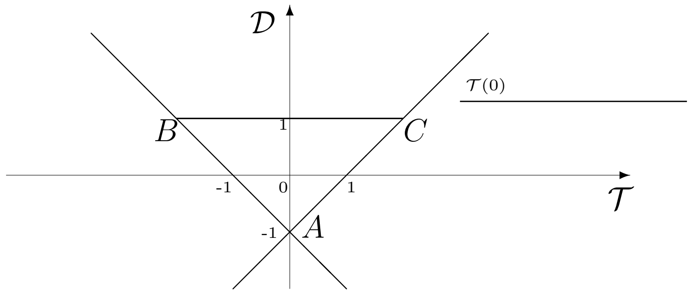


Figure 9: $\epsilon_{lw}<\underline{\epsilon}_{lw}$

As $\mathcal{D}$ does not depend on ϵ_{cc} and, when $\epsilon_{lw}<\underline{\epsilon}_{lw}$, the determinant $\mathcal{D}$ satisfies $\mathcal{D}>1$ or $\mathcal{D}\in(0,1)$ depending on the values of Θ_l and $\epsilon_{l\lambda}$, and we get an horizontal line characterizing the variation of $\mathcal{T}(\epsilon_{cc})$ when ϵ_{cc} is varied over $[0,+\infty)$. Obviously, this line cannot cross the line BC. Moreover, as $1-\mathcal{T}(0)+\mathcal{D}<0$, the starting point when $\epsilon_{cc}=0$ is located below the line AC and we have $\lim_{\epsilon_{cc}\to+\infty}\mathcal{T}(\epsilon_{cc})=+\infty$. The steady state is then a saddle-point for any $\epsilon_{cc}\geq 0$.

Case 2 - Let us consider now the case with a high wage elasticity for the labor supply, i.e. $\epsilon_{lw}>\underline{\epsilon}_{lw}$, and low capital externalities, i.e. $\Theta_k\in[0,\underline{\Theta}_k)$. We get the following graphical configuration:

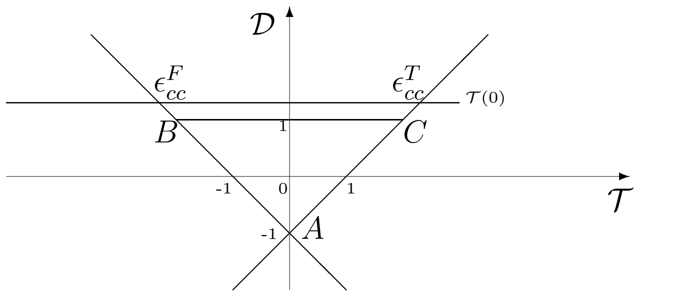


Figure 10: $\epsilon_{lw}>\underline{\epsilon}_{lw}$ and $\Theta_k\in[0,\underline{\Theta}_k)$, with $\Theta_l<\underline{\Theta}_l$ or $\Theta_l\in(\underline{\Theta}_l,\bar{\Theta}_l)$ and $\epsilon_{l\lambda}<\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$

When $\Theta_l<\underline{\Theta}_l$ or $\Theta_l\in(\underline{\Theta}_l,\bar{\Theta}_l)$ and $\epsilon_{l\lambda}<\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, we have $\mathcal{D}>1$ but now $\lim_{\epsilon_{cc}\to+\infty}\mathcal{T}(\epsilon_{cc})=-\infty$. Local indeterminacy cannot arise but the steady state is not always a saddle-point and can be a source. Indeed, the steady state is saddle-point stable for any $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$ and locally unstable when $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.

When $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ and $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, we get $\mathcal{D} \in (0,1)$ with $\lim_{\epsilon_{cc} \to +\infty} \mathcal{T}(\epsilon_{cc}) = -\infty$. It follows that the line now crosses the triangle ABC and we get indeterminacy:

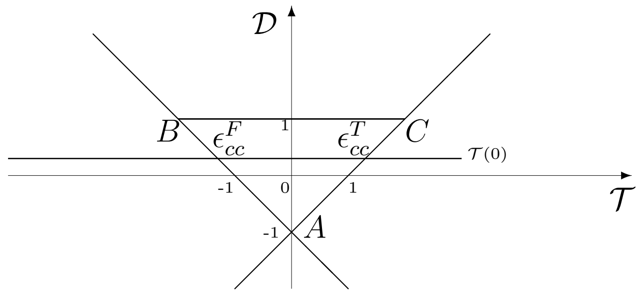


Figure 11: $\epsilon_{lw} > \underline{\epsilon}_{lw}$ and $\Theta_k \in [0, \underline{\Theta}_k)$, with $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ and $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

As $1 - \mathcal{T}(0) + \mathcal{D} < 0$, the starting point when $\epsilon_{cc} = 0$ is located below the line AC and the steady state is saddle-point stable. As ϵ_{cc} increases, $\mathcal{T}(\epsilon_{cc})$ will cross the line AC when $\epsilon_{cc} = \epsilon_{cc}^T$, implying the existence of a degenerate transcritical bifurcation since the steady state is unique. When $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$, the steady state is locally indeterminate. When $\epsilon_{cc} = \epsilon_{cc}^F$, a flip bifurcation generically occurs leading to the existence of period-two cycles in a right or left neighborhood of ϵ_{cc}^F. Finally, when $\epsilon_{cc} > \epsilon_{cc}^F$, the steady state is again saddle-point stable.

Case 3 - Let us finally consider the case with a high wage elasticity for the labor supply, i.e. $\epsilon_{lw} > \underline{\epsilon}_{lw}$, and large capital externalities, i.e. $\Theta_k \in (\underline{\Theta}_k, \bar{\Theta}_k)$. Since in this case $\underline{\Theta}_l > \bar{\Theta}_l$, we have necessarily $\Theta_l \in (0, \underline{\Theta}_l)$ and thus $\mathcal{D} > 1$. We then get the same configuration as Figure 10. The steady state is saddle-point stable for any $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$ and locally unstable when $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.

6.8 Proof of Proposition 5

We easily conclude that under Assumptions 1-4, local indeterminacy is ruled out for any $\sigma > 0$ in the following cases:

i) when $\Theta_k = \Theta_l = \Theta$, we get

$$\mathcal{D} = \frac{1}{\beta}\left\{1 + \theta \frac{\Theta\left(1+\frac{\epsilon_{lw}}{\sigma}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma} - \Theta_l(1-s)\right] + \epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}\right\} > \frac{1}{\beta}$$

ii) when $\epsilon_{lw} = 0$ we get

$$\lim_{\epsilon_{lw} \to 0} 1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} = -\infty$$

iii) when $\epsilon_{l\lambda} = 0$ we get

$$\mathcal{D} = \frac{1}{\beta}\left\{1 + \theta \frac{\Theta_k + \frac{\epsilon_{lw}}{\sigma}\left[s\Theta_k + (1-s)\Theta_l\right]}{1+\epsilon_{lw}\left[\frac{s}{\sigma} - \Theta_l(1-s)\right]}\right\} > \frac{1}{\beta}$$

iv) when $\Theta_l = 0$ we get

$$\mathcal{D} = \frac{1}{\beta}\left\{1+\theta\frac{\Theta_k\left[1+\frac{s\epsilon_{lw}}{\sigma}+(1-s)\epsilon_{l\lambda}\right]}{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{\epsilon_{l\lambda}\theta(1-s)}{\sigma}}\right\} > \frac{1}{\beta}$$

