# Social Shock Sharing and Stochastic Dominance

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Aix-Marseille Univ, CNRS, EHESS, Centrale Marseille, AMSE, Marseille, France.

E-mail: christophe.muller@univ-amu.fr.

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**Abstract:**

Since the seminal paper of Atkinson and Bourguignon (1982), little decisive progress has been achieved in developing empirically efficient stochastic dominance criteria for multidimensional social welfare analysis. By proposing new axioms of 'Social Shock Sharing', this paper provides new intuitive justifications to imposing sign restrictions on partial derivatives of individual von Neumann-Morgenstern utility functions. These new breakthrough findings are exploited to derive necessary and sufficient stochastic dominance criteria for multidimensional social welfare comparisons, up to the sixth order, at least. Equivalent results are derived in terms of multidimensional poverty conditions. Empirically powerful discriminatory criteria are obtained by combining all social shock sharing axioms up to some high order and by deriving a dimension reduction property. An application to Egypt at the beginning of the $XXI^{st}$ century demonstrates the practical substantial gain in discriminating power of the approach by revealing a unambiguous continual improvement in bivariate income-education social welfare over the studied period.

# 1  Introduction

This investigation will deal with comparisons of inequality, poverty and social welfare across society situations that are characterised by several individual well-being attributes. For example, income and education are typically invoked as distinct relevant dimensions of individual well-being. The aim of this research is to tackle the poor performance of the currently available comparison criteria, from theoretical and empirical points of view, which has so far limited their practical use in economics.

In this context, robust 'dominance' judgments are used in order to attain a consensus among people who may accept different norms. For this, stochastic dominance theorems have been available for a long time (Hardy, Littlewood and Polya, 1929), although their usefulness for economic and social comparisons of one-dimensional income distributions only emerged much later (Atkinson, 1970; Kolm, 1969). In this approach, a utilitarian social welfare function is generally specified and used as an evaluation benchmark, and consensual normative hypotheses can be introduced through variational properties of the

individual utility functions (e.g., aversion to income inequality). The theorems that have been derived in this literature provide a complete toolbox of efficient criteria to conduct empirically normative comparisons of one-dimensional income distributions, and they have been used extensively. Yet, even for one-dimensional problems, it is fair to say that only first and second order stochastic dominance theorems have been accepted as normatively justified, and perhaps third order theorems by some authors (e.g., Zheng, 1999) on the grounds of diverse transfer sensitivity axioms.

However, the current state of affairs is much less satisfactory for comparisons based on multidimensional attributes of individual well-being. Despite considerable efforts in the literature, it has proven to be much harder to design empirically powerful comparison criteria. This is related to some kind of 'multidimensionality curse' as each such used normative condition, for example income inequality aversion, fails to account for the variety of interactions (in the generation of the individual wellbeing index) between the different dimensions of wellbeing, and this variety extends fast with the number of dimensions. One primary objective of this research is to break this wall of discriminatory power that prevents a general use of robust dominance criteria in multidimensional settings.

In typical multidimensional welfare analyses, the marginal utility functions, with respect to each attribute, are assumed to be identical across agents; for example owing to some 'anonymity' axioms[1]. In that case, these functions are, therefore, convenient instruments that can be used to impose normative theoretical restrictions liable to be generally accepted. These marginal utility functions are generally supposed to be non-negative and non-increasing, which, respectively, reflect, on the one hand, the well-being benefits brought by the attributes, and, on the other hand, some plausible hypotheses of inequality aversion associated with each attribute. However, these assumptions alone

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[1]In some cases, this can be justified by controlling explicitly for some differences in needs or in 'types', such as in Atkinson and Bourguignon (1987), Jenkins and Lambert (1993), Chambaz and Maurin (1998), Moyes (2013).

have not allowed researchers to achieve stochastic dominance criteria that are discriminating enough to make them efficient guides for empirical economic policy. This is why some researchers have attempted to reinforce these decision rules through additional hypotheses on signs of higher derivatives of utility functions.

Multidimensional stochastic dominance criteria for social welfare and inequality analyses were put forward by Kolm (1977) and Atkinson and Bourguignon (1982). Their criteria[2] were based on utility functions that were constrained by the signs of some of their partial derivatives up to the fourth order. However, as rightly stated in Atkinson (2003), the conditions involving fourth-order derivatives have been recognised as not being easy to interpret, and therefore to justify. On the other hand, criteria based on lower order derivatives have been found to have insufficient discriminating power in empirical work (e.g., in Muller and Trannoy, 2011, with some third-order derivatives).

Another relevant, and somewhat isomorphic, literature is that of decisions under uncertainty modelled with the expected utility criterion. In that case, multidimensional risks for an individual decision maker are considered instead of wellbeing attributes in a population. Important generalisations of risk aversion notions and technical advances have been produced in this literature that allows for a variety of stochastic dominance results depending on assumed properties of the von Neuman-Morgenstern utility function[3]. However, it seems even harder to justify broadly accepted high order conditions on the shape of the utility function in the risk context, as found empirically in Deck and Schlesinger (2014).

Besides, returning to the social welfare context, only criteria based on partial derivatives up to the third order, at most, are typically used empirically[4]. As mentioned previously, this happens because normative justifications have been found lacking to

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[2]Their main criterion is stated below in Theorem 3 of subsection 4.3.

[3]Fishburn and Vickson (1978), Gollier and Pratt (1996), Eeckhoudt and Schlesinger (2006), Eeckhoudt, Rey and Schlesinger (2007).

[4]For example, in Bazen and Moyes (2003), Muller and Trannoy (2011, 2012), Gravel and Moyes (2012).

justify pushing the analysis to higher orders. Despite this, Duclos and Echevin (2008), Duclos, Sahn and Younger (2006, 2011) conduct robust social welfare empirical comparisons, including with Atkinson and Bourguignon conditions based on up to fourth-order derivatives, even though without precise normative justifications[5]. This suggests that improved discriminatory power obtained this way can be attractive for empirical researchers. However, although enhancing the performance of the available empirical decision rules is important, only limited progress in this direction has been achieved over the last thirty six years.

The solution here proposed to escape this predicament is to make the expression of social solidarities explicit through a new definition of 'social welfare shocks', or social harm caused by individual shocks, and how they can be considered normatively. For this, a theoretical setting is needed that specifies how social solidarities may operate normatively face to shocks that may affect diverse dimensions of individual well-being. Individual shocks may affect health, education, income, physical safety, employment, family issues, environment, prices, and so on.

Considering adverse shocks of distinct nature may be particularly relevant for the poor or the vulnerable who may not be able to cope on their own with all these shocks. In that case, institutions implementing social solidarity, whether they are traditional or modern ones, are needed that can deal with mixtures of shocks. Examples of such institutions include public systems of social security, social networks, clans and families. These institutions can assist households by sharing shocks among citizens, whether these shocks are random or not, either through ex-post compensation devices, such as cash

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[5]Some authors (e.g., Kolm, 1976a,b, Fishburn and Willig, 1984) have long ago proposed empirical stochastic dominance applications based on even higher derivatives orders; for example, by multiplying poverty headcounts or poverty gap indices at the individual level and aggregating them. This is also typically the case in the specific literature dealing with one-dimensional problems, which even uses up to an infinite sequence of derivative conditions (e.g., Gayant and Le Pape, 2017). Again, normative justifications are needed in order to better settle these practices.

transfers and emergency assistance, or through ex-ante insurance or protection policies. Modern social security systems have become increasingly complex and sophisticated over the history. In particular, they can now simultaneously address many different kinds of risks, handicaps, inequalities and other shocks. Besides, such extensive capacity may also have been the case for traditional solidarity mechanisms that are typically not specialized into dealing with one specific kind of shocks exclusively. Therefore, investigating the extent to which multidimensional shocks should be socially shared, and how to account for them normatively in welfare analysis, is likely to respond to practical policy issues, as well.

More generally, for social welfare analysis, a new method is put forward here to specify normative restrictions that allow for social solidarity. One could consider that some limited intuitions about social solidarity in welfare analysis are already somewhat depicted by diverse axioms of transfer and compensation-substitution that have been proposed in the literature (e.g., in Silber, 1999). In this work, the analysis is carried one step further by introducing intuitions about social solidarity that are stated in terms of the social sharing of some individual shocks. To do this, some new normative axioms of 'welfare shock sharing' are proposed. These axioms are then used to justify *separately* many normative conditions expressed in terms of variational properties of the utilities of individuals faced by random or non-random shocks. In a generalised utilitarian setting, proofs are provided for the equivalence of these conditions with restrictions on signs of partial derivatives of the utility functions, including high-order derivatives. In particular, this allows for the characterisation of normative conditions on utilities that have never been considered before.

In a second stage, several natural sets of utility functions are considered that are defined in terms of signs of partial derivatives up to the fourth order, and later to the sixth order. For each of these sets, necessary and sufficient stochastic dominance conditions are derived that will allow the decision maker to compare the levels of multidimensional social welfare between two arbitrary empirical situations. Equivalent results are also

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reported in the form of generalised poverty gap conditions.

In addition to all these general results, I also propose a specific solution to the limited empirical power of multidimensional stochastic dominance tests. This is based on combining all social shock sharing axioms at the fourth order, and even at the fifth and sixth order if needed. In this case, I can prove a property of dimension reduction. Therefore, not only the multidimensionality curse, but also high order conditions can be addressed, which is what delivers the empirical power of the obtained stochastic dominance tests.

The next section presents the setting. Section 3 discusses the new normative axioms of welfare shock sharing. Section 4 reports the new stochastic dominance theorems. Section 5 proposes an empirical application to changes in bivariate income-education social welfare in Egypt at the beginning of the twenty-first century. Finally, Section 6 will conclude this paper. The proofs not discussed in the text are relegated in the appendix.

# 2   The Setting

An analytical setting is now presented that describes the distribution of the effects of welfare shocks in the population. In the literature, shocks are generally assumed to be additive to the attributes, and this approach is followed. Shocks are often seen as being random, although this is not necessary. Assuming randomness of shocks may be justified for accounting for current ignorance, individuals' anticipations of future uncertain events, or some similar ignorance or anticipations by a social planner, or even by analysts.

The specification of the analytical setting is presented in several stages. First, the attributes and the corresponding shocks are defined along with their distributions. Second, for any given individual, a utility level is specified that represents its wellbeing,

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possibly up to a positive affine transformation[6]. This utility level is a function of the attribute levels and of the shocks, thus accounting for the impact of the shocks on the individual well-being. Third, a social welfare evaluation function is chosen to aggregate the individual utility levels over the whole population.

To simplify the presentation, the discussion focuses on the bivariate case, while most of it is also clearly valid for higher dimensions. A simple example is when the first argument, $x$, of the individual utility function is the income, while the second argument, $y$, is the education level. Assume that these vectors take values on the rectangle $[0, a_1] \times [0, a_2]$, where $a_1$ and $a_2$ are in $\bar{\mathbb{R}}_+^*$. Let $F(x, y)$ denote the joint cumulative distribution function of $x$ and $y$. The cdf $F(x, y)$, which is assumed to be continuous, describes the distribution of the realised wellbeing attributes over a population of interest, rather than a distribution of risks. $F_x(x)$ and $F_y(y)$ denote the respective marginal cdfs of $x$ and $y$.[7]

As mentioned above, some information about shocks will be included in the individual utility. In that case, for example, the ex-ante individual well-being can be described by an expected von Neuman-Morgenstern (VNM) utility index, $EU(x, y)$. The use of the notation $EU(x, y)$ here means that the random shocks on $x$ and $y$ are incorporated in the individual well-being index, through the usual expected utility model. However, other approaches are also possible that do not require the expected utility hypothesis. To simplify the discussion, let us also assume that all individuals have the same individual utility function, for example by accounting for individual heterogeneity in this function through equivalence scales.

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[6]Most notions or results in this paper, including the welfare premium function, are therefore subject to this cardinality property.

[7]Instead of assuming continuous cdfs, it would be possible to work with general Lebesgue integrals. However, this would imply tedious discussions of the derivability in diverse Lebesgue integrals occurring in the proof. Moreover, using Lebesgue integrals implies to allow for the possible presence of exceptional discrete terms when using integration by parts, without bringing anything substantial to the argument. Therefore, I choose to stick to the continuous specification that avoids being distracted by unsubstantial technical details.

Let us now examine how welfare shocks may affect the social evaluation of the situation of the society through their impact on the distribution of the individual utility levels that depend on these shocks. Let $W_F$ be an additively separable social welfare function, associated with a joint cdf $F$ of well-being attributes and with some welfare shocks, among which some may be random. In this case, the risks associated with the random shocks have to be incorporated within the social evaluation criterion. A natural way to do this is through an ex ante individual utilitarian criterion.