□

6.9 Proof of Proposition 6

The critical value ϵ_{cc}^T given in Lemma 3 provides a lower bound on ϵ_{cc} to get local indeterminacy. Since $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, we can derive the following lower bound for ϵ_{cc}^T:

$$\epsilon_{cc}^T > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})\frac{\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}s\Theta_k}{1-s}\right)+\Theta_l\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}}{\sigma}\right)+\mathcal{C}\frac{\epsilon_{l\lambda}(\epsilon_{lw})}{\epsilon_{lw}}\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)}{\frac{\epsilon_{lw}}{\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)-\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)} \equiv \underline{\epsilon}_{cc}^T(\epsilon_{lw})$$

$\underline{\epsilon}_{cc}^T(\epsilon_{lw})$ is a decreasing function of ϵ_{lw} over $(\underline{\epsilon}_{lw}, +\infty)$ with $\lim_{\epsilon_{lw}\to\underline{\epsilon}_{lw}} = +\infty$. Straightforward computations then show that under Assumption 5, $\underline{\epsilon}_{cc}^T > 2$ when $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$. As a result local indeterminacy is ruled out and the steady state is always a saddle-point. □

6.10 Proof of Proposition 7

From (45) we derive

$$\widehat{w}_t = \frac{s}{\sigma}\left(\widehat{k}_t - \widehat{l}_t\right) \quad (67)$$

Using this into (55) and (56) then gives

$$\begin{aligned} \widehat{l}_t &= \frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{\lambda}_t + \frac{s\epsilon_{lw}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t \\ \widehat{c}_t &= \left[\mathcal{C}\left(1-\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} - \epsilon_{cc}\right]\widehat{\lambda}_t + \frac{\mathcal{C}\frac{s}{\sigma}(\epsilon_{lw}-\epsilon_{l\lambda})}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t \end{aligned} \quad (68)$$

Equation (67) then becomes

$$\widehat{w}_t = \frac{s}{\sigma}\left[\frac{1}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t - \frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{\lambda}_t\right] \quad (69)$$

From the prices r_t and p_t as given by (44) and (46) we finally derive:

$$\begin{aligned} \widehat{r}_t &= \frac{(1-s)}{\sigma}\left[\frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{\lambda}_t - \frac{1}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t\right] \\ \widehat{p}_t &= -\frac{\Theta}{s\beta\delta}\left\{\frac{s\theta\left[1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}\right]}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t + \left[\frac{\theta(1-s)\epsilon_{l\lambda}^2}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right]\widehat{\lambda}_t\right\} \end{aligned}$$

Tedious computations based on these results allow to get from the system of difference equations (42)-(43):

$$\begin{pmatrix} 0 & 1 \\ A_{21} & -A_{22} \end{pmatrix}\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & -B_{22} \end{pmatrix}\begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix}$$

with

$$
\begin{aligned}
A_{21} &= 1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} - \frac{(1-\delta)\Theta}{s\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right] \\
A_{22} &= \frac{\frac{\theta(1-s)}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \frac{\theta(1-\delta)\Theta}{\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}} \\
B_{11} &= \frac{1+\Theta}{s\beta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right] \\
B_{12} &= \frac{1}{\beta}\left[1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \theta\Theta\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}\right] \\
B_{21} &= 1 - \frac{\Theta}{s\beta\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right] \\
B_{22} &= \frac{\theta\Theta}{\beta\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}
\end{aligned}
$$

The Proposition follows.

□

6.11 Proof of Lemma 4

We easily derive from Proposition 5 the Determinant and Trace:

$$
\begin{aligned}
\mathcal{D} &= \frac{B_{11}B_{22}+B_{12}B_{21}}{A_{21}} \\
\mathcal{T} &= 1 + \mathcal{D} + \frac{(B_{21}-A_{21})(1-B_{12})+B_{11}(A_{22}-B_{22})}{A_{21}}
\end{aligned}
$$

The characteristic polynomial is then

$$
\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) \tag{70}
$$

with

$$
\begin{aligned}
\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= \frac{1}{\beta}\left\{1 + \Theta\theta\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{\theta(1-s)\epsilon_{l\lambda}}{\sigma}-\frac{\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left[\frac{\mathcal{C}\epsilon_{l\lambda}^2}{\epsilon_{lw}}+\epsilon_{cc}(1+\frac{s\epsilon_{lw}}{\sigma})\right]}\right\} \\
\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) \\
&+ \frac{\frac{\theta(\theta-s\beta\delta)}{s\beta}\left\{\epsilon_{l\lambda}\left[\frac{(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}}{\sigma}\right]+\frac{\mathcal{C}\epsilon_{l\lambda}^2}{\epsilon_{lw}}\left[\frac{1-s}{\sigma}-\Theta\left(1-\frac{1-s}{\sigma}\right)\right]+\epsilon_{cc}\left[\frac{1-s}{\sigma}-\Theta\left(1-\frac{1-s}{\sigma}\right)-\Theta\frac{s}{\sigma}\epsilon_{lw}\right]\right\}}{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{\theta(1-s)\epsilon_{l\lambda}}{\sigma}-\frac{\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left[\frac{\mathcal{C}\epsilon_{l\lambda}^2}{\epsilon_{lw}}+\epsilon_{cc}(1+\frac{s\epsilon_{lw}}{\sigma})\right]} \\
&\equiv 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{X}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)
\end{aligned}
$$

It is easy to derive that as the parameter ϵ_{cc} is varied over the interval $(0, +\infty)$, $\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)$ and $\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)$ are linked through a linear relationship $\Delta(\mathcal{T})$ such that

$$
\mathcal{D} = \Delta(\mathcal{T}) = \mathcal{S}(\epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)\mathcal{T} + \mathcal{M}
$$

with

$$
\mathcal{S}(\epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = \frac{\partial\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \Theta)/\partial\epsilon_{cc}}{\partial\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \Theta)/\partial\epsilon_{cc}}
$$

which does not depend on ϵ_{cc}.

Straightforward computations show that $\partial\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)/\partial\epsilon_{cc} > 0$ and, under Assumptions 1, 3, 4, 6 and 7, $\partial\mathcal{T}(\epsilon_{cc},\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)/\partial\epsilon_{cc} > 0$. It follows that $\mathcal{S}(\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta) > 0$ and thus $\Delta(\mathcal{T})$ is a line in the space $(\mathcal{T},\mathcal{D})$ with a positive slope. In order to locate this line, we need to compute the starting and end points $(\mathcal{T}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta),\mathcal{D}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta))$ and $(\mathcal{T}(+\infty,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta),\mathcal{D}(+\infty,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta))$. We easily get

$$\begin{aligned}\mathcal{D}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta) &= \frac{1}{\beta}\left\{1+\Theta\theta\frac{1+\frac{s}{\sigma}\epsilon_{lw}+\frac{(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s}{\sigma}\epsilon_{lw}+\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}-\frac{\Theta(1-\delta)\theta(1-s)}{s\delta}\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}\right\}\equiv\mathcal{D}_0\\ \mathcal{X}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta) &= \frac{\frac{\theta(\theta-s\beta\delta)\epsilon_{l\lambda}[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}{s\beta\sigma\epsilon_{lw}}}{1+\frac{s}{\sigma}\epsilon_{lw}+\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}-\frac{\Theta(1-\delta)\theta(1-s)}{s\delta}\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}\end{aligned}$$

with

$$\hat{\epsilon}_{lw}\equiv\frac{\Theta\mathcal{C}\epsilon_{l\lambda}(\sigma-\sigma_{sup})}{(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}}\text{ and }\sigma_{sup}\equiv\frac{(1-s)(1+\Theta)}{\Theta}$$