Several interesting cases can be examined specifically. In the case of the expected individual utility, $EU(x, y)$, being the individual well-being index, these indices may still be integrated over the population joint cdf $F$, which in that case is a cdf over a population domain of random vectors, $(x, y)$, of bivariate random wellbeing characteristics that may incorporate some welfare shocks.

$$W_F = \sum_{i=1}^{N} EU(x_i, y_i) = \int EU(x, y) \, dF(x, y),$$

where $U$ is the considered (cardinal) VNM utility function, and $N$ is the number of individuals in the displayed discrete representation. When there is no randomness, or when one chooses an ex-post perspective instead, one obtains the usual utilitarian social welfare function over the population joint cdf $F$ of fixed wellbeing characteristics:

$$W_F = \int_{[0,a_1] \times [0,a_2]} U(x, y) \, dF(x, y).$$

Although the main application will be the case of the individual well-being index defined as an ex-ante expected utility, some of the axioms proposed below will also be useful in other cases. Using the expected utility $EU$ as a normative basis for measuring individual well-being is far from mandatory. In some close variants, some other increasing transformations of $EU$ can be used instead to separate the normative individual components of the social evaluation function from the norms of the social planner. However, directly including expected utility indices in the utilitarian social evaluation

function is still the normalisation that is used in most empirical and theoretical work when dealing simultaneously with risk and social welfare issues. Then, I shall follow this common practice to avoid mixing the main argument of this investigation with other issues of the social welfare analysis under uncertainty. However, similar arguments and results than those developed in this paper can easily be developed with more general specifications, such as sums of increasing function of expected utilities, if wished.

The starting point in the discussion is the case of no randomness. From this benchmark, the incorporation of social welfare shocks, including some random shocks, will be examined. For this, all of the partial derivability properties of $U$ are assumed that will be needed to express the results, in each case. Moreover, throughout this paper, it is assumed that all of the considered integrals are bounded to avoid absurdities.[8]

Let $\triangle W_U := W_F - W_{F^*}$ be the change in social welfare between any two joint distributions $F$ and $F^*$. Then, $\triangle W_U = \int \int U(x,y) d\triangle F(x,y)$, where $\triangle F$ denotes $F - F^*$. Social welfare dominance corresponds to unanimity over a given set $\boldsymbol{U}$ of utility functions $U$. Stochastic dominance of distribution $F$ over distribution $F^*$ is now defined in this context.

**Definition 1** *F dominates $F^*$ for a family $\boldsymbol{U}$ of utility functions if and only if $\triangle W_U \geq 0$, for all utility functions $U$ in $\boldsymbol{U}$. This is denoted $F\ D_{\boldsymbol{U}}\ F^*$.*

To be more specific about dominance relationships, a few relevant sets $\boldsymbol{U}$ of utility functions must be specified. This is done in the next sub-section.

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[8]If $a_1 = +\infty$ or $a_2 = +\infty$, then some integrals may not be well-defined for certain theoretical distributions, even with non-random variables $x_1$ and $x_2$, such as if $F(x_1, x_2)$ has heavy tails. This may also be the case for some integrals of some partial derivatives of utility arising in the expansions in our proofs. However, these cases are of no or little empirical relevance.

## 2.1  A few utility sets of interest

From now, in order to alleviate notations, the partial derivatives will be denoted by using as subscripts the indices of the attributes (1 and 2), repeated as many times as there are derivations with respect to the considered attribute. For example, $U_{1122}$ denotes $\frac{\partial^4}{\partial x^2 \partial y^2} U$.

Conditions on signs of partial derivatives were introduced progressively in the literature. Levy and Paroush (1974) were the first, to the best of my knowledge, to propose to use the condition $U_{12} \leq 0$. Atkinson and Bourguignon (1982) proposed several classes of utility functions. Their largest class is defined by functions satisfying $U_1, U_2 \geq 0, U_{12} \leq 0$, while their smallest class is defined by the same restrictions to which are added: $U_{11} \leq 0, U_{22} \leq 0$, $U_{112} \geq 0, U_{221} \geq 0$ and $U_{1122} \leq 0$. Several authors have proposed classes with intermediate sets of restrictions.[9]

To increase the power of the stochastic dominance tests, one may want to assume as many restrictions as possible. Let $\boldsymbol{U}$ be the class of the increasing utility functions that satisfy the following restrictions of signs for the partial derivatives.

$$
\begin{aligned}
\boldsymbol{U} \;=\; & \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{22} \leq 0,\ U_{12} \leq 0, \\
U_{111} \;\geq\; & 0, U_{222} \geq 0, U_{112} \geq 0, U_{221} \geq 0, & (1) \\
U_{1111} \;\leq\; & 0, U_{2222} \leq 0, U_{1122} \leq 0, U_{1112} \leq 0, U_{1222} \leq 0\}. & (2)
\end{aligned}
$$

This class involves a complete set of sign restrictions on partial derivatives up to the fourth order. Other conditions with opposite signs could also be considered, although they would yield rather counter-intuitive meanings (e.g., individuals loving income inequality with $U_{11} \geq 0$). Therefore, only these most relevant signs are considered for the analysis.

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[9]Moyes (1999a), Bazen and Moyes (2003), Gravel and Moyes (2012), Muller and Trannoy (2011, 2012).

A short comment on what these restrictions are typically believed to mean may be useful. The non-negativity of the first-order derivatives of $U$ implies that each attribute positively contributes to utility, or at least is not noxious to it. The non-positivity of each direct second-order derivatives may be seen as expressing inequality aversion, with respect to each attribute, in utilitarian social welfare settings, or expressing one-dimensional risk aversion when there are risky shocks assessed by using expected utility. These second-order conditions correspond to the concavity of the utility function in the direction of each attribute separately.

The hypotheses $U_{12} \leq 0$, $U_{112} \geq 0$ and $U_{122} \geq 0$ can be justified, as in Muller and Trannoy (2011, 2012), by invoking normative compensation arguments. In this case, one attribute can be assumed to serve as a compensating variable to redress inequality with respect to the needs in the other attribute. For example, assuming that the first argument is income, with $U_{12} \leq 0$, then the more destitute in education an individual is, the higher the claim for compensating income transfers. Moreover, with $U_{112} \geq 0$, this claim is all the more vindicated that potential transfer beneficiaries are poorer. There are other possible normative justifications of these hypotheses. For example, $U_{12} \leq 0$ can be seen as embodying aversion for correlations between attributes, as in Tsui (1999). So far, normative justifications of signs of the fourth-order or higher derivatives are missing in the literature. This paper will fill this gap.

Class $\boldsymbol{U}$ can be seen as corresponding to some 'maximum requirement' in terms of the signs of the partials because its definition gathers a complete set of restrictions on signs up to the fourth order. However, other utility classes can be considered that involve fourth-order partial derivatives, without including all the conditions that appear in the above definition of Class $\boldsymbol{U}$. For example, the hypotheses in Atkinson and Bourguignon (1982) correspond to the class: $\boldsymbol{U}^{--} = \{U_1, U_2 \geq 0, U_{11}, U_{22}, U_{12} \leq 0, U_{121}, U_{212} \geq 0, U_{1122} \leq 0\}$. The next section normatively justifies all these restrictions by introducing new axioms that are based on the innovative notion of welfare shock sharing.

# 3   Normative Justifications

## 3.1   Welfare shock sharing

First, some damages will be specified, which are called 'shocks', that affect individuals and that the social planner should consider while assessing social situations. Then, it will be stated when sharing these shocks across individuals should be judged as normatively good. In that sense, these shocks can be seen as 'social shocks' because they have social welfare consequences, notably in terms of protection or redistribution policies. The shocks may occur randomly or not.

To fix these ideas, let us consider a social situation that is described by some endowments $(x, y)$ to individuals in a population. For example, $x$ may be the income and $y$ the education level, which are both non-negative variables. Of course, any other well-being attributes could be considered if wished[10]. Let us further assume that society is only composed of two individuals, and let us examine the social planner's preferences for equity across individuals. For example, a planner who is reluctant to see the same individual bearing all the shocks would prefer the social situation, or 'society', $\{(x - c, y); (x, y - d)\}$, where the first individual has the endowments $x - c$ in the first attribute and $y$ in the second attribute, and the second individual has the, respective, endowments equal to $x$ and $y - d$, with $c$, $d > 0$ being fixed losses, to the social situation, in which the same individuals have the respective endowments $\{(x, y); (x - c, y - d)\}$, such as one of the two individuals would suffer all the losses. That is: in that case, one would naturally like to state normatively that the social planner prefers a situation where the allocation of the shocks is 'shared' among individuals. This specific normative inclination is denoted below by 'welfare correlation aversion'. However, other types of shocks can be considered, which correspond to other ways of sharing shocks among the

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[10]This may also include the variables describing individual needs instead, although in the latter case the signs of derivatives we use in this paper would have to be changed accordingly, as in Bourguignon (1989) for example.

individuals.

For understanding this, a few axiom definitions are now stated that describe how the social planner preferences should prefer shock sharing in several specific cases. Let us start again with two individuals with the same bivariate endowments, $(x, y)$. The effects of diverse individual shocks are considered from the point of view of the preferences of the social planner. Each of the following definitions may be seen as a plausible normative axiom about social shock sharing. The last axiom involves four individuals.

**Definition 2** *Welfare Shock Sharing Axioms: Let $(x, y) \in R_+^2$ be any non-random buvariate well-being endowments. Let $c$ and $d > 0$. Let $\varepsilon$ be a centred numerical random variable and $\delta$ be a centred numerical random variable independent of $\varepsilon$.*

*($i$) A social planner is said to be welfare correlation averse if $x - c > 0$ and $y - d > 0$ implies that the social planner prefers the society $\{(x - c, y); (x, y - d)\}$ to the society $\{(x, y); (x - c, y - d)\}$. That is: 'sharing fixed losses affecting different attributes improves social welfare'.*

*($ii$) A social planner is said to be welfare prudent in $x$ if $x + \varepsilon > 0$ and $x - c > 0$ implies that the planner prefers the society $\{(x - c, y); (x + \varepsilon, y)\}$ to the society $\{(x - c + \varepsilon, y); (x, y)\}$. That is: 'sharing a fixed loss and a centred risk affecting the same first attribute improves social welfare'.*

*($iii$) A social planner is said to be welfare cross-prudent in $x$ if $y + \delta > 0$ and $x - c > 0$ implies that the planner prefers the society $\{(x, y + \delta); (x - c, y)\}$ to the society $\{(x, y); (x - c, y + \delta)\}$. That is: 'sharing a fixed loss in the first attribute and a centred risk in the second attribute improves social welfare'.*

*($iv$) A social planner is said to be welfare temperate in $x$ if $x + \varepsilon > 0$ , $x + \delta > 0$ and $x + \delta + \varepsilon > 0$ implies that the planner prefers the society $\{(x + \delta, y); (x + \varepsilon, y)\}$ to the society $\{(x, y); (x + \delta + \varepsilon, y)\}$. That is: 'sharing centred risks affecting the same first attribute improves social welfare'.*

*($v$) A social planner is said to be welfare cross-temperate if $x + \varepsilon > 0$ and $y + \delta > 0$*

*implies that the planner prefers the society $\{(x+\varepsilon, y); (x, y+\delta)\}$ to the society $\{(x, y); (x+\varepsilon, y+\delta)\}$. That is: 'sharing centred risks affecting different attributes improves social welfare'.*

*(vi) A social planner is said to be welfare-premium correlation averse in $x$, if $x+\varepsilon > 0, x-c+\varepsilon > 0$ and $y-d > 0$ implies that the planner prefers the society $\{(x-c, y); (x, y-d); (x+\varepsilon, y); (x+\varepsilon-c, y-d)\}$ to the society $\{(x, y); (x-c, y-d); (x+\varepsilon-c, y); (x+\varepsilon, y-d)\}$. That is: 'sharing fixed losses affecting different attributes improves social welfare, while less so under background risk in the first attribute'.*

In addition, the usual definitions for monotonicity and inequality aversions, with respect to each attribute, could be explicitly stated and included in the list of axioms. However, they are omitted here since they are trivial. Symmetric definitions of those above can, of course, also be obtained by substituting the roles of $x$ and $y$, and they are also omitted. Furthermore, weaker conditions than the independence could be stated for the random shocks $\varepsilon$ and $\delta$, as in Brown (2017) for example, but doing so would rather obscure the intuitive argument in the discussion, and I prefer to refrain from this. Finally, the list of axioms could be extended by considering more complex shocks, such as composite lotteries, and/or a higher number of individuals in society.

In Axiom (vi), society can be split into two subgroups of two individuals each. The welfare of the first two individuals may be compared across the two considered societies according to the criterion of welfare correlation aversion, which induces a preference for sharing fixes losses. In that case, the situation of this first couple of individuals is clearly preferable in the first society than in the second. However, in this case, the last two individuals are given the opposite ranking for the same reason, because in the second society they share the same fixed losses (as those for the first two individuals), while they do not do it in the first society. Even though the last two individuals suffer from the same respective levels of losses as the first couple, and they have the same initial well-being attributes, they also both suffer from an identical centered random shock on the

first attribute. The result of the comparisons of the two 4-individual societies is deemed to be in favour of the first society, as a definition of the axiom. The intuition behind this is that the presence of the random shocks makes feeling the losses less severely. For example, in the case of the expected utility criterion, integrating $U$ with respect to the random shock makes the resulting 'indirect utility' less concave in $x$ than the initial $U$, and therefore less sensitive to income inequality.