Under Assumption 7, we obviously have $\sigma < 2 < \sigma_{sup}$ so that the bound $\hat{\epsilon}_{lw} < 0$ is no longer relevant. It follows that $\mathcal{D}_0$ satisfies:

- $\mathcal{D}_0 > 1/\beta$ if and only if $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})(\epsilon_{lw})$ with

$$\epsilon_{l\lambda}^0(\epsilon_{lw})\equiv\frac{\frac{\theta(1-s)s\delta}{\sigma}\epsilon_{lw}+\sqrt{\left[\frac{\theta(1-s)s\delta}{\sigma}\epsilon_{lw}\right]^2+4\Theta(1-\delta)\theta(1-s)\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}}}{2\Theta(1-\delta)\theta(1-s)}$$

- $\mathcal{D}_0 \in (-\infty,1)$ if and only if $\epsilon_{l\lambda}\in(\epsilon_{l\lambda}^0(\epsilon_{lw}),\bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$ with

$$\bar{\epsilon}_{l\lambda}(\epsilon_{lw})\equiv\frac{\frac{\theta(1-s)s\delta\epsilon_{lw}(1-\beta+\Theta)}{\sigma}+\sqrt{\left[\frac{\theta(1-s)s\delta\epsilon_{lw}(1-\beta+\Theta)}{\sigma}\right]^2+4\Theta(1-\beta)(1-\delta)\theta(1-s)\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}(1-\beta+\Theta\theta)}}{2\Theta(1-\beta)(1-\delta)\theta(1-s)}>\epsilon_{l\lambda}^0(\epsilon_{lw})$$

- $\mathcal{D}_0\in(1,1/\beta)$ if and only if $\epsilon_{l\lambda}>\bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

We also immediately conclude that

$$1-\mathcal{T}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)+\mathcal{D}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)=-\mathcal{X}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)<0$$

if and only if $\epsilon_{l\lambda}<\epsilon_{l\lambda}^0(\epsilon_{lw})$. Moreover, we easily derive that

$$1+\mathcal{T}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)+\mathcal{D}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)>0$$

if and only if $\epsilon_{l\lambda}\in(0,\epsilon_{l\lambda}^0(\epsilon_{lw}))\cup(\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}),+\infty)$ with

$$\tilde{\epsilon}_{l\lambda}(\epsilon_{lw})\equiv\frac{\frac{\theta\delta\epsilon_{lw}}{\sigma}\left\{2(1-s)s(1+\beta+\Theta)+(\theta-s\beta\delta)[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}]\right\}+\sqrt{\Delta}}{2\Theta\theta(1-s)\left[2(1-\delta)(1+\beta)+\frac{\theta\delta(\sigma-\sigma_{sup})}{\sigma}\right]}\in(\epsilon_{l\lambda}^0(\epsilon_{lw}),\bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$$

and

$$\begin{aligned}\Delta =& \left(\frac{\theta\delta\epsilon_{lw}\left\{2(1-s)s(1+\beta+\Theta)+(\theta-s\beta\delta)[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}]\right\}}{\sigma}\right)^2\\ &+8\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}(1+\beta+\Theta\theta)\Theta\theta(1-s)\left[2(1-\delta)(1+\beta)+\frac{\theta\delta(\sigma-\sigma_{sup})}{\sigma}\right]\end{aligned}\tag{71}$$

while

$$1+\mathcal{T}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)+\mathcal{D}(0,\epsilon_{lw},\epsilon_{l\lambda},\sigma,\Theta)<0$$

if and only if $\epsilon_{l\lambda}\in(\epsilon_{l\lambda}^0(\epsilon_{lw}),\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

Finally, we have

$$\mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = \frac{1}{\beta} \equiv \mathcal{D}_\infty \in (1, \mathcal{D}_0)$$
$$\mathcal{X}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = \frac{\theta\delta}{\sigma\beta} \frac{s(\epsilon_{lw} - \underline{\epsilon}_{lw})}{(1-\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)}$$

with

$$\underline{\epsilon}_{lw} \equiv \frac{\sigma_{sup} - \sigma}{s}$$

We conclude here that since $\sigma < 2 < \sigma_{sup}$, $\mathcal{X}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ and thus

$$1 - \mathcal{T}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = -\mathcal{X}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \Theta) < 0$$

if and only if $\epsilon_{lw} > \underline{\epsilon}_{lw}$.

We then get the following conclusions:

- if $\epsilon_{l\lambda} < \epsilon^0_{l\lambda}(\epsilon_{lw})$ then $1 - \mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) < 0$ for any $\epsilon_{lw} \geq 0$ while $1 - \mathcal{T}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ if and only if $\epsilon_{lw} < \underline{\epsilon}_{lw}$;
- if $\epsilon_{l\lambda} > \epsilon^0_{l\lambda}(\epsilon_{lw})$ then $1 - \mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ for any $\epsilon_{lw} \geq 0$ while $1 - \mathcal{T}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ if and only if $\epsilon_{lw} < \underline{\epsilon}_{lw}$.

Let us compute the critical values $\epsilon^H_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$, $\epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$ and $\epsilon^F_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$ respectively associated with Hopf, transcritical and flip bifurcations. The first one $\epsilon^H_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$ is obtained solving the equality $\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = 1$ with respect to ϵ_{cc}. Straightforward computations yield the value

$$\epsilon^H_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) = \frac{\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)(1-\beta+\Theta\theta) + \frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}(1-\beta+\Theta) - \Theta\theta(1-\beta)(1-\delta)(1-s)\frac{\epsilon^2_{l\lambda}}{s\delta\epsilon_{lw}}}{\Theta(1-\beta)\frac{(1-\delta)}{s\delta}(\theta - s\beta\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)}$$

It follows obviously that for any given ϵ_{lw}, $\epsilon^H_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) \geq 0$ if and only if $\epsilon_{l\lambda} \leq \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

The critical value $\epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$ is obtained as the solution of the equality $\mathcal{X}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = 0$, namely

$$\epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) \equiv \frac{\epsilon_{l\lambda}[(1-s)(1-\mathcal{C}) + \Theta s\mathcal{C}](\epsilon_{lw} - \hat{\epsilon}_{lw})}{\Theta s \epsilon_{lw}(\epsilon_{lw} - \underline{\epsilon}_{lw})}$$

We have $\hat{\epsilon}_{lw} < 0$ and thus $\epsilon_{lw} - \hat{\epsilon}_{lw} > 0$. So, obviously, $\epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) > 0$ if and only if $\epsilon_{lw} > \underline{\epsilon}_{lw}$. By convention, we will consider that $\epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) = +\infty$ when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$. Note that since the steady state is generically unique, $\epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$ corresponds to a degenerate transcritical bifurcation.