As will be shown in Subsection 3.2, under the expected utility hypothesis, stating Axiom *(vi)* corresponds to stating Axiom *(i)* with the utility function replaced by the *welfare premium function* in $x$, that is: $p^x(x, y, \varepsilon) \equiv U(x, y) - EU(x + \varepsilon, y)$.

Some of the stated definitions are formally akin to notions that have been used in risk analysis (prudence, temperance). That is why a similar vocabulary is proposed, even though there are notable differences between welfare and risk contexts. First, social welfare comparisons are examined instead of individual decisions under risk. Second, the notions are here initially defined in terms of preferences over society situations characterised by the joint distribution of well-being attributes, without invoking, at that stage, any representative function of individual preferences (such as utility functions). Although the particular application that will be later examined is in terms of expected utility, there is no reason why welfare shock sharing axioms should be considered and used only under this hypothesis. Third, the kind of risk apportionment that has been used in the literature on risk, is in many cases, formally different from the shock sharing formulae, even by combining 'good' and 'bad' states[11]. Fourth, for some proposed notions (welfare cross-temperance, welfare premium correlation aversion), it has not been possible to find any analogue in the risk literature, even with broadened interpretation[12].

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[11]e.g., in Eeckhoudt and Schlesinger (2006), Tsetlin and Winkler (2009), Denuit and Eeckhoudt (2010), Li, Liu and Wang (2016).

[12]However, a working paper by Crainich, Eeckhoudt and Le Courtois (2013) was signalled to me by Pr. Eeckhoudt. In this paper, which deals with risky choices, for example portfolio choices, the authors propose and analyse an index of absolute correlation aversion (analog of $-u_{12}/u_1$) and an index of cross downside risk aversion (respectively, $u_{122}/u_1$), and they assume that these indices are decreasing

Fifth, I am not aware of any risk notion based on the fifth or the sixth orders, as are developed below in sub-section 2.4 for social welfare analyses. Sixth, the anonymity axioms that are typically used in the social welfare literature imply that the notions must be robust to some specific changes in the positions of the individuals. This is expressed by describing social alternatives about sets of individual situations instead of n-uplets in some risk problems in which present and future states may not be substitutable. Seventh, the above definitions of the welfare premium and of the social cost of shocks involve comparing individuals (e.g., under a veil of ignorance) rather than comparing random states. Eighth, all of the attributes are supposed to be non-negative, as is typical for welfare attributes, but is not the case for financial returns, for example. Finally, by invoking feelings of social justice, the proposed axioms seem to be much more likely to gather agreement among researchers and practitioners than similar axioms that are defined for risk apportionment, which look more arbitrary (e.g., temperance). This is the relevance of these 'social' feelings that makes possible to climb up the orders of utility derivative conditions.

Note that some axioms (i.e., $(ii)$ and $(iv)$) can be defined independently of the presence of the other attribute, provided that the latter is fixed at a given level. Further, the axioms of welfare shock sharing do not depend on assuming specific levels of the endowments $(x, y)$ since they are defined for any such endowments above minimal thresholds. Note that for each of the defined axioms, some specific initial endowment levels must be excluded in order to have the ex-post endowments reaching a minimal threshold. Indeed, almost all of the well-being attributes that one can think of have a natural non-negative lower bound. For example, education levels cannot be negative and consumption expenditure cannot be below some subsistence minimum. Then, the distributions of the corresponding shocks in the axioms must be truncated accordingly.

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in wealth. Therefore, it seems clear that it should be possible to develop indices of risk attitudes even with higher order. Whether the indices can be broadly accepted and used by risk analysts is another matter.

The axioms are defined for all $c, d, \varepsilon, \delta$ that satisfy these domain constraints. However, only one given version of these parameters and shocks is necessary in these definitions, which correspond therefore to broader preferences sets, than if they were stated for all $c, d, \varepsilon, \delta$. Yet, no substantial consequences follow from this in differentiable cases: what matters is the 'marginality' of the variations, not the way it is expressed in terms of parameters, unless there is a special interest in some given specific shocks.

As mentioned previously, a few of the new welfare shock sharing axioms can be somewhat connected to risk-apportionment notions in the literature. Eeckhoudt and Schlesinger (2006) introduce risk-apportionment techniques to characterise one-dimensional prudence and temperance notions. Eeckhoudt, Rey and Schlesinger (2007) and Jokung (2011) extend these notions to bivariate settings. Eeckhoudt and Schlesinger (2006), and Eeckhoudt, Rey and Schlesinger (2007) characterise von Neuman-Morgenstern utility functions for expected utility criteria by using prudence, temperance, correlation aversion, cross-prudence and cross-temperance notions. However, in Eeckhoudt and Schlesinger (2006) and in Jokung (2011), 'high-order' risk preferences are constructed through a binary recurrence process over lotteries. While this is clearly distinct from the approach in this paper, some formulae coincide for low order notions such as risk aversion. Indeed, Eeckhoudt and Schlesinger (2006), and Jokung (2011) define their risk-apportionment notions from a sequence of risks, each of which is defined recursively. In contrast, the welfare shock sharing axioms here are directly defined without using a recurrent sequence of risks, which leads to different formulae.

Note that there is no compelling reason why expected utility criteria of individual welfare, or monotonous transformations of them, should be chosen as necessary building blocks of the social welfare setting, even when some shocks are random. However, this special case should of course be included in many plausible settings and it is pursued from now on. Besides, there may be additional specific reasons for considering simple criteria that are linear in some kind of utility functions. For example, a von Neumann-Morgenstern ranking of socially risky situations that is consistent, in the Pareto sense,

with individual VNM utilities, may be seen as resulting from comparing the sum of the individual's VNM expected utility functions across alternatives (Harsanyi, 1955). Other aggregation theorems (Weymark, 1991 and 1993, Danan, Gajdos and Tallon, 2015) yield similar linear constructions, including for incomplete preferences. Therefore, from now on, an aggregate social decision criterion will be used that is based on sums of the expected utilities of the individuals (or sums of given monotone transformations of these expected utilities).

The first result of social welfare analysis is as follows. In social welfare contexts with the social evaluation function additive in the individual expected utility functions (or a given monotone transformation of them), the axioms of welfare shocks sharing can be characterised in terms of restrictions of the signs of the partial derivatives of the utility function, up to the fourth order, as shown in the next theorem. Of course, such a setting, as in Atkinson and Bourguignon (1982), relies on an inevitable subjacent cardinality hypothesis of the utility.

**Theorem 1** *If some expected utility represents the individual welfare with a VNM utility function $U(x, y)$, and with a social welfare function additive in utilities, one has:*

*(a) Inequality aversion in $x$ is equivalent to $U_{11} \leq 0$. An alternative interpretation that is now proposed as a consequence of the welfare shock sharing perspective is that of preference for sharing fixed losses in the first attribute.*

*(b) Welfare correlation aversion is equivalent to $U_{12} \leq 0$.*

*(c) Welfare cross-prudence in $x$ is equivalent to $U_{122} \geq 0$.*

*(d) Welfare prudence in $x$ is equivalent to $U_{111} \geq 0$.*

*(e) Welfare cross-temperance is equivalent to $U_{1122} \leq 0$.*

*(f) Welfare temperance in $x$ is equivalent to $U_{1111} \leq 0$.*

*(g) Welfare-premium correlation aversion in $x$ is equivalent to $U_{1112} \leq 0$.*

*(h) The expected utility hypothesis is not necessary for obtaining results (a) and (b); and in the other cases this hypothesis needs only apply to the attributes for which there*

*is at least two derivations of the VNM utility function in the stated sign condition.*

Obviously, similar properties can be obtained by substituting $x$ and $y$, and their statements are omitted. The proofs are given in the appendix. They rely on the fact that, on an interval, the corresponding finite variations and derivatives have the same sign when the sign of the derivatives is constant. Similar equivalence can be proven for higher order notions. While they are omitted here for the sake of shortness, some of them will be briefly analysed in subsection 4.4. Note that the properties of differentiability that have been assumed are less restrictive that it appears once these generalised concavity assumptions are made. Indeed, a numerical convex function of a real variable has only a counting number of non-differentiability points (Rockafeller, 1970, p. 244). This property extends to the above assumptions.

In all these axioms, the feelings about social shock sharing that they state imply features of the shapes of the utility functions that have consequences for the comparison of society situations even when there are no observed shocks. The interpretation in terms of shock sharing in (a) does not seem to have appeared so far in the social welfare literature as an explicit axiom. Indeed, this literature rather invokes inequality aversion motives or transfer axioms. Note also that it is quite possible that $U_{11} \leq 0$ holds and not $U_{22} \leq 0$ (or the opposite) when the two attributes have distinct normative roles. For example, one could imagine a society prone to redistribute income while ignoring education differences for designing social policies.

Results (b) and (c), in the specific example with income and education, imply that cross-prudence in income, $U_{122} \geq 0$, along with correlation aversion, $U_{12} \leq 0$, can be seen as depicting a motive for compensation to alleviate inequity in education through income transfers in favour of little educated persons, as argued in Muller and Trannoy (2012).

Justifying normative restrictions in social welfare analysis by invoking Pigou-Dalton transfer-type arguments may be controversial when some attributes cannot be practically transferred, such as education status. Using instead normative assumptions based

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on welfare shock sharing axioms diminishes this difficulty. What the concrete mechanisms or institutions through which the shocks could be shared are is another matter, which is not examined in this article.

Although they were not pursued in the literature, a few other possible justifications of the above signs of partial derivatives of utility in Theorem 1 are now briefly reviewed, for the fourth order. First, the signs of some utility derivatives may be related to variations in aversion to inequality. For example, $U_{1122} \leq 0$ is equivalent to $U_{22}$ concave in $x_1$. In that case and in the example, decreasing income transfers increase aversion to education inequality (in terms of the concavity of $U$ in $x_2$). Second, the condition $U_{1111} \leq 0$ can be related to one-dimensional temperance and other analogue notions that have already been developed in the risk literature. For example, increasing 'outer inequality' could naturally be defined as corresponding to such a negative fourth derivative, as a generalisation and translation of Menezes and Wang (2005) for increasing outer risk. Other analogue notions that can be invoked to justify this sign restriction are: proper risk aversion (Pratt and Zeckauser, 1987), decreasing absolute prudence (Kimball, 1993), and risk vulnerability (Gollier and Pratt, 1996).

The next subsection returns to a new normative condition that was introduced above, and which is based on a fourth-order partial derivative that was not used before in social welfare analysis.

## 3.2   Welfare-premium correlation aversion

The welfare premium function that will be used corresponds to comparing two individuals whose individual well-being is assessed by using their expected utility (or any increasing affine linear transformation of it). The welfare premium is the amount of cardinal ('welfare') utility that an individual would accept to give up for avoiding being someone that suffers the random shock $\varepsilon$ as compared to being someone not bearing this shock. For example, this comparison can be seen as performed by the individual

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herself or himself under a veil of ignorance, so as to guarantee that individual idiosyncratic characteristics do not affect the result. Under expected utility with a concave VNM utility function, the welfare premium, in that sense 'socially', compensates for a possible well-being loss from the random shock $\varepsilon$ affecting the first attribute.

The conditions $U_{1222} \leq 0$ and $U_{1112} \leq 0$ have been left out in the social welfare literature. I now pursue their analysis that began with Theorem 1. Without loss of generality, let us consider $U_{1112} \leq 0$. The proof of result (g) in Theorem 1 is now spelt out.

Let $c$ be any fixed positive loss amount in the first attribute, and $\varepsilon$ be any given centred random shock in the same attribute, such that $x - c + \varepsilon > 0$ and $x - c > 0$, for all $x$. Let the *welfare loss function* be $w(x, y; c) \equiv U(x, y) - U(x - c, y)$, which describes the utility loss due to a fall in the first attribute. Let the Jensen's gap corresponding to function $w$ be: $v(x, y) \equiv w(x, y; c) - Ew(x + \varepsilon, y; c)$.

Consider the condition $v_2(x, y) \leq 0$, which is equivalent to saying that the *welfare premium* function $p^x(x, y, \varepsilon) \equiv U(x, y) - EU(x + \varepsilon, y)$ is subject to correlation aversion. Indeed, $v_2 = U_2(x, y) - U_2(x - c, y) - EU_2(x + \varepsilon, y) + EU_2(x + \varepsilon - c, y) = p^x_2(x, y, \varepsilon) - p^x_2(x - c, y, \varepsilon) \leq 0$, which is equivalent to $p^x_{12} \leq 0$. This condition is now described in two equivalent ways: first, by stating the sign of a fourth-order partial derivative of utility; and second, by comparing the total welfare outcomes for two specific societies that differ according to their patterns of shock sharing.

Indeed, $v_2(x, y) = w_2(x, y; c) - Ew_2(x + \varepsilon, y; c) \leq 0$ if and only if $w_2$ is concave in $x$. Assuming derivability when needed, this condition is equivalent to $w_{112} \leq 0$, for any levels of the attributes, that is: $U_{1112} \leq 0$, since finite variations and derivatives have the same constant sign on an interval and $w$ can be seen as expressing discrete variations of $U$ in $x$. Then, the condition $U_{1112} \leq 0$, which is to be characterised, is equivalent to $v_2(x, y) \leq 0$, for all $x, y$.

The next step consists in noting that one has $v_2(x, y) \leq 0$ for all $x, y, c, \varepsilon$ such that $y > 0, x - c + \varepsilon > 0$ and $x - c > 0$ if and only if

$w(x, y; c) - Ew(x + \varepsilon, y; c) - w(x, y - d; c) + Ew(x + \varepsilon, y - d; c) \leq 0$, for all such $x, y, c, \varepsilon$ and $d$, through replacing $v$ and finite variation approximation in $y$. This yields, by replacing $w$: $U(x, y) - U(x - c, y) - EU(x + \varepsilon, y) + EU(x + \varepsilon - c, y)$

$-U(x, y - d) + U(x - c, y - d) + EU(x + \varepsilon, y - d) - EU(x + \varepsilon - c, y - d) \leq 0$, for all such $x, y, c, \varepsilon$ and $d$. By reordering terms, this condition can be rewritten as

$U(x - c, y) + U(x, y - d) + EU(x + \varepsilon, y) + EU(x + \varepsilon - c, y - d)$

$\geq U(x, y) + U(x - c, y - d) + EU(x + \varepsilon - c, y) + EU(x + \varepsilon, y - d).$

That is, providing that one uses the expected utility, or any increasing affine linear transformation of it, as the individual welfare measure, the four-individuals society $\{(x - c, y); (x, y - d); (x + \varepsilon, y); (x + \varepsilon - c, y - d)\}$ is preferred to the four-individuals society $\{(x, y); (x - c, y - d); (x + \varepsilon - c, y); (x + \varepsilon, y - d)\}$. This ends the proof of (g) in Theorem 1.

The transformation of function $U$ through the inclusion of the background risk $\varepsilon$ and the expectation operation make the concavity in $x$ of the expected utility function $EU(. + \varepsilon, y)$ less pronounced than the concavity of the original utility function $U(., y)$. This makes inequality issues look less severe for the individuals that are subjected to the background risk. This is the distinction of the considerations pertaining to risk (through the expectation of utility and the corresponding utility premium) and to inequality (through welfare shock sharing and the normative representation of welfare by cardinal utility in a utilitarian setting) that allows the elicitation of this feature.

Therefore, a rigorous characterisation of the condition $U_{1112} \leq 0$ has been provided. Of course, a similar normative justification can be obtained for the symmetrical condition: $U_{1222} \leq 0$. However, some readers may think that four-individual societies may be harder to grasp intuitively than two-individual societies. So, an equivalent two-individual society characterisation is now provided.

The result (g) in Theorem 1 implies that the condition $U_{1112} \leq 0$ could be interpreted as "sharing fixed losses affecting different attributes improves social welfare, while less so under background risk on the first attribute". As pointed out previously, under the

hypotheses, the planner considers that the degradation in the situation of the second couple of individuals is more than compensated by the improvement in the situation of the first couple. Returning to the welfare premium, one has equivalently: $p^x(x - c, y, \varepsilon) + p^x(x, y - d, \varepsilon)$ is preferred to $p^x(x, y, \varepsilon) + p^x(x - c, y - d, \varepsilon)$. In that sense, the welfare premium function summarises the potential social compensations of the random shocks within each of the two subgroups and allows the social planner to focus on correlation aversion relative to non-random losses. Let us now turn to a few stochastic dominance theorems that can be derived by assuming these new normative justifications of signs of high-order derivatives of utility.

# 4 Stochastic Dominance

## 4.1 A few definitions

First, a few stochastic integrals need be defined that will be used to state the results.

**Definition 3** *Let*

$$
F_x^{k_1+1}(x) \;=\; \int_0^x \underset{(k_1\ times)}{\cdots} \int_0^{s_1} F_x(s_1)ds_1...ds_{k_1} \ \ and
$$

$$
F^{k_1+1,k_2+1}(x,y) \;=\; \int_0^y \underset{(k_2\ times)}{\cdots} \int_0^{t_1}\int_0^x \underset{(k_1\ times)}{\cdots} \int_0^{s_1} F(s_1,t_1)ds_1...ds_{k_1}dt_1...dt_{k_2}.
$$

In particular, one has $F^{1,1}(x,y) = F(x,y)$ and $F_x(x) = F_x^1(x) = F^{1,1}(x,a_2) = F(x,a_2)$. The gaps of any of these stochastic integrals for any two distributions $F$ and $F^*$ is also denoted by using the operator $\triangle$, as in Section 2.

Some generalised concave functions of $(x,y)$ are now defined that will be used later. These classes will be useful to facilitate the derivation of stochastic dominance results, particularly since their generators can often been found in the literature. Denuit, Lefèvre and Mesfioui (1999), Denuit and Mesfioui (2010) and Denuit, Eeckhoudt, Tsetling and

Walker (2013) provide generator functions for these classes, which are used to derive new stochastic dominance conditions in Sub-Section 4.2.

**Definition 4** *Consider the functions of $(x, y)$ from $[0, a_1] \times [0, a_2]$ to $\mathbb{R}$. The $(s_1, s_2)-$increasing concave $((s_1, s_2) - icv)$ functions are the appropriately derivable functions $g$ such that*

$$(-1)^{k_1+k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} g = (-1)^{k_1+k_2+1} g_{x...xy...y} \geq 0,$$

*with index $x$ $(y)$ appearing $k_1$ $(k_2)$ times,*

*and where $k_i = 0, .., s_i; i = 1, 2; s_1$ and $s_2$ are two non-negative integers and $1 \leq k_1 + k_2$. The corresponding class of functions is denoted by $U_{(s_1,s_2)-icv}$.*

*The $4-$increasing directionally concave functions (4-idircv) are the appropriately derivable functions $g$ such that*

$$(-1)^{k_1+k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} g = (-1)^{k_1+k_2+1} g_{x...xy...y} \geq 0,$$

*with index $x$ $(y)$ appearing $k_1$ $(k_2)$ times, and where $k_1$ and $k_2$ are two non-negative integers and $1 \leq k_1 + k_2 \leq 4$. The corresponding class of functions is denoted by $\boldsymbol{U}_{4-idircv}$.*

In this paper, attractive normative interpretations will be discussed for this class. The next theorem links these different classes.

**Theorem 2** *Let $R = \{(r_1, r_2) \in N^2 | \ r_1 + r_2 = 4\}$. Then,*

$$\boldsymbol{U}_{4-idircv} = \bigcap_{(r_1,r_2) \in R} \boldsymbol{U}_{(r_1,r_2)-icv}$$

The 4-idircv classes are based on symmetrical restrictions that make them particularly liable to be characterised by asymptotic expansions through symmetric derivations across all variables. In turn, these expansions can be used to identify generator functions, which can then be mobilised to obtain the necessary and sufficient conditions of stochastic dominance results.

Let us now discuss the link of these classes of functions with the utility sets of interest. Class $\boldsymbol{U}^{--}$ in Atkinson and Bourguignon (1982) can also be described as Class $(2,2) - icv$. The following classes that involve fourth-order derivatives are also considered:

Class $(3,1) - icv$ corresponds to the conditions: $U_1, U_2 \geq 0$; $U_{11}, U_{12} \leq 0$; $U_{112}, U_{111} \geq 0$; $U_{1112} \leq 0$, which have never been considered jointly in the literature.

Class $(4,0) - icv$ corresponds to the conditions: $U_1 \geq 0$; $U_{11} \leq 0$; $U_{111} \geq 0$; $U_{1111} \leq 0$ and no condition on the second attribute. Classes $(1,3) - icv$ and $(0,4) - icv$ can be easily obtained by symmetry.

Finally, the above class $\boldsymbol{U}$ of main interest coincides with Class 4-idircv. As its name may suggests, it requires symmetric restrictions on the marginal variations in all directions.

## 4.2   Stochastic dominance results

To derive some of the stochastic dominance theorems, recent results on multidimensional stochastic orderings will be used, notably Theorem 7(i) in Denuit, Eeckhoudt, Tsetlin and Walker (2010). The first result brought to the fore is that the class $\boldsymbol{U}^{--}$ in Atkinson and Bourguignon (1982) may now become legitimately available to empirical researchers because its fourth order condition ($U_{1122} \leq 0$) can now be associated with a clear normative interpretation, namely with imposing welfare cross-temperance. The result obtained in Atkinson and Bourguignon (1982) is recalled in the following theorem. It corresponds to classes of utility functions that are (2, 2)-increasing concave.[13] The new

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[13]We provide the first explicit complete proof for this theorem since in Atkinson and Bourguignon (1982) only the proof of the sufficient condition is given. In this seminal article, the necessary condition is omitted on the grounds that it is an obvious generalisation of the one-dimensional case. As a matter of fact, it should not be seen as a mere generalisation. Instead, obtaining the necessary condition is often the main difficulty in proofs of necessary and sufficient conditions for multidimensional stochastic dominance results. More discussions of proofs of necessary conditions can be found in Atkinson and

complete proof is provided in the Appendix.

**Theorem 3** *(Atkinson and Bourguignon): For the* $(2,2)-icv$ *Class, that is:* $U_1, U_2 \geq 0$; $U_{11}, U_{12}, U_{22} \leq 0$; $U_{112}, U_{221} \geq 0$; $U_{1122} \leq 0$, *the following conditions are necessary and sufficient for the corresponding stochastic dominance between the two continuous joint distributions.*

(a) $\triangle F^{2,2}(x,y) \leq 0$, *for all* $x, y$.

(b) $\triangle F_x^2(x) = \triangle F^{2,1}(x, a_2) \leq 0$, *for all* $x$.

(c) $\triangle F_y^2(y) = \triangle F^{1,2}(a_1, y) \leq 0$, *for all* $y$.

The first equality in (b) and (c) merely points out two equivalent notations for the used stochastic integrals, respectively with one and two dimensions. It is worth reminding how the new axioms of social welfare shock sharing illuminate the normative conditions assumed for this class. The conditions on the first-order derivatives of the utility function indicate monotonicity with respect to the two attributes. In this example, the condition $U_{11} \leq 0$ (respectively, $U_{22} \leq 0$) can be interpreted as assuming some aversion for income (respectively, for education) inequality, or alternatively, in the new interpretation in this paper, preference for social sharing of income (respectively, education) fixed losses. $U_{12} \leq 0$ assumes welfare correlation aversion, that is: preference for social sharing of fixed losses for different attributes. $U_{111} \geq 0$ (respectively, $U_{222} \geq 0$) is associated with welfare prudence in income (respectively, in education), while $U_{112} \geq 0$ (respectively, $U_{122} \geq 0$) means welfare cross-prudence in education (respectively in income). In all cases, these conditions correspond to preference for social sharing of a fixed loss and of a centered risk. Finally, $U_{1122} \leq 0$ is equivalent to welfare cross temperance, which is associated with preference for social sharing of centered risks on distinct attributes.

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Bourguignon (1987), for an analog problems involving specification of needs, although only for lower order problems. In contrast, in our case, the proof directly follows from eliciting the generators of the class of the $(2,2) - icv$ functions.

In contrast with the previous class of utility functions, which had already been investigated by Atkinson and Bourguignon (1982), the following class has never (to the best of my knowledge) previously been studied. This class satisfies in particular the property of welfare-premium correlation aversion in income. However, it does not include restrictions involving deriving more than once with respect to the second argument. This setting may be appropriate for welfare problems involving an ordinal second attribute, as in Bazen and Moyes (2003), and Gravel and Moyes (2012). One obtains.

**Theorem 4** *For the* $(3,1) - icv$ *Class, that is:* $U_1, U_2 \geq 0$; $U_{11}, U_{12} \leq 0$; $U_{112}, U_{111} \geq 0$; $U_{1112} \leq 0$, *the following conditions are necessary and sufficient for the corresponding stochastic dominance between the two continuous joint distributions.*

*(a)* $\triangle F^{3,1}(x,y) \leq 0$, *for all* $x, y$.

*(b)* $\triangle F^{2,1}(a_1; y) \leq 0$, *for all* $y$.

*(c)* $\triangle F_y(y) = \triangle F(a_1, y) \leq 0$, *for all* $y$.

Condition (c) corresponds to first-order stochastic dominance on the second attribute, often a demanding condition with typical data. This reflects the fact that only first-order derivations with respect to the second attribute have been used in the definition of the utility function class. The second condition (b) characterises the sign of a mixed stochastic dominance term, in which the joint cdf is aggregated according to the first attribute, up to its upper bound $x = a_1$. In particular, at the bound $y = a_2$, it implies $\triangle F_x^2(a_1) = \triangle F^{2,1}(a_1, a_2) \leq 0$, which can be seen as a negative difference in a specific inequality measure that is defined in terms of the first attribute, and allows the two situations to be compared using this inequality measure. In particular, when the marginal distributions of the second attribute are fixed, Condition (b) corresponds to the sequential generalised Lorenz criterion. In the general case, it can also be expressed using projected generalised Lorenz tests, as shown in Muller and Trannoy (2012).