The critical value $\epsilon^F_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$ is finally obtained as the solution of the equality $1 + \mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = 0$, namely

$$\epsilon^F_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) = \frac{\theta(1-s)\left[2(1-\delta)(1+\beta) + \frac{\theta\delta(\sigma-\sigma_{sup})}{\sigma}\right]\left(\epsilon^-_{l\lambda}(\epsilon_{lw}) - \epsilon_{l\lambda}\right)(\epsilon_{l\lambda} - \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))}{(\theta - s\beta\delta)\epsilon_{lw}\left[2(1+\beta)(1-\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right) + \frac{\theta s\delta}{\sigma}(\epsilon_{lw} - \underline{\epsilon}_{lw})\right]}$$

with

$$\epsilon^-_{l\lambda}(\epsilon_{lw}) = \frac{\frac{\theta\delta\epsilon_{lw}}{\sigma}\left\{2(1-s)s(1+\beta+\Theta) + (\theta - s\beta\delta)[(1-s)(1-\mathcal{C}) + \Theta s\mathcal{C}]\right\} - \sqrt{\Delta}}{2\Theta\theta(1-s)\left[2(1-\delta)(1+\beta) + \frac{\theta\delta(\sigma-\sigma_{sup})}{\sigma}\right]}$$

and Δ as given by (71). Obviously, $\epsilon^-_{l\lambda}(\epsilon_{lw}) < 0$ and we conclude that $\epsilon^F_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) \geq 0$ if and only if $\epsilon_{l\lambda} \leq \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. This critical value corresponds to a flip bifurcation giving rise to the existence of period-two cycles.

As this will become obvious later on, we need now to check that $\epsilon^H_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) \leq \epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda})$. This inequality is satisfied if and only if $\epsilon_{l\lambda} \geq \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$ with

$$\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = \frac{-\sigma(1-s)\left\{(1-\beta+\Theta\theta)(1-\beta)(1-\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)-\frac{\theta}{\sigma}(1-\beta+\Theta)s\delta(\epsilon_{lw}-\underline{\epsilon}_{lw})\right\}+\sigma\sqrt{\hat{\Delta}}}{2\Theta\theta(1-\beta)(1-\delta)(1-s)\sigma_{sup}}$$

and

$$\begin{aligned}\hat{\Delta} &= (1-s)^2\left\{(1-\beta+\Theta\theta)(1-\beta)(1-\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)-\frac{\theta}{\sigma}(1-\beta+\Theta)s\delta(\epsilon_{lw}-\underline{\epsilon}_{lw})\right\}^2 \\ &+ \frac{4\Theta\theta(1-\beta)(1-\delta)(1-s)\sigma_{sup}}{\sigma}\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)(1-\beta+\Theta\theta)s\delta(\epsilon_{lw}-\underline{\epsilon}_{lw})\end{aligned}$$

Note that $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) > 0$ if and only if $\epsilon_{lw} > \underline{\epsilon}_{lw}$. By convention, we will consider that $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0$ when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$.

When $\epsilon_{l\lambda} < \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$, the Hopf bifurcation is always ruled out. In order to locate the $\Delta(\mathcal{T})$ line we need to check whether $\mathcal{D} = -1$ can occur, and if yes, we need to know the sign of $\mathcal{T}$ when $\mathcal{D} = -1$. If the sign is positive then the $\Delta(\mathcal{T})$ line is located below the triangle ABC and local indeterminacy is ruled out. On the contrary, if the sign is negative then the $\Delta(\mathcal{T})$ line may cross the triangle ABC leading to the possible existence of local indeterminacy. Solving $\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = -1$ with respect to ϵ_{cc} gives

$$\bar{\epsilon}_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) = \frac{\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)(1+\beta+\Theta\theta)+\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}(1+\beta+\Theta)-\Theta\theta(1+\beta)(1-\delta)(1-s)\frac{\epsilon_{l\lambda}^2}{s\delta\epsilon_{lw}}}{\Theta(1+\beta)\frac{(1-\delta)}{s\delta}(\theta-s\beta\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)}$$

Straightforward computations show that $\bar{\epsilon}_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) > 0$ if and only if $\epsilon_{l\lambda} > \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ with

$$\begin{aligned}\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}) &\equiv \frac{\frac{\theta(1-s)s\delta\epsilon_{lw}(1+\beta+\Theta)}{\sigma}+\sqrt{\left[\frac{\theta(1-s)s\delta\epsilon_{lw}(1+\beta+\Theta)}{\sigma}\right]^2+4\Theta(1+\beta)(1-\delta)\theta(1-s)\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}(1+\beta+\Theta\theta)}}{2\Theta(1+\beta)(1-\delta)\theta(1-s)} \\ &\in \left(\epsilon_{l\lambda}^0(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw})\right)\end{aligned}$$

If $\epsilon_{l\lambda} > \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ we derive that

$$\mathcal{T}(\bar{\epsilon}_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = \chi(\bar{\epsilon}_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)$$

and straightforward computations yield $\mathcal{T}(\bar{\epsilon}_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) \geq 0$ if and only if $\epsilon_{l\lambda} \geq \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ with

$$\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) \equiv \frac{-(1-s)\left\{\frac{(1-\beta+\Theta\theta)(1+\beta)(1-\delta)}{\delta}+\frac{\theta(1+\beta+\Theta)(\sigma_{sup}-\sigma)}{\sigma}+\frac{s\epsilon_{lw}}{\delta\sigma}\left[(1+\beta)[\theta(1-\delta)-\delta]+\Theta\theta[(1+\beta)(1-\delta)-\delta]\right]\right\}+\sqrt{\underline{\Delta}}}{\frac{2\Theta(1+\beta)(1-\delta)\theta(1-s)\sigma_{sup}}{\delta\sigma}} < \epsilon_{l\lambda}^0(\epsilon_{lw})$$

and

$$\begin{aligned}\underline{\Delta} &= (1-s)^2\left\{\frac{(1-\beta+\Theta\theta)(1+\beta)(1-\delta)}{\delta}+\frac{\theta(1+\beta+\Theta)(\sigma_{sup}-\sigma)}{\sigma}+\frac{s\epsilon_{lw}}{\delta\sigma}\left[(1+\beta)[\theta(1-\delta)-\delta]\right.\right. \\ &\left.\left.+\Theta\theta[(1+\beta)(1-\delta)-\delta]\right]\right\}^2+4\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)(1+\beta+\Theta\theta)s(\epsilon_{lw}-\underline{\epsilon}_{lw})\frac{\Theta(1+\beta)(1-\delta)\theta(1-s)\sigma_{sup}}{\delta\sigma}\end{aligned}$$

Note that $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ is obviously such that $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda}) = \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ when $\epsilon_{l\lambda} = \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$. □

6.12 Proof of Theorem 2

Under Assumptions 1, 3, 4, 6 and 7, straightforward computations show that if $\epsilon_{lw} > \underline{\epsilon}_{lw}$, $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$. We need now to check whether $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) \gtrless \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. Since $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0 < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$ when $\epsilon_{lw} = \underline{\epsilon}_{lw}$, obvious computations then show that there exists a unique value

$\bar{\epsilon}_{lw} > \underline{\epsilon}_{lw}$ such that $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$ if and only if $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$ if and only if $\epsilon_{lw} = \bar{\epsilon}_{lw}$.