Finally, Condition (a) again involves a mixed stochastic dominance term, in which, this time, the joint cdf is aggregated twice according to the first attribute, and this

for any levels of the two attributes. At the bound $y = a_2$, this implies $\triangle F_x^3(x) = \triangle F_x^{3,1}(x, a_2) \leq 0$, which corresponds to the well-known one-dimensional third-order stochastic dominance term. The case of $(1, 3) - icv$ is obviously symmetric. The next theorem corresponds to the well-known results of one-dimensional fourth-order stochastic dominance that are just recalled, while linking them with the two-dimensional notations.

**Theorem 5** *For the $4 - icv$ Class, that is: $U_1 \geq 0, U_{11} \leq 0, U_{111} \geq 0; U_{1111} \leq 0$, the following conditions are necessary and sufficient for the corresponding stochastic dominance between the two continuous joint distributions.*

*(a) $\triangle F_x^4(x) = \triangle F^{4,1}(x, a_2) \leq 0$, for all $x$.*

*(b) $\triangle F_x^3(a_1) = \triangle F^{3,1}(a_1, a_2) \leq 0$.*

*(c) $\triangle F_x^2(a_1) = \triangle F^{2,1}(a_1, a_2) \leq 0$.*

Beyond being a reminder, this theorem points out that the second attribute can be neglected in the analysis, as long as the imposed normative conditions do not involve utility derivatives with respect to this attribute. Importantly, with the new normative approach (assuming notably here that sharing centered risks improves social welfare), it is now possible to provide a good reason to use fourth-order one-dimensional stochastic dominance tests, which have been neglected so far in the empirical literature because of the lack of normative interpretation. The reason why there is no first-order condition of the type $\triangle F_x(a_1) \leq 0$ in this sequence is that this difference cancels out since $F_x(a_1) = 1 = F_x^*(a_1)$ for any two distributions $F_x$ and $F_x^*$. A similar proposition can of course be stated with the other attribute $y$. Finally, the class of the fourth-order increasing directionally concave functions is dealt with in the next theorem.

**Theorem 6** *Consider the $4 - idircv$ Class, that is: $U_1, U_2 \geq 0; U_{11}, U_{12}, U_{22} \leq 0; U_{112}, U_{221}, U_{111}, U_{222} \geq 0; U_{1122}, U_{1112}, U_{1222}, U_{1111}, U_{2222} \leq 0$.*

Let be the change in variable from the algebraic form to the trigonometric form of complex numbers: $z = x + iy = \rho e^{i\theta}$ with $\rho = \sqrt{x^2 + y^2}$ and $\theta = Arg(z)$, where

$x, y \in R_+, \rho \in R_+$ and $\theta \in [0, \pi/2]$ in the case $a_1 = a_2 = +\infty$, so as to impose the restrictions $x \geq 0$ and $y \geq 0$. Let $F_\rho$ be the cdf of $\rho$. Then,

( $\alpha$ ) 4-idircv in (x,y) is equivalent to 4-icv in $\rho$.

( $\beta$ ) The necessary and sufficient conditions of stochastic dominance for the 4-idircv class are, in the case $a_1 = a_2 = +\infty$:

(a) $\triangle F_\rho^4(\rho) \leq 0$, for all $\rho$.

(b) $\triangle F_\rho^3(+\infty) \leq 0$.

(c) $\triangle F_\rho^2(+\infty) \leq 0$.

For finite levels of $a_1$ or $a_2$, the conditions in the expressions (b) and (c) may be expressed at a finite bound $a_\rho = \sqrt{a_1^2 + a_2^2}$ instead of $+\infty$.

( $\gamma$ ) The generators of the $4 - idircv$ class are the functions of $x$ and $y$ defined by: $\left( \max\{c - \sqrt{x^2 + y^2}, 0\} \right)^{k-1}$, for all $c \in [0, a_\rho]$, if $k = 4$, and $c = a_\rho$ if $k = 1, 2, 3$.

The proof is given in the appendix. In ( $\alpha$ ), a remarkable property of dimension reduction is exhibited. It stems from the symmetry of the class of utility functions, notably expressed in Theorem 2, and from the non-negativity of the attributes. This result may be the elusive simplification researched by LeBreton (1999) in a multidimensional welfare context, and it allows for straightforward use of the well-known corpus of equivalence representation results in one-dimensional setting. Note that there is no problem of incompatibility of domain definitions when changing in variables between $(x, y)$ and $(\rho, \theta)$, even if some of these variables are bounded. Indeed, the dominance conditions can be trivially adjusted, if wished, by defining the corresponding appropriate bounds of the joint domain in both representations.

Could there be other convenient ways of aggregating the multi-dimensional information than using quadratic norms like $\rho$? Even though this may be possible, there is little hope of a fruitful research direction here. To understand this, we need to return to the principle of the proof of Theorem 6. What makes this proof work is the transformation of the classes of multi-dimensional-attributes utility functions into a unique

one-dimensional-attribute class whose generators are known. Indeed, the classical series of one-dimensional stochastic dominance orderings seems to be recognized as the most appropriate way to order one-dimensional distributions in social welfare analysis. Therefore, it does not seem promising to turn instead to other kind of one-dimensional aggregates that may correspond to little attractive multi-dimensional orderings. Moreover, fierce technical difficulties may arise when trying to transform multi-dimensional utility classes into another kind of one-dimensional utility class. Indeed, what makes the proof work is the polar symmetry of the s-idircv classes and of their link with $(s_1, s_2) - icv$ classes. Before being able to propose another kind of aggregation, other subjacent principles should be discovered first that would play the same role as these symmetries.

As for the one-dimensional fourth-order stochastic dominance, there is a fourth-order term in Condition (a) of $(\beta)$ in Theorem 6, although it is now in terms of the modulus variable $\rho$. One can therefore hope for substantial gain in the discriminatory power of empirical tests, as compared with the typical applications limited to the use of second-order stochastic dominance. In the next subsection, the stochastic dominance results are translated into poverty ordering results.

## 4.3   Poverty orderings

Foster and Shorrocks (1988) showed that one-dimensional stochastic dominance orderings are equivalent to some one-dimensional poverty orderings. The same elementary translation into multi-dimensional poverty measures is now performed.

**Definition 5** *(i) A distribution F is said to 'poverty dominate' a distribution G for a poverty measure P, and a range of poverty lines Z, 'F P(Z) G', if and only if:*

*$P(F, z) \leq P(G, z)$, for all $z \in Z$, and $P(F, z) < P(G, z)$ for at least a $z \in Z$.*

*This definition applies for multiple attributes and multidimensional poverty lines.*

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*(ii) The FGT Poverty measure of order $\alpha > 0$ is:*

$$P^\alpha(F, z) = \tfrac{1}{z^\alpha} \int_0^{F(z)} \left( z - F^{-1}(p) \right)^{\alpha-1} dp,$$

*where $F$ is the cdf of incomes and $z$ is the poverty line, while the integration is displayed here over the quantile index of the incomes. The positive parameter $\alpha$ is typically chosen equal to 1 ('head-count index'), 2 ('poverty gap') or 3 ('poverty severity index').*

*The 'poverty ordering $P^\alpha(Z)$' is defined as follows for two distributions $F$ and $G$: $F\ P^\alpha(Z)\ G$ if and only if*

*$P^\alpha(F, z) \leq P^\alpha(G, z)$ for all $z \in Z$, and $P^\alpha(F, z) < P^\alpha(G, z)$ for at least a $z \in Z$.*

*Cumulative integrals of a one-dimensional cdf $F$ are defined recursively as follows: $F^1 \equiv F$ and $F^\alpha(s) \equiv \int_0^s F^{\alpha-1}(t)dt$, for any integer $\alpha \geq 2$.*

*Similarly, $F^{k_1, k_2}$ is the joint density function $F(x, y)$ integrated $k_1$ times with respect to the first attribute and $k_2$ times with respect to the second attribute.*

*(iii) Let $z_i$ be an absolute poverty line for the $i^{th}$ attribute, $i = 1, 2$.*

*The 'joint FGT poverty measure of order $(k_1, k_2)$', for a population deprived in $x$ below a level $z_1$ and deprived in $y$ below a level $z_2$, is:*

$$P^{k_1, k_2}(z_1, z_2) = \frac{1}{z_1^{k_1} z_2^{k_2}} \int\limits_{[0, z_2]} \int\limits_{[0, z_1]} (z_1 - x)^{k_1 - 1} (z_2 - y)^{k_2 - 1} dF(x, y).$$

Note that the results derived in this paper involve multidimensional poverty indices, for example, with $k_1 = 3$ or $k_2 = 3$, which have not been used before with a clear normative basis and become now available. With these notations, Foster and Shorrocks (1988) have shown that: $z^{\alpha-1} P^\alpha(F, z) = \int_0^s (z - y)^{\alpha-1} dF(y) = (\alpha - 1)! F^\alpha(z)$. Thus, there is equivalence between the poverty ordering $P^\alpha(Z)$ and the $\alpha^{th}$-order one-dimensional stochastic dominance ordering. In particular, for all $\alpha \leq \beta$, $F\ P^\alpha(Z)\ G$ implies $F\ P^\beta(Z)\ G$. Foster and Shorrocks also note that $F\ P^2(Z)\ G$ is equivalent to $F$ 'generalised Lorenz dominates' $G$. In the next theorem, Foster and Shorrocks's (1988) results are straightforwardly extended to the multi-dimensional setting. The proof is given in the appendix.

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**Theorem 7:**

*The $F^{k_1,k_2}$ stochastic ordering is equivalent to the $P^{k_1,k_2}$ stochastic ordering.*

Obviously, this result can be extended to any number of attributes. Equipped with these results, let us now consider successively a few classes of utility functions of interest.

### 4.3.1 4-icv

The 4-icv class is the class generating the classical results of the one-dimensional fourth-order stochastic dominance. Therefore, based on Foster and Shorrocks's (1988) results, it is known that this dominance ordering is equivalent to the poverty ordering $P^4(Z_x) = P^{4,1}(Z_x, \{a_2\})$, where $Z_x$ is the set of poverty lines for the first attribute; with two additionally conditions at the bounds corresponding to the third and second order stochastic dominance orderings, here respectively translated into $P^3(\{a_1\}) = P^{3,0}(\{a_1, a_2\})$ and $P^2(\{a_1\}) = P^{2,1}(\{a_1, a_2\})$ poverty orderings. Although this ordering has not been typically used in the empirical one-dimensional stochastic dominance literature, there is now a welfare shock sharing motivation for using it. This is important because, providing the social planner accepts that socially sharing centred income risks is good, which justifies the use of $P^4$, this criterion should give much more importance to extreme poverty than even what is typically done by using the popular poverty severity index $P^3$.

### 4.3.2 4-idircv

For the 4-idircv case, one can still draw on Foster and Shorrocks's (1988) classical results to state that the corresponding stochastic dominance ordering is equivalent to the poverty dominance ordering $P^4(Z_\rho)$, here calculated for the modulus variable $\rho$, and for poverty lines $z_\rho \in Z_\rho$ defined in terms of this variable, where $Z_\rho$ is the corresponding range of these poverty lines; and in addition the second- and third-order stochastic dominance conditions in $\rho$ at the bounds, which are equivalent to the poverty orderings

$P^2(a_\rho)$ (i.e., poverty gap index for $\rho$) and $P^3(a_\rho)$ (i.e., poverty severity index for $\rho$), where $a_\rho$ is the upper bound of poverty lines $z_\rho$ for $\rho$. This is a direct consequence of Theorem 6. Note that in the case in which the bounds $a_1$ and $a_2$ for $x$ and $y$, are not infinite, the bound $a_\rho$ for $\rho$ may be inferior, while not necessarily strictly equal, to $\sqrt{(a_1)^2 + (a_2)^2}$, depending on the shape of the support of $F(x,y)$.

### 4.3.3 (3,1)-icv and (2,2)-icv

From Theorem 7, it is known that the $F^{k_1,k_2}$ stochastic dominance is equivalent to $P^{k_1,k_2}$ stochastic dominance. Thus, the following necessary and sufficient conditions are obtained for the (3,1)-icv class of utility functions:

For all $x$ and $y$, $\triangle P^{3,1}(x,y) \le 0$; $\triangle P^{2,1}(a_1, y) \le 0$; and $\triangle P^{1,1}(a_1, y) = \triangle P^1(y) \le 0$.

Respectively, the following necessary and sufficient conditions for the (2,2)-icv class utility functions are: For all $x$ and $y$, $\triangle P^{2,2}(x,y) \le 0$, $\triangle P^2_x(x) = \triangle P^{2,1}(x, a_2) \le 0$, and $\triangle P^2_y(y) = \triangle P^{1,2}(a_1, y) \le 0$.

## 4.4 Higher orderings

I now briefly discuss orderings beyond the fourth-order. In order to limit the size of the discussion and to avoid notation clutter, the axioms, their equivalent characterizations and the proofs will be given only for the conditions $U_{11111} \ge 0$ and $U_{111111} \le 0$. It should now be clear how to deal with the case with derivations with respect to several attributes, after all the practice performed with smaller orders.[14]

However, I now sketch how stochastic dominance results can be derived for the fifth and sixth orders. The same principles as above can be applied to extend stochastic

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[14]Another reason to postpone a more detailed study is that examining higher orders opens the possibility of more varied and subtle normative justifications that need to be fully explored. For example, complex mixtures of shocks, large numbers of individuals and of attributes could be considered while analysing these cases, which may not be straightforward to interpret normatively. Moreover, it may be opportune to develop new operators to deal with this increasing complexity.

dominance results to multi-dimensional partial derivative conditions at higher orders. This will yield higher-order one-dimensional stochastic dominance results in $\rho$. As little fundamental intuition is lost by looking only at them, once the previously exposed principles have been understood, we limit the discussion to Axioms (vii) and (viii) below.

**Definition 6**

Axiom (vii): ('*Bivariate-Risk Priority of the Poorest*'). *Let* $(x, y) \in R^2$ *be any two fixed non-random bivariate wellbeing endowments. Let* $e > 0$, *and let* $\varepsilon, \eta, \delta$ *be three independent centred random shocks, such that all the considered atttibute levels remain non-negative.*

*The 4-individual society* $\{x, , x + \varepsilon + \eta, x + \varepsilon - e, x + \eta - e\}$ *is preferred to the society* $\{x + \varepsilon, x + \eta, x - e, x + \varepsilon + \eta - e\}$ *by the social planner.*

Axiom (viii): ('*Bivariate-Risk-Sharing Priority of the Most Vulnerable*'). *The society* $\{x, x + \varepsilon + \eta, x + \varepsilon + \delta, x + \eta + \delta\}$ *is preferred to society* $\{x + \varepsilon, x + \eta, x + \delta, x + \varepsilon + \eta + \delta\}$ *by the social planner.*

In each of the two societies compared in Axiom (vii), the first couple of terms correspond to no background fixed loss, while the second couple to a background fixed loss. This axiom can be interpreted as the social planner giving more weight to sharing socially the two independent centred risks $\varepsilon$ and $\eta$ when there is already a background fixed loss. Alternatively, one can interpret it as a higher tendency to protect the poor against the two independent risks $\varepsilon$ and $\eta$. In that sense, this extends naturally the intuition of the Pigou-Dalton transfer axiom. We denote this axiom: *Bivariate-Risk Priority of the Poorest.*

In the first society in Axiom (viii), the bivariate shock $(\varepsilon, \eta)$ is shared by the couple of individuals under background risk $\delta$, while in the second society this shock is shared by the couple of individuals that do not face this background risk. Then, the social planner

would choose to protect from the two additional independent risks more those who are already bearing some risks. We denote this axiom: *Bivariate-Risk-Sharing Priority of the Most Vulnerable.* I now prove the respective equivalence of Axioms (vii) and (viii) with sign conditions for $U_{11111}$ and $U_{111111}$.

**Theorem 7**

(a) *Axiom (vii) is equivalent to $U_{11111} \geq 0$.*

(b) *Axiom (viii) is equivalent to $U_{111111} \leq 0$.*

Proof of (a):

Let $e > 0$, and let $\varepsilon, \eta, \delta$ be three independent centred random shocks, such that all the considered attribute levels remain non-negative.

Let $t(x) \equiv U_{111}(x)$. Then, $U_{11111}(x) \geq 0$ is equivalent to $t_{11}(x) \geq 0$, that is: $t$ is a convex function of $x$. Applying Jensen's inequality to $t$, implies $U_{111}(x) - EU_{111}(x+\varepsilon) \leq 0$, for any $x$ and $\varepsilon$.

Let $v(x) \equiv U(x) - EU(x+\varepsilon)$. Then, $v_{111}(x) \leq 0$, for any $x$. On the considered domain, which is an interval, this is equivalent to $v_{11}(x) - v_{11}(x-e) \leq 0$, for any fixed loss $e > 0$.

Let $m(x,e) \equiv v(x) - v(x-e)$. Then, $m_{11}(x) \leq 0$, that is: $m$ is concave in $x$. Jensen's inequality applied to $m$ gives: $m(x,e) - Em(x+\eta,e) \geq 0$, for all $\eta$.

Then, $v(x) - v(x-e) - Ev(x+\eta) + Ev(x-e+\eta) \geq 0$.

This gives

$U(x) - EU(x+\varepsilon) - U(x-e) + EU(x+\varepsilon-e) - EU(x+\eta)$

$+EU(x+\varepsilon+\eta) + EU(x+\eta-e) - EU(x+\varepsilon+\eta-e) \geq 0.$

Rearranging delivers the result

$\{U(x) + EU(x+\varepsilon+\eta)\} + \{EU(x+\varepsilon-e) + EU(x+\eta-e)\}$

$\geq \{EU(x+\varepsilon) + EU(x+\eta)\} + \{U(x-e) + EU(x+\varepsilon+\eta-e)\}.$

That is, the 4-individual society $\{x, , x + \varepsilon + \eta, , x + \varepsilon - e, x + \eta - e\}$ is prefered to the society $\{x + \varepsilon, x + \eta, x - e, x + \varepsilon + \eta - e\}$. QED.

Proof of (b):

Let $t(x) \equiv U_{1111}(x)$. Then, $U_{111111}(x) \leq 0$ is equivalent to $t_{11}(x) \leq 0$, that is: $t$ is a concave function of $x$. Applying Jensen's inequality to $t$, implies $U_{1111}(x) - EU_{1111}(x + \varepsilon) \geq 0$, for any $x$ and $\varepsilon$.

Let $v(x) \equiv U(x) - EU(x + \varepsilon)$. Then, $v_{1111}(x) \geq 0$, for any $x$. Let $w(x) \equiv v_{11}(x)$. Then, the previous inequality is equivalent to $w_{11}(x) \geq 0$, for all $x$, i.e. $w$ convex in $x$. Then, from Jensens's inequality: $w(x) \leq Ew(x + \eta)$ from any centred random shock $\eta$ independent of $\varepsilon$. Replacing yields $v_{11}(x) - Ev_{11}(x + \eta) \leq 0$, which can be written

$U_{11}(x) - EU_{11}(x + \varepsilon) - EU_{11}(x + \eta) + EU_{11}(x + \varepsilon + \eta) \leq 0$.

Let $m(x) \equiv U(x) - EU(x + \varepsilon) - EU(x + \eta) + EU(x + \varepsilon + \eta)$.

Then, $m_{11} \leq 0$, which is equivalent to $m$ concave in $x$. Therefore, Jensen's inequality implies $m(x) \geq Em(x + \delta)$, where $\delta$ is any centred random shock independent from $\varepsilon$ and $\eta$. By replacing terms, one obtains

$U(x) - EU(x + \varepsilon) - EU(x + \eta) + EU(x + \varepsilon + \eta)$

$-EU(x + \delta) + EU(x + \varepsilon + \delta) + EU(x + \eta + \delta) - EU(x + \varepsilon + \eta + \delta) \geq 0$. Finally, by rearranging:

$U(x) + EU(x + \varepsilon + \eta) + EU(x + \varepsilon + \delta) + EU(x + \eta + \delta)$

$\geq EU(x + \varepsilon) + EU(x + \eta) + EU(x + \delta) + EU(x + \varepsilon + \eta + \delta)$.

The society $\{x, x + \varepsilon + \eta, x + \varepsilon + \delta, x + \eta + \delta\}$ is preferred to society $\{x + \varepsilon, x + \eta, x + \delta, x + \varepsilon + \eta + \delta\}$. QED.

Clearly, normative justification of the other fifth and sixth order cross partial derived conditions can be similarly obtained by combining the principles of the interpretation and of the proofs that have been developed previously. We omit them to keep the presentation short.

Regrouping all these kinds of multidimensional partials' sign conditions in a utility class up to the fifth or to the sixth order, will correspond to the 5-idircv and 6-idircv classes. The derivation of the associated SD theorems generates the classical one-dimensional SD orderings for the composite variable $\rho$, respectively at the fifth and the sixth order. Let us now turn to a brief empirical application.

# 5  Empirical Application

As an application of the new method, let us now investigate changes in the bivariate 'income-education' social welfare and inequality in Egypt at the beginning of the twenty-first century. Two well-being attributes are considered for each household: the deflated income per capita and the education level of the head. The question that is investigated is what the main features of the changes in the corresponding bi-dimensional social welfare are in Egypt over 1999–2012. In particular, can we see some general improvement in these social aspects, along with economic development and growth, or can we detect instead the occurrence of some social welfare crisis which could have contributed to the 2011 Egyptian revolution?

The information on household income and education are taken from the Egyptian Household Income, Expenditure and Consumption Survey. The used survey rounds are for the years 1999, 2004, 2008, 2010 and 2012. The respective samples of surveyed households are reasonably large, ranging from about 8,000 to 40,000 households. As mentioned before, the social shock sharing axioms imply characterizations of the shapes of the utility functions that have consequences for the comparison of society situations even without observed shocks, which fits well the available data.

One practical issue is that the measurement units of the two attributes may have different orders of magnitude. In that case, all the above theoretical results are still valid since the domains over which the stochastic integrals must be calculated are fully covered by the complete set of conditions. However, with an empirically finite sample, any gross

unbalance in measurement units may imply the occurrence of most observations within an elongated small domain that gives more justice to the variations in the attribute that has the smaller measurement unit. For example, if the measured levels of $y_{it}$ are much smaller than the measured levels of $x_{it}$, because of gross disparity of measurement units, one would have $\rho_{it} \simeq x_{it}$, somewhat falling back to an approximate one-dimensional analysis. Therefore, a balanced and fair representation of the multidimensional features of the data requires a wise choice of comparable measurement units. Moreover, this avoids that the results of empirical tests should depend on clumsy and arbitrary choices of the measurement units. Indeed, there is no reason why the choice of the units should affect normative conclusions of the analysis.

As a matter of fact, these reflexions on the choice of measurement units apply to any empirical multidimensional stochastic dominance test, although they do not seem to have been stated before. Therefore, to make the process rigorous, one may consider that an additional axiom of scale invariance of the marginal distribution for each attribute, or alternatively an axiom justifying the use of specific units, could be imposed as soon as there is an empirical confrontation of the theory with some data. This may be akin to the use of the (relative) Lorenz curve instead of the (non-normalized) generalised Lorenz curve in inequality analysis (e.g., in Le Breton, 1999). Of course, imposing an axiom of scale invariance may have also normative theoretical consequences, which could for example be exploited for deriving some practical formulae of aggregate social welfare indices, or inequality indices. However, I find it better for the analysis here to keep apart the axioms regarding theoretical concerns about social shock sharing from axioms adopted for merely practical empirical purposes.

Back to the empirical analysis, pragmatically, each attribute is rescaled to obtain comparable magnitudes. A normalization of $x_{it}$ and $y_{it}$ by their respective global means over all observed households and years, $\bar{x}$ and $\bar{y}$, is implemented. Then, consistent estimators of the diverse stochastic integrals that appear in the above theorems are chosen to be their corresponding empirical analogueues, provided the sample size in

each year is large enough, which seems to be the case. More sophisticated smoothed and trimmed kernel estimators could be used to better deal with efficiency, robustness and even cosmetic concerns. However, since the focus is on theoretical issues, I prefer to stick to the transparent formulae of the empirical analogues that directly reflect the empirical distributions.

Figure 1 and Figure 2 show the estimated dominance curves, according to Theorem 6, across the compared years. They are based on the modulus $\rho$ constructed from the two normalised attributes, $x_{it}/\bar{x}$ and $y_{it}/\bar{y}$, and the preferred class of 4-idircv utility functions. The curves are displayed only for limited ranges of $\rho$ in order to better exhibit the locations of the distributions of $\rho$ where some crossing of curves may occur. That is, there is an obvious unambiguous dominance ranking of all the curves, at any order, for the ranges that are not shown. There are six shown panels. The first and second panels display, respectively, the estimated stochastic dominance curves for the second and third order stochastic dominance comparisons. The third panel shows the estimated curves for the fourth order. The fourth, fifth and sixth panels again show the estimated curves for the fourth order, while zooming on possibly contentious areas with respect to curve crossing. Figures 2 and 3 display the estimated stochastic dominance curves for the fifth and sixth orders, respectively, only for the contentious ranges of $\rho$.

I only discuss the ranges of variation of the modulus for which crossings seem to occur. Obviously, some crossings of the estimated curves occur for the second and third orders. Therefore, no unambiguous bivariate dominance result can be obtained at these orders. In contrast, for the same ranges of the modulus variable $\rho$, the estimated dominance curve does not seem to cross at the fourth order, except perhaps marginally, and certainly not at the fifth or sixth orders.

Given these precautions, the results imply that no worsening or no crisis of the income-education social welfare in Egypt between 1999 and 2012 can be revealed by the dominance analysis, at least when using the two measured attributes and the years for which there are data. On the opposite, if one accepts the normative restrictions of our

main class, there appears to have been a continual improvement of the bivariate social welfare from 1999 to 2012. In contrast, usual analyses only based on second degree dominance, or even third degree, would not allow the analyst to conclude.