Let us consider now the value of the Trace when $\epsilon_{cc} = \epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$. We get

$$\mathcal{T}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \Theta) = 2 + \mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \Theta)$$

with

$$\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = -\frac{\frac{(1-\beta)(\theta - s\beta\delta)\epsilon_{l\lambda}}{s\beta} \frac{[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}{\epsilon_{lw}} \left\{1 + \frac{\Theta s \epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})\epsilon_{lw}(\underline{\epsilon}_{lw}-\epsilon_{lw})}{\epsilon_{l\lambda}[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}\right\}}{\Theta\theta\left[1 + \frac{s}{\sigma}\epsilon_{lw} + \frac{(1-s)}{\sigma}\epsilon_{l\lambda}\right]}$$

We easily derive that $\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) < 0$ and thus $\mathcal{T}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \Theta) < 2$ if and only if $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

We need now to provide a condition to get $\mathcal{T}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > -2$ or equivalently $\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > -4$ for any $\epsilon_{lw} \geq 0$. Straightforward computations give

$$\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = \frac{\frac{\theta(\theta - s\beta\delta)}{s\beta}\left\{\epsilon_{lw}\epsilon_{l\lambda}\left[\frac{(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}}{\sigma}\right] + \frac{\Theta\mathcal{C}\epsilon_{l\lambda}^2}{\sigma}(\sigma_{sup}-\sigma) + \frac{\Theta s \epsilon_{lw}\epsilon_{cc}^H}{\sigma}(\underline{\epsilon}_{lw}-\epsilon_{lw})\right\}}{\epsilon_{lw}\left(1 + \frac{s\epsilon_{lw}}{\sigma} + \frac{\theta(1-s)\epsilon_{l\lambda}}{\sigma}\right) - \frac{\Theta(1-\delta)(\theta - s\beta\delta)}{s\delta}\left[\mathcal{C}\epsilon_{l\lambda}^2 + (\underline{\epsilon}_{lw}-\epsilon_{lw})\epsilon_{cc}^H(1 + \frac{s\epsilon_{lw}}{\sigma})\right]}$$

and thus

$$\mathcal{X}(\epsilon_{cc}^H(0, \epsilon_{l\lambda}), 0, \epsilon_{l\lambda}, \sigma, \Theta) = -\frac{\theta\delta(\sigma_{sup}-\sigma)}{\beta\sigma(1-\delta)}$$

We then derive that if $\sigma > \sigma_{inf}$ with

$$\sigma_{inf} = \frac{\delta\theta(1-s)(1+\Theta)}{\Theta[4\beta(1-\delta)+\delta\theta]}$$

then $\mathcal{X}(\epsilon_{cc}^H(0, \epsilon_{l\lambda}), 0, \epsilon_{l\lambda}, \sigma, \Theta) > -4$ and $\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > -4$ for any $\epsilon_{lw} \geq 0$.

Case **1** - Let us consider in a first step the case $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ where $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0$. We get the following geometrical characterizations of the $\Delta(\mathcal{T})$ line depending on the value of $\epsilon_{l\lambda}$. When $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})$ or $\epsilon_{l\lambda} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$, we have:

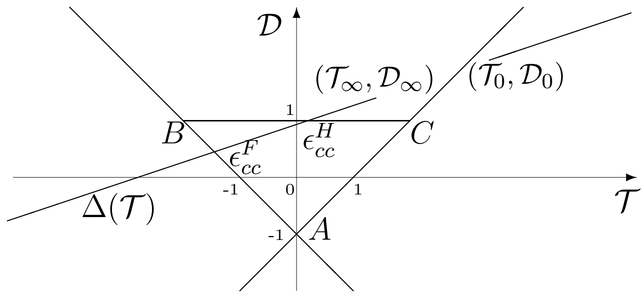


Figure 12: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})$.

As shown by these Figures, increasing ϵ_{cc} from 0, the steady state is first saddle-point stable. Still increasing ϵ_{cc} leads to the existence of a flip bifurcation giving rise to the existence of period-two cycles when ϵ_{cc} crosses $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$. Above $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ the steady state then becomes locally indeterminate and when ϵ_{cc} crosses the bound $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, a Hopf bifurcation occurs giving rise to the existence of periodic cycles. Above $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, the steady state is totally unstable for any $\epsilon_{cc} \in (\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), +\infty)$.

When $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$, we obviously get the following case where the flip

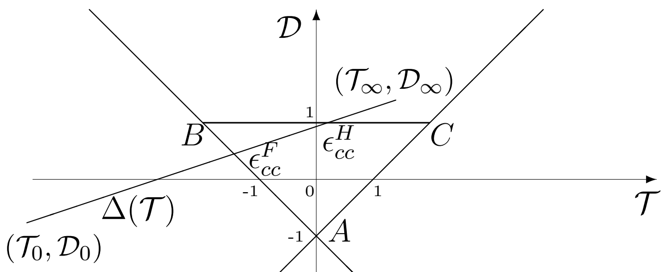


Figure 13: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

bifurcation no longer exists, i.e. $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$:

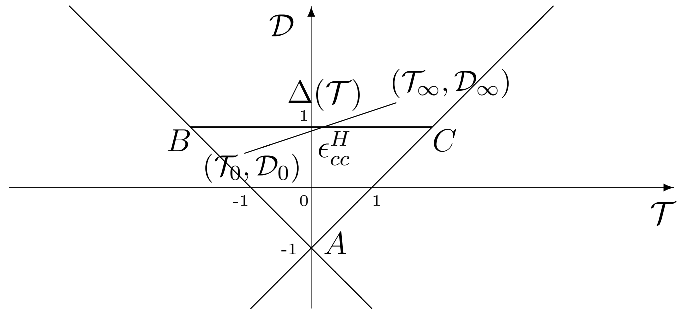


Figure 14: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

Finally, when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$, we get the following case where the Hopf bifurcation no longer exists, i.e. $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$:

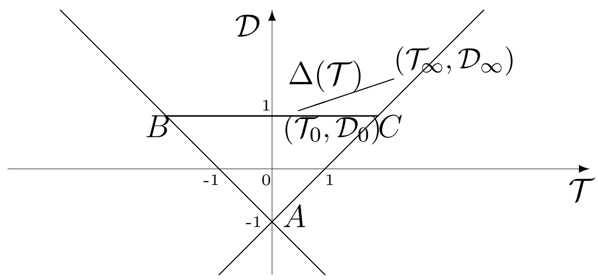


Figure 15: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

Case **2** - Let us consider now the case $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ where $0 < \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. We get $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) < \epsilon_{l\lambda}^0(\epsilon_{lw}) < \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}) < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$ and $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) < \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. However, the bounds $\epsilon_{l\lambda}^0(\epsilon_{lw})$ and $\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ may be lower or larger than $\hat{\epsilon}_{l\lambda}(\epsilon_{lw})$ but this does not really impact the local stability results. It is worth noticing that when $\epsilon_{lw} = \underline{\epsilon}_{lw}$, $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) = \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0$. We then obtain the following geometrical characterizations of the $\Delta(\mathcal{T})$ line depending on the value of $\epsilon_{l\lambda}$ and ϵ_{lw}. Recall that $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and consider for now that $\epsilon_{l\lambda}^0(\epsilon_{lw}) < \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}) < \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$. When $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ we have

When $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$ we get the following three cases:

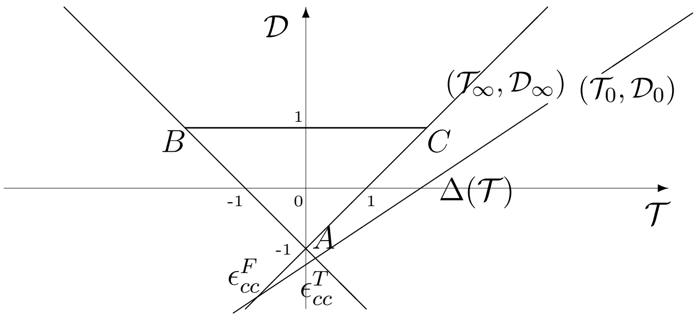


Figure 16: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

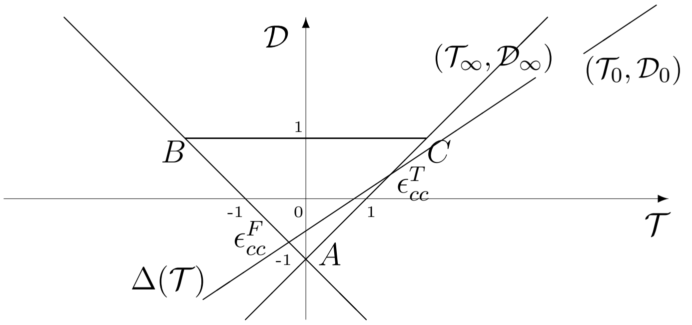


Figure 17: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}(\epsilon_{lw}), \epsilon_{l\lambda}^0(\epsilon_{lw}))$.