In the interest of focussing on the theoretical argument of this paper, we do not deal with many approximation and measurement issues that are important in full empirical work. In particular, confidence bands may make the comparisons of the curves less distinct, although the measurement errors and specification errors in well-being attributes are probably more worrying than sampling errors in such large samples. Even without these errors, one may contest the normative relevant of rejecting dominance due to a few special individuals. In that sense, Zheng (2018) provides a method to analyse curves that are hardly distinguishable graphically. On the whole, these issues seem to matter less in practice than the specification choices for the variables representing the welfare attributes. Since the objective in this paper is to elicit new theoretical results in stochastic dominance and not to do a fully-fledged empirical application, these issues are not further discussed.

However, even with these caveats, the empirical results clearly show that social unrest in Egypt, notably around the period of the Arab Spring and of the 2011 Egyptian Revolution, cannot be the consequence of some worsening of the income-education social welfare. Indeed, any axiomatically-valid aggregate indicator of social welfare in the full fourth order sense, accounting simultaneously for both income and education, has continuously improved over the studied period. It may be that changes in aspirations or frustrations with corruption and bad governance may matter more for people than objective changes in household situations.

# 6 Conclusion

Since the seminal paper by Atkinson and Bourguignon (1982), developing powerful empirical multi-dimensional dominance criteria has been pursued by researchers, usually

with little success so far, much because it has been found hard to justify normatively criteria based on high-degree partial derivatives of individual utility functions. In this paper, a new method is proposed to impose normative requirements in multidimensional social welfare analyses by specifying welfare shock sharing axioms. It is shown that the conditions in these new axioms are equivalent to imposing specific restrictions on the signs of partial derivatives of Von Neuman-Morgenstern individual utility functions in an expected utility setting. These findings are exploited to derive new multidimensional stochastic dominance theorems with complete proofs of necessary and sufficient conditions, and to justify normatively the empirical use of some already known theorems. Empirically powerful discriminatory criteria are obtained by combining all social shock sharing axioms up to high orders, and by proving a dimension reduction property. Finally, an application to Egypt illustrates the empirical power of these new dominance rules.

The new approach should aid policy design and evaluation. Availing of powerful robust multi-dimensional decision criteria is likely to assist the evaluation of economic and social development policies that often rely on multi-dimensional wellbeing indicators. For example, the indicators of the Millennium Development Goals and the Human Development Index (HDI) are routinely invoked by organisations and policy makers, even though these indicators are subject to fierce criticisms owing to their arbitrariness.

There are, nonetheless, remaining obstacles to the fully fledged use of multidimensional dominance criteria in welfare economics. Notably, measurement issues are pervasive and even defining reasonable mixes of attributes covering most well-being effects is a challenge. However, using powerful multi-dimensional criteria should help researchers to sort out what matters empirically, and what does not, when defining these variables and measuring them.

Some extensions of this work look natural and are being investigated. First, some attributes of the utility function can be seen as describing components of needs and discrete variations rather than direct and continuous contributions to well-being, such

as by introducing equivalence scales within the analytical setting. Second, conditions characterizing still higher-order derivatives of utility and large numbers of wellbeing attributes can be further studied, notably to explore the empirical limits of the new approach and the mathematical regularities obtained in large dimensions. Risk studies, without social welfare background, may also benefit from similar intuitions to the ones developed in this research, although this seems to be a less promising field of applications for them since one cannot take advantage of solidarity feelings to justify normative assumptions in that case.

Finally, an open problem remains how to extend the benefits of the intuition brought by social shock sharing axioms to social decision criteria that are not linear in utilities or that involve anticipations of some characteristics of the shocks and of their correlations with individual characteristics.

# References

[1] Atkinson, A.B. (1970): "On the Measurement of Inequality," Journal of Economic Theory, 2, 244-263.

[2] Atkinson, A.B. (2003): "Multidimensional Deprivation: Contrasting Social Welfare and Counting Approaches," Journal of Economic Inequality, 1, 51–65.

[3] Atkinson, A.B., and F. Bourguignon (1982): The Comparison of Multi-dimensioned Distributions of Economic Status, Review of Economic Studies, 49, 181–201.

[4] Atkinson, A.B., and F. Bourguignon (1987): Income Distribution and Differences in Needs, in: Arrow and the Foundation of the Theory of Economic Policy,ed. by G.R. Feiwel. London: Macmillan, 350–370.

[5] Bazen, S., and P. Moyes (2003): International Comparisons of Income Distributions, Research in Economic Inequality, 9, 85–104.

[6] Bourguignon, F. (1989): Family Size and Social Utility: Income Distribution Dominance Criteria, Journal of Econometrics, 42, 67–80.

[7] Brown, D.P. (2017): New characterizations of increasing risk, Journal of Mathematical Economics, 69, 7-11.

[8] Chambaz, C. and E. Maurin (1998): Atkinson and Bourguignon's Dominance Criteria. Extended and Applied to the Measurement of Poverty in France. Review of Income and Wealth. Serie 44, N. 4, December.

[9] Crainich, D., L. Eechkoudt and O. Le Courtois (2013): An Index of (Absolute) Correlation Aversion: Theory and Some Implications. Working Paper 2013-ECO-12, IESEG School of Management, Lille, France.

[10] Danan, E., T. Gajdos and J.-M., Tallon (2015): Harsanyi's Aggregation Theorem with Incomplete Preferences, American Economic Journal: Microeconomics, vol. 7, no. 1, 61-69, February.

[11] Deck, C. and H. Schelsinger (2014): Consistency of Higher Order Risk Preferences, Econometrica, 42(5), 1913-1943.

[12] Denuit, M. and L. Eeckhoudt (2010): Stronger Measures of Higher Order Risk Attitudes, Journal of Economic Theory, 145, 2027-2036.

[13] Denuit, M., L. Eeckhoudt, I. Tsetlin, and R.L. Walker (2013): Multivariate Concave and Convex Stochastic Dominance, in F. Biagini et al. (eds.), Risk Measures and Attitudes, EAA Series, Springer-Verlag London.

[14] Denuit, M., C. Lefevre, and M. Mesfioui (1999): A Class of Bivariate Stochastic Orderings, with Applications in Actuarial Sciences, Insurance: Mathematics and Economics, 29, 31–50.

[15] Denuit, M., and M. Mesfioui (2010): Generalized Increasing Convex and Directionnally Convex Orders, Journal of Applied Probability, 47, 264–276.

[16] Duclos, J-Y, D. Echevin (2008): Health and Income: A Robust Comparison of Canada and the US, Journal of Health Economics, Vol. 30(2), 293-302, March.

[17] Duclos, J-Y, D.E. Sahn, and S.D. Younger (2010a): Partial Multidimensional Inequality Orderings, Journal of Public Economics, 95 (3-4) , 225–238.

[18] Duclos, J-Y, D.E. Sahn, and S.D. Younger (2010b): Robust Multidimensional Poverty Comparisons, The Economic Journal, vol. 116, 943-968, October.

[19] Eeckhoudt, L., B. Rey, and H. Schlesinger (2007): A Good Sign for Multivariate Risk Taking, Management Science 53, 117–124.

[20] Eeckhoudt, L. and H. Schlesinger (2006): Putting Risk in Its Proper Place. American Economic Review, vol. 96, issue 1, 280-289.

[21] Fishburn, P.C., and R.C. Vickson (1978): Theoretical Foundations of Stochastic Dominance, in G.A. Whitmore and M.C. Finley (Eds.), Stochastic Dominance, Lexington Books, Toronto.

[22] Fishburn, P.C., and R.D. Willig (1984): Transfer Principles in Income Redistribution, Journal of Public Economics, 25, 323–328.

[23] Foster, J. E., and A.F. Shorrocks (1988): Poverty Orderings and Welfare Dominance, Social Choice and Welfare, 5, 179–198.

[24] Gayant, J.-P. and N. Le Pape (2017): Increasing $N$th degree inequality, Journal of Mathematical Economics, 70, 185-189.

[25] Gollier, C. and J.W. Pratt (1996): Risk Vulnerability and the Tempering Effect of Background Risk, Econometrica, 64(5), 1109–1123

[26] Gravel, N., and P. Moyes (2012): Ethically Robust Comparisons of Bidimensional Distributions with an Odinal Attribute, Journal of Economic Theory, 147(4), 1384–1426.

[27] Hardy, G.H., Littlewood, J.E. and Polya, G. (1929): Inequalities. Cambridge University Press.

[28] Harsanyi, J. (1955): Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility, Journal of Political Economy, 63, 309–321.

[29] Jenkins, S.P. and P. Lambert (1993): Ranking Income Distribution When Needs Differ, Review of Income and Wealth, Ser. 39, No. 4, 337-56.

[30] Jokung, O. (2011): Risk apportionment via bivariate stochastic dominance. Journal of Mathematical Economics, 47, 448–452.

[31] Kimball, M. S. (1993): Standard Risk Aversion, Econometrica, 61 , 589–611.

[32] Kolm, S. (1969): The Optimal Production of Social Justice, in Public Economics, ed. by J. Margolis, and S. Guitton, London. MacMillan.

[33] Kolm, S. (1976a): Unequal Inequalities, I, Journal of Economic Theory, 12, 416–442.

[34] Kolm, S. (1976b): Unequal Inequalities, I, Journal of Economic Theory, 13, 82–111.

[35] Kolm, S. (1977): Multidimensional Egalitarianisms, Quarterly Journal of Economics, 91, 1–13.

[36] Le Breton (1999): The Mathematical Foundations of Inequality Analysis, Chapter 5 in J. Silber, Handbook of Income Inequality Measurement, Boston, MA: Kluwer.

[37] Levy, H., and J. Paroush (1974): Multi-Period Stochastic Dominance, Management Science, 21, 428–435.

[38] Li, J., D. Liu and J. Wang (2016): Risk aversion with two risks: A theoretical extension, Journal of Mathematical Economics, 63, 100-105.

[39] Menezes, C.F., and X.H. Wang (2005): Increasing Outer Risk, Journal of Mathematical Economics, 41, 875–886.

[40] Moyes, P. (1999a): Comparaisons de distributions hétérogènes et critères de dominance, Économie & Prévision, n°138-139, 1999-2-3.

[41] Moyes, P. (1999b): Stochastic Dominance and the Lorenz Curve, in J. Silber, Handbook of Income Inequality Measurement, Boston, MA: Kluwer.

[42] Moyes, P. (2012): Comparisons of Heterogeneous Distributions and Dominance Criteria, Journal of Economic Theory. 147, 1351-1383.

[43] Moyes, P. (2013): Rearrangements and Sequential Rank Order Dominance, Cahiers du GREThA, n. 2013-10, March.

[44] Muller, C., and A. Trannoy (2011): A Dominance Approach to the Appraisal of the Distribution of Well-being Across Countries, Journal of Public Economics, 95, 239–246.

[45] Muller, C., and A. Trannoy (2012): Multidimensional Inequality Comparisons: A Compensation Perspective, Journal of Economic Theory, 147, 4, 1427-1449.

[46] Pratt, J. and Zeckauser, R. (1987): Proper risk aversion. Econometrica, 55:143–154.

[47] Rockafeller, T. (1970): Convex Analysis, Princeton University Press, Princeton.

[48] Shorrocks, A. F. (1983): Ranking Income Distributions, Economica 50, 1-17.

[49] Silber, J (1999): Handbook of Income Inequality Measurement, Boston, MA: Kluwer.

[50] Tsetlin, I. and R.L. Winkler (2009): Multiattribute Utility Satisfying a Preference for Combining Good with Bad, Management Science, Vol. 55, No. 12, 1942-1952, December.

[51] Tsui, K.-Y. (1999): Multidimensional Inequality and Multidimensional Generalized Entropy Measure: An Axiomatic Derivation, Social Choice and Welfare, 16, 145–157.

[52] Weymark, J. A. (1991): A Reconsideration of the Harsanyi-Sen Debate on Utilitarianism, in Interpersonal Comparisons of Well-Being, ed. by J. Elster, and J. Roemer, pp. 255–320. Cambridge, UK: Cambridge University Press.

[53] Weymark, J. A., (1993): Harsanyi's Social Aggregation Theorem and the Weak Pareto Principle, Social Choice and Welfare, 10, 209–221.

[54] Zheng, B. (1999): On the Power of Poverty Orderings, Social Choice and Welfare, 16: 349-371.

[55] Zheng, B. (2018): Almost Crossing Lorenz Curves, Social Choice and Welfare, 51, 51-63.