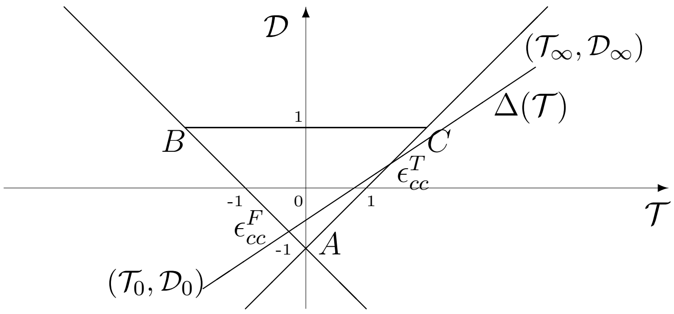


Figure 18: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

In these three cases, local indeterminacy arises if $\epsilon_{cc} \in (\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}))$ and saddle-point stability holds outside this interval. The same conclusion would be obtained is $\epsilon_{l\lambda}^0(\epsilon_{lw})$ and/or $\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ were larger than $\hat{\epsilon}_{l\lambda}(\epsilon_{lw})$.

When $\epsilon_{l\lambda} > \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$, the Hopf bifurcation may again occur as long as $\epsilon_{l\lambda} < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$. We get indeed:

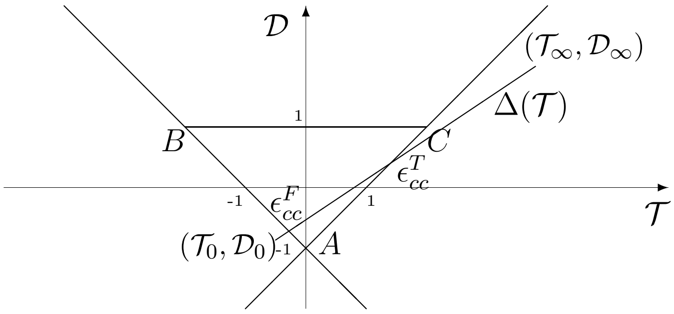


Figure 19: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

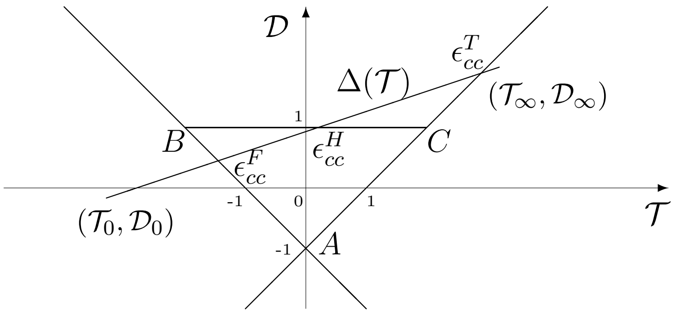


Figure 20: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

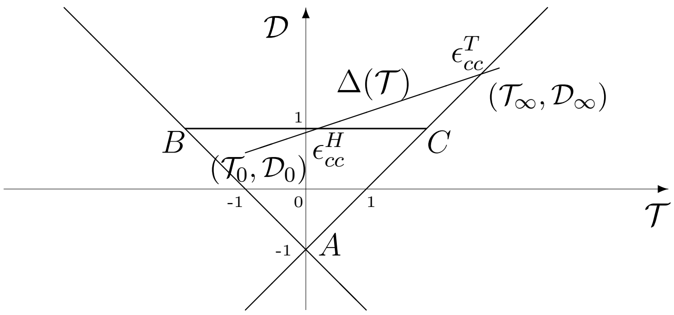


Figure 21: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

As shown by these Figures, when $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$, increasing ϵ_{cc} from 0, the steady state is first saddle-point stable. Still increasing ϵ_{cc} leads to the existence of a flip bifurcation giving rise to the existence of period-two cycles when ϵ_{cc} crosses $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$. Above $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ the steady state then becomes locally indeterminate and when ϵ_{cc} crosses the bound $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, a Hopf bifurcation occurs giving rise to the existence of periodic cycles. Above $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, the steady state is locally unstable when $\epsilon_{cc} \in (\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}))$ and saddle-point stable when $\epsilon_{cc} \in (\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}), +\infty)$. Since the steady state is generically unique, $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ corresponds to a degenerate transcritical bifurcation. When $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$, we get $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$ and when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$ we get $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$.

Case **3** - Let us consider finally the case $\epsilon_{lw} > \bar{\epsilon}_{lw}$ where $\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}) < \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$. Therefore, using all our previous results, we get the following geometrical characterizations of the $\Delta(\mathcal{T})$ line depending on the value of $\epsilon_{l\lambda}$. When $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, local indeterminacy is ruled out as the $\Delta(\mathcal{T})$ does not cross the triangle ABC as in Figure 16.

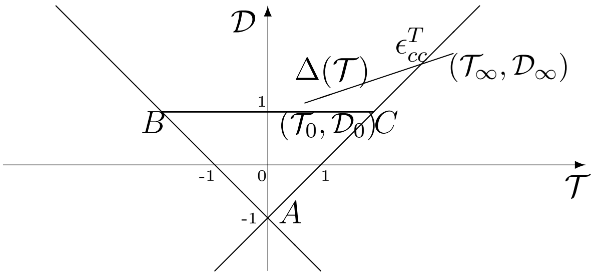


Figure 22: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

On the contrary, when $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$, local indeterminacy can arise as we get the following three cases where the $\Delta(\mathcal{T})$ crosses the triangle ABC as in Figures 17, 18 and 19.

When $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$ the flip bifurcation no longer exists.

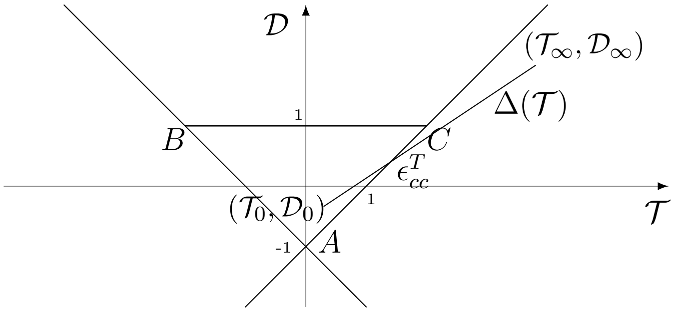


Figure 23: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} > \bar{\epsilon}_{lw}$ and $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

When $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$, the Hopf bifurcation exists as the same configuration as Figure 21 occurs. When $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$, the Hopf bifurcation no longer exists and local indeterminacy is again ruled out as the same configuration as Figure 22 occurs.

□

6.13 Proof of Corollary 1

With KPR preferences such that $\epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}$, we get

$$\begin{aligned}
\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) &= \tfrac{1}{\beta}\left\{1 + \Theta\theta \frac{1+\frac{s\epsilon_{lw}}{\sigma}[s+(1-s)\epsilon_{cc}]}{1+\frac{s\epsilon_{lw}}{\sigma}[s+\theta(1-s)\epsilon_{cc}] - \frac{\Theta(1-\delta)(\theta-s\beta\delta)\epsilon_{cc}}{s\delta}\left[\mathcal{C}\epsilon_{cc}\epsilon_{lw}+1+\frac{s\epsilon_{lw}}{\sigma}\right]}\right\} \\
\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) &= 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) \\
&+ \frac{\frac{\theta(\theta-s\beta\delta)\epsilon_{cc}}{s\beta}\left\{\epsilon_{lw}\left[\frac{(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}}{\sigma}\right]+\frac{\mathcal{C}\epsilon_{cc}\epsilon_{lw}\Theta(\sigma_{sup}-\sigma)}{\sigma}+\frac{s\Theta}{\sigma}(\underline{\epsilon}_{lw}-\epsilon_{lw})\right\}}{1+\frac{s\epsilon_{lw}}{\sigma}[s+\theta(1-s)\epsilon_{cc}] - \frac{\Theta(1-\delta)(\theta-s\beta\delta)\epsilon_{cc}}{s\delta}\left[\mathcal{C}\epsilon_{cc}\epsilon_{lw}+1+\frac{s\epsilon_{lw}}{\sigma}\right]} \\
&\equiv 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) + \mathcal{X}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta)
\end{aligned}$$

with

$$\sigma_{sup} \equiv \tfrac{(1-s)(1+\Theta)}{\Theta} \text{ and } \underline{\epsilon}_{lw} \equiv \tfrac{\sigma_{sup}-\sigma}{s}$$

Under Assumption 7 we obviously have $\sigma < 2 < \sigma_{sup}$. Applying the same technique as in the Proof of Theorem 2 we can compute the Hopf bifurcation value

$$\epsilon_{cc}^H = \frac{\frac{(1-s)\epsilon_{lw}}{\Theta\sigma}(1-\beta+\Theta) - \frac{(1-\beta)(1-\delta)(\theta-s\beta\delta)}{\theta s\delta}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)+\sqrt{\Delta^H}}{\frac{2(1-\beta)(1-\delta)(\theta-s\beta\delta)}{\theta s\delta}\mathcal{C}\epsilon_{lw}}$$

with

$$\begin{aligned}\Delta^H &= \left[\frac{(1-\beta)(1-\delta)(\theta-s\beta\delta)}{\theta s\delta}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right) - \frac{(1-s)\epsilon_{lw}}{\Theta\sigma}(1-\beta+\Theta)\right]^2 \\ &+ 4\frac{(1-\beta)(1-\delta)(\theta-s\beta\delta)}{\theta s\delta}\mathcal{C}\epsilon_{lw}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)\left(1+\frac{1-\beta}{\Theta\theta}\right)\end{aligned}$$

and the flip bifurcation value

$$\epsilon_{cc}^F = \frac{\frac{2(1-s)\theta\epsilon_{lw}}{\sigma}(1+\beta+\Theta)+\frac{\theta(\theta-s\beta\delta)}{s}\left[\frac{\epsilon_{lw}(1-\mathcal{C})s}{\sigma}\left(\frac{1-s}{s}-\Theta\right)+\frac{\Theta}{\sigma}(\sigma_{sup}-\sigma)\right]-\frac{2(1+\beta)\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)+\sqrt{\Delta^F}}{\frac{2\Theta(\theta-s\beta\delta)\mathcal{C}\epsilon_{lw}}{s}\frac{2(1+\beta)(1-\delta)+\theta\delta}{\delta\sigma}(\sigma-\sigma^F)}$$

with

$$\begin{aligned}\Delta^F &= \left\{\frac{2(1+\beta)\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right) - \frac{2(1-s)\theta\epsilon_{lw}}{\sigma}(1+\beta+\Theta) + \frac{\theta(\theta-s\beta\delta)}{s}\left[\frac{\epsilon_{lw}(1-\mathcal{C})s}{\sigma}\left(\frac{1-s}{s}-\Theta\right)+\frac{\Theta}{\sigma}(\sigma_{sup}-\sigma)\right]\right\}^2 \\ &+ 8\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)(1+\beta+\Theta\theta)\frac{\Theta(\theta-s\beta\delta)\mathcal{C}\epsilon_{lw}}{s}\left[\frac{2(1+\beta)(1-\delta)}{\delta}-\frac{\theta}{\sigma}(\sigma_{sup}-\sigma)\right]\end{aligned}$$

and

$$\sigma^F \equiv \frac{\delta\theta(1-s)(1+\Theta)}{\Theta[2(1+\beta)(1-\delta)+\theta\delta]}$$

We obviously need $\sigma > \sigma^F$ to get $\epsilon_{cc}^F > 0$. We easily show that $\mathcal{T}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) < 2$ and that solving $\epsilon_{cc}^H = \epsilon_{cc}^F$ allows to prove the existence of $\tilde{\sigma}_{inf} > \sigma^F$ as given by

$$\tilde{\sigma}_{inf} \equiv \frac{(1-\beta+\Theta\theta)\delta(1-s)(1+\Theta)}{\Theta[4(1-\delta)\Theta\beta+\delta(1-\beta+\Theta\theta)]}$$

such that $\mathcal{T}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) > -2$ if $\sigma > \tilde{\sigma}_{inf}$. Notice that the bound obtained here is different from the one derived in the Proof of Theorem 2 as we consider a specific value $\epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}$ which modifies the computations.

□

6.14 Proof of Corollary 2

With GHH preferences such that $\epsilon_{l\lambda} = 0$, we get

$$\begin{aligned}\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) &= \frac{1}{\beta}\left\{1+\frac{\Theta\theta}{1-\frac{\epsilon_{cc}\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}}\right\} \\ \mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) &= 1+\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta)+\frac{\frac{\theta(\theta-s\beta\delta)}{\beta}\epsilon_{cc}\frac{\Theta}{\sigma}(\underline{\epsilon}_{lw}-\epsilon_{lw})}{\left(1+\frac{\epsilon_{lw}}{\sigma}\right)\left[1-\frac{\epsilon_{cc}\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\right]} \\ &\equiv 1+\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta)+\mathcal{X}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta)\end{aligned}$$

with

$$\sigma_{sup} \equiv \frac{(1-s)(1+\Theta)}{\Theta} \text{ and } \underline{\epsilon}_{lw} \equiv \frac{\sigma_{sup}-\sigma}{s}$$

Applying the same technique as in the Proof of Theorem 2 we can compute the Hopf bifurcation value

$$\epsilon_{cc}^H = \frac{(1-\beta+\Theta\theta)s\delta}{\Theta(1-\beta)(1-\delta)(\theta-s\beta\delta)}$$

and the flip bifurcation value

$$\epsilon_{cc}^F = \frac{2\delta\sigma(1+\beta+\Theta\theta)\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)}{\Theta(\theta-s\beta\delta)[2(1+\beta)(1-\delta)+\theta\delta](\epsilon_{lw}-\epsilon_{lw}^F)}$$

with

$$\sigma^F \equiv \frac{\delta\theta(1-s)(1+\Theta)}{\Theta[2(1+\beta)(1-\delta)+\theta\delta]} \text{ and } \epsilon_{lw}^F \equiv \frac{\sigma^F-\sigma}{s}$$

Assuming $\sigma > \sigma^F$ implies that $\epsilon_{cc}^F > 0$ for any $\epsilon_{lw} > 0$. However, we easily get

$$\mathcal{T}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) = 2 + \frac{\frac{\theta(\theta-s\beta\delta)}{\beta}\epsilon_{cc}^H\frac{\Theta}{\sigma}(\epsilon_{lw}-\underline{\epsilon}_{lw})}{\left(1+\frac{\epsilon_{lw}}{\sigma}\right)\Theta\theta}$$

so that $\mathcal{T}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) < 2$ if and only if $\epsilon_{lw} < \underline{\epsilon}_{lw}$. We need now to provide a condition to get $\mathcal{T}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) > -2$ or equivalently $\mathcal{X}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) > -4$ for any $\epsilon_{lw} \geq 0$. Straightforward computations give

$$\mathcal{X}(\epsilon_{cc}^H, 0, \epsilon_{l\lambda}, \sigma, \Theta) = -\frac{\delta(1-\beta+\Theta\theta)(\sigma_{sup}-\sigma)}{\beta\sigma(1-\delta)\Theta}$$

We then derive that if $\sigma > \tilde{\sigma}_{inf}$ with

$$\tilde{\sigma}_{inf} = \frac{(1-\beta+\Theta\theta)\delta(1-s)(1+\Theta)}{\Theta[4(1-\delta)\Theta\beta+\delta(1-\beta+\Theta\theta)]} \in (\sigma^F, 2)$$

then $\mathcal{X}(\epsilon_{cc}^H, 0, \sigma, \Theta) > -4$ and $\mathcal{X}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) > -4$ for any $\epsilon_{lw} \geq 0$. Notice that the bound obtained here is different from the one derived in the Proof of Theorem 2 as we consider a specific value $\epsilon_{l\lambda} = 0$ which modifies the computations.

□

6.15 Proof of Corollary 3

Under generalized Hansen preferences such that $\epsilon_{l\lambda} = \epsilon_{lw}$, we get

$$\begin{aligned}
\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) &= \frac{1}{\beta}\left\{1 + \Theta\theta\frac{1+\frac{\epsilon_{lw}}{\sigma}}{\frac{\Theta(1-\delta)\theta(1-s)(\sigma-\bar{\sigma})(\bar{\epsilon}_{lw}-\epsilon_{lw})}{\sigma s\delta} - \epsilon_{cc}\frac{\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)}\right\} \\
\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) &= 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) \\
&+ \frac{\frac{\theta(\theta-s\beta\delta)}{s\beta}\left\{\epsilon_{lw}\left[\frac{(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}}{\sigma} + \frac{\mathcal{C}\Theta}{\sigma}(\sigma_{sup}-\sigma)\right] + \epsilon_{cc}\frac{\Theta s}{\sigma}(\underline{\epsilon}_{lw}-\epsilon_{lw})\right\}}{\frac{\Theta(1-\delta)\theta(1-s)(\sigma-\bar{\sigma})(\bar{\epsilon}_{lw}-\epsilon_{lw})}{\sigma s\delta} - \epsilon_{cc}\frac{\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)} \\
&\equiv 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta) + \mathcal{X}(\epsilon_{cc}, \epsilon_{lw}, \sigma, \Theta)
\end{aligned}$$

with

$$\bar{\sigma} \equiv \frac{s\delta[s+\theta(1-s)]}{\Theta(1-\delta)\theta(1-s)}, \quad \bar{\epsilon}_{lw} \equiv \frac{\sigma s\delta}{\Theta(1-\delta)\theta(1-s)(\sigma-\bar{\sigma})}$$
$$\sigma_{sup} \equiv \frac{(1-s)(1+\Theta)}{\Theta}, \quad \underline{\epsilon}_{lw} \equiv \frac{\sigma_{sup}-\sigma}{s}$$

Applying the same technique as in the Proof of Theorem 2 we can compute the Hopf bifurcation value

$$\epsilon_{cc}^H = \frac{\frac{(1-\beta)(1-\delta)(1-s)}{\sigma s\delta}(\sigma-\sigma^H)(\epsilon_{lw}^H-\epsilon_{lw})}{\frac{1-\beta}{\theta}\frac{(1-\delta)(\theta-s\beta\delta)}{s\delta}\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)}$$

the flip bifurcation value

$$\epsilon_{cc}^F = \frac{\theta(1-s)(\sigma-\sigma^F)(\epsilon_{lw}^F-\epsilon_{lw})}{s(\theta-s\beta\delta)(\epsilon_{lw}-\hat{\epsilon}_{lw})}$$

and the transcritical bifurcation value

$$\epsilon_{cc}^T = \frac{\epsilon_{lw}[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}+\mathcal{C}\Theta(\sigma_{sup}-\sigma)]}{s\Theta(\epsilon_{lw}-\underline{\epsilon}_{lw})}$$

with

$$\hat{\sigma} \equiv \frac{\delta\theta(1-s)(1+\Theta)}{\Theta[2(1+\beta)(1-\delta)+\theta\delta]}, \quad \hat{\epsilon}_{lw} \equiv \frac{\hat{\sigma}-\sigma}{s}$$

$$\sigma^H \equiv \frac{s\delta[\Theta\theta+(1-\beta)[s+\theta(1-s)]]}{\Theta(1-\beta)(1-\delta)\theta(1-s)}, \quad \epsilon_{lw}^H \equiv \frac{\left(1+\frac{1-\beta}{\Theta\theta}\right)\sigma s\delta}{(1-\beta)(1-\delta)(1-s)(\sigma-\sigma^H)}$$

$$\sigma^F \equiv \frac{2[(1+\beta)[s+\theta(1-s)]+\Theta\theta]+\frac{\theta(\theta-s\beta\delta)\left[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}+\mathcal{C}\Theta\sigma_{sup}\right]}{s}}{\Theta\frac{\theta(1-s)}{s\delta}[2(1+\beta)(1-\delta)+\theta\delta]}, \quad \epsilon_{lw}^F \equiv \frac{2(1+\beta+\Theta\theta)\sigma s\delta}{\Theta\theta(1-s)[2(1+\beta)(1-\delta)+\theta\delta](\sigma-\sigma^F)}$$

Assuming $\sigma > \hat{\sigma}$ implies $\hat{\epsilon}_{lw} < 0$ so that the existence of ϵ_{cc}^F relies only on the bounds σ^F and ϵ_{lw}^F. Straightforward computations show that as long as $\sigma < \sigma^H$, $\mathcal{X}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) < 0$ and thus $\mathcal{T}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) < 2$ for any $\epsilon_{lw} \geq 0$. We need finally to provide a condition to get $\mathcal{T}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) > -2$ or equivalently $\mathcal{X}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) > -4$ for any $\epsilon_{lw} \geq 0$. Straightforward computations give

$$\mathcal{X}(\epsilon_{cc}^H, 0, \epsilon_{l\lambda}, \sigma, \Theta) = -\frac{\delta(1-\beta+\Theta\theta)(\sigma_{sup}-\sigma)}{\beta\sigma(1-\delta)\Theta}$$

We then derive that if $\sigma > \tilde{\sigma}_{inf}$ with

$$\tilde{\sigma}_{inf} = \frac{(1-\beta+\Theta\theta)\delta(1-s)(1+\Theta)}{\Theta[4(1-\delta)\Theta\beta+\delta(1-\beta+\Theta\theta)]} \in (\sigma^F, 2)$$

then $\mathcal{X}(\epsilon_{cc}^H, 0, \sigma, \Theta) > -4$ and $\mathcal{X}(\epsilon_{cc}^H, \epsilon_{lw}, \sigma, \Theta) > -4$ for any $\epsilon_{lw} \geq 0$. Notice that the bound obtained here is different from the one derived in the Proof of Theorem 2 as we consider a specific value $\epsilon_{l\lambda} = \epsilon_{lw}$ which modifies the computations.

□

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