# Appendix

**Proof of Theorem 1:**

(b) For the condition $U_{12} \leq 0$, let us denote $v(x,y) \equiv U_{12}(x,y)$ as an ancillary function. Then, consider finite variations as approximations of the partial derivatives of $U$ embodied in $v$. This is relevant here because on the whole considered domain, the fixed sign of these finite variations will also be the sign of the corresponding derivatives. Let $c > 0$ and $d > 0$ be any fixed constants such that $x - c > 0$ and $y - d > 0$. First, $U(x,y) - U(x-c,y)$ approximates $U_1$. Then, $U(x,y) - U(x-c,y) - U(x,y-d) + U(x-c,y-d)$ approximates $U_{12}(x,y)$. As a result, $U_{12} \leq 0$ over the whole domain is equivalent to $U(x,y) + U(x-c,y-d) \leq U(x,y-d) + U(x-c,y)$ over the whole

domain. Then, provided the social welfare criterion is additive in utility functions, such as for utilitarianism, one has: society $\{(x-c,y); (x,y-d)\}$ is weakly preferred to society $\{(x,y); (x-c,y-d)\}$. Sharing shocks that are fixed losses among individuals is a social improvement even if the shocks affect different attributes.

(a) Starting instead from the condition $U_{11} \leq 0$, and using the same approximation method with $x - c - d > 0$, and only the first attributes, one obtains $U(x,y) + U(x-c-d,y) \leq U(x-c,y) + U(x-d,y)$. With social welfare criteria that are additive in utilities, society $\{(x-c,y); (x-d,y)\}$ is weakly preferred to society $\{(x,y); (x-c-d,y)\}$. Of course, $U_{22} \leq 0$ is liable to the same kind of interpretation for the second attribute. Sharing shocks that are fixed losses affecting the same attribute among individuals is a social improvement.

(c) Let us now turn to the condition $U_{112} \geq 0$. Let $\varepsilon$ be any centred shock and $d$ any positive constant such that $x + \varepsilon > 0$ and $y - d > 0$. Define the welfare premium function by $v(x,y) \equiv p^x(x,y,\varepsilon) = U(x,y) - EU(x+\varepsilon,y)$. By deriving once with respect to the second attribute, one obtains $v_2(x,y) = U_2(x,y) - EU_2(x+\varepsilon,y)$. Let us now consider the following fixed sign over the whole domain: $v_2 \leq 0$. On the one hand, using Jensen's inequality with respect to the first attribute, this condition is equivalent to $U_2$ convex in $x_1$, which is equivalent to $U_{112} \geq 0$, the condition that is studied. On the other hand, $v_2 \leq 0$ over the whole domain is equivalent to $U(x,y) - EU(x+\varepsilon,y) - U(x,y-d) + EU(x+\varepsilon,y-d) \leq 0$, through finite variation approximation. Rearranging yields $U(x,y) + EU(x+\varepsilon,y-d) \leq U(x,y-d) + EU(x+\varepsilon,y)$, which implies that $\{(x,y-d); (x+\varepsilon,y)\}$ is weakly preferred to $\{(x,y); (x+\varepsilon,y-d)\}$. Of course, the case $U_{122} \geq 0$ can be dealt with similarly. If one shock is a fixed loss and the other is a random centered shock on the other attribute, sharing them among individuals improves social welfare.

(d) I now consider the condition $U_{111} \geq 0$. Using the same reasoning as before, while allocating the fixed loss to the first attribute instead, one obtains

$U(x,y) + EU(x+\varepsilon-c,y) \leq U(x-c,y) + EU(x+\varepsilon,y)$, with $c$ any positive constant

and $\varepsilon$ any centred shock such that $x + \varepsilon - c > 0, x - c > 0$ and $x + \varepsilon > 0$. Society $\{(x - c, y); (x + \varepsilon, y)\}$ is weakly preferred to $\{(x, y); (x + \varepsilon - c, y)\}$.

Again, one obtains an interpretation in terms of welfare shock sharing between two individuals of a fixed loss shock and a random shock on the same attribute, which is considered as a social welfare improvement. Note that even in the risk context, such interpretation does not seem to have emerged so far from the literature. The case $U_{222} \geq 0$ is similar.

(e) The condition $U_{1122} \leq 0$ is examined by starting again with the welfare premium function $v(x, y) \equiv p^x(x, y, \varepsilon) = U(x, y) - EU(x + \varepsilon, y)$, with $\varepsilon$ any centred shock such that $x + \varepsilon > 0$. However, let us now derive twice with respect to the second argument to obtain $v_{22}(x, y) = U_{22}(x, y) - EU_{22}(x + \varepsilon, y)$. In these conditions, $v_{22}(x, y) \geq 0$ is equivalent to $U_{22}$ concave in $x$, that is: $U_{1122} \leq 0$. On the other hand, $v_{22}(x, y) \geq 0$ can be characterised by the Jensen's inequality with respect to the second argument, as applied to the welfare premium function: $v(x, y) - Ev(x, y + \delta) \leq 0$, where $\delta$ is a random centred shock independent of $\varepsilon$. By replacing the formula for $v$, one gets $U(x, y) - EU(x + \varepsilon, y) - EU(x, y + \delta) + EU(x + \varepsilon, y + \delta) \leq 0$. Rearranging leads to $U(x, y) + EU(x + \varepsilon, y + \delta) \leq EU(x + \varepsilon, y) + EU(x, y + \delta)$. This yields: the society $\{(x, y + \delta); (x + \varepsilon, y)\}$ is weakly preferred to the society $\{(x, y); (x + \varepsilon, y + \delta)\}$. In that case, this is the sharing of two random shocks between individuals, each on a different attribute, that is seen as enhancing social welfare.

(f) For the condition $U_{1111} \leq 0$, the proof of the previous case can be replicated by allocating the random centred shock $\delta$ to the first attribute instead, still with $\delta$ and $\varepsilon$ any centred random shocks mutually independent, while now such that $x + \varepsilon > 0, x + \varepsilon + \delta > 0$ and $x + \delta > 0$. This leads to $U(x, y) + EU(x + \varepsilon + \delta, y) \leq EU(x + \varepsilon, y) + EU(x + \delta, y)$, which can be interpreted in terms of preferences for shock sharing as before, with here the random shocks affecting the same attribute. The case $U_{2222} \leq 0$ is similar.

(g) The proof of this case is in the text in Sub-section 3.2. (h) is obvious from the proofs. QED.

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**Proof of Theorem 2**: Recall $R = \{(r_1, r_2) \in N^2 | \ r_1 + r_2 = 4\}$. Any function $g \in \bigcap_{(r_1, r_2) \in R} U_{(r_1, r_2) - icv}$ is such that $(-1)^{k_1 + k_2 + 1} \frac{\partial^{k_1 + k_2}}{\partial x^{k_1} \partial y^{k_2}} g \geq 0$, which is denoted by property $P(k_1, k_2)$, and is satisfied for $k_1 = 0, ..., r_1; k_2 = 0, ..., r_2; k_1 + k_2 \geq 1$; for any $r_1 + r_2 = 4$.

In particular, it is now shown that for any $g \in \bigcap_{(r_1, r_2) \in R} U_{(r_1, r_2) - icv}$, one has also $P(k_1, k_2)$ true for any $(k_1, k_2)$ such that $1 \leq k_1 + k_2 \leq 4$. Indeed, there exist some $(r_1, r_2) \in R$ such that the $k_1$ in the range $0, .., r_1$, and the $k_2$ is in the range $0, .., r_2$, with $k_1 + k_2 \geq 1$. By construction the sum $k_1 + k_2 \leq r_1 + r_2 = 4$. An example of such $(r_1, r_2)$ is $r_1 = k_1$ and $r_2 = 4 - r_1$. Therefore, $g \in U_{4 - idircv}$. It has then been shown that $U_{4 - idircv} \supset \bigcap_{(r_1, r_2) \in R} U_{(r_1, r_2) - icv}$.

Reciprocally, let $g \in U_{4 - idircv}$ and any given $(r_1, r_2) \in R$. It must be shown that $g \in U_{(r_1, r_2) - icv}$. It is known that $P(k_1, k_2)$, for any $k_1$ and $k_2$ non-negative integers such that $1 \leq k_1 + k_2 \leq 4$. In particular, this is satisfied for all $(k_1, k_2)$ such that $k_1 \leq r_1, k_2 \leq r_2$ since in that case $k_1 + k_2 \leq r_1 + r_2 = 4$. Therefore, $g \in U_{(r_1, r_2) - icv}$. Since this reasoning can apply for any $(r_1, r_2) \in R$, this implies $U_{4 - idircv} \subset \bigcap_{(r_1, r_2) \in R} U_{(r_1, r_2) - icv}$. QED.

**Proof of Theorems 3 and 4:**

The results can be obtained by using the following result in terms of expectations from Denuit, Eeckhoudt, Tsetlin and Walker (2013), which we expose with their notations. Let $\tilde{x}, \tilde{y} \in [\underline{x}, \bar{x}]$ be two real random vectors of dimension $s$. Then, $\tilde{x} \succ_{s - icv} \tilde{y}$ if and only if

$$E\left[\prod_{i=1}^{N} (c_i - \tilde{x}_i)_+^{k_i - 1}\right] \leq E\left[\prod_{i=1}^{N} (c_i - \tilde{y}_i)_+^{k_i - 1}\right],$$

for all $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$; and $c_i = \bar{x}_i$ if $k_i = 1, ..., s_i - 1; i = 1, ..., n$.

The transformation of these conditions by using successive integrations by parts yields the results of the Theorems 3 and 4. Note that the stated result for Theorem 4 simplifies because $\triangle F(\bar{x}_1, \bar{x}_2) = 0$, since the common value of any cdf at the joint upper bound is 1.

**Proof of Theorem 5:**

This is a classic result of one-dimensional stochastic dominance, although I could not find the exact reference for its explicit statement, as authors (e.g., Fishburn and Vickson, 1978, Moyes, 1999b) typically stop at the third order. Here, it is derived from the Denuit-Eeckhoudt-Tsetlin-Walker's (2013) formula and by integrations by parts as in the proof of Theorems 3 and 4.

**Proof of Theorem 6:**

($\alpha$) Consider the representation of a couple of real numbers, $(x, y)$, in the complex plan: $z \equiv x + iy \equiv \rho e^{i\theta}$, with the modulus of the complex number $z$ defined as $\rho = \sqrt{x^2 + y^2}$ and its complex argument defined as $\theta = Arg(z)$, here restricted to $[0, \pi/2]$ so as to impose $x \geq 0$ and $y \geq 0$. The inverse transformation yields $x = \rho \cos \theta$ and $y = \rho \sin \theta$.

Then, the derivatives of a utility function $u(x, y)$, when transformed as a function of $(\rho, \theta)$, can be obtained by using the chain rule. Let $u(\rho, \theta) \equiv u(x, y)$, allowing for a slight abuse of notation to alleviate clutter[15]. Excluding the uninteresting case $\rho = 0$, where there are no two-side derivatives, we have $\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta = \frac{1}{\rho}\left( x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right)$.

At the second order, by keeping the $\theta$ constant, since partial derivatives are being calculated and $\theta$ is independent from $\rho$ in the formula of $u(\rho, \theta)$, one obtains $\frac{\partial^2 u}{\partial \rho^2} = \frac{\partial\left[ \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \right]}{\partial \rho} = \cos \theta \frac{\partial\left[ \frac{\partial u}{\partial x} \right]}{\partial \rho} + \sin \theta \frac{\partial\left[ \frac{\partial u}{\partial y} \right]}{\partial \rho} = \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2}$ by replacing, respectively, $u$ with $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in the previous calculus and rearranging. Iterating yields $\frac{\partial^3 u}{\partial \rho^3} = \cos^3 \theta \frac{\partial^3 u}{\partial x^3} + 3 \sin \theta \cos^2 \theta \frac{\partial^3 u}{\partial x^2 \partial y} + 3 \sin^2 \theta \cos \theta \frac{\partial^3 u}{\partial x \partial y^2} + \sin^3 \theta \frac{\partial^3 u}{\partial y^3}$. Finally, $\frac{\partial^4 u}{\partial \rho^4} = \cos^4 \theta \frac{\partial^4 u}{\partial x^4} + 4 \sin \theta \cos^3 \theta \frac{\partial^4 u}{\partial x^3 \partial y} + 6 \sin^2 \theta \cos^2 \theta \frac{\partial^4 u}{\partial x^2 \partial y^2} + 4 \sin^3 \theta \cos \theta \frac{\partial^4 u}{\partial x \partial y^3} + \sin^4 \theta \frac{\partial^4 u}{\partial y^4}$.

Then, it is clear in these formulae that a $4 - idircv$ utility function $u$ in $(x, y)$, which has all its considered partial derivatives, with respect to $x$ and $y$, alternatively non-positive and non-negative as the order of derivation is raised with respect $x$ and $y$, has also alternating non-positive and non-negative derivatives as the derivation order

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[15]That is: denoting the function $u$ in the same way in the two systems.

is raised with respect to $\rho$ (i.e., positive for first-order derivatives, negative for second-order derivatives, etc). Indeed, all the coefficients of these partials in these formulae are non-negative due to $\theta \in [0, \pi/2]$. Therefore, if function $u$ is 4-idircv in $(x, y)$, then it is 4-icv in $\rho$.

Let us now prove the reciprocal statement by recurrence, starting with the first-order derivatives. Let be a function $g(\rho, \theta)$ of $(\rho, \theta)$ and consider its variations after change in variables into $(x, y)$. Assume that $\frac{\partial g}{\partial \rho} \geq 0$ for all $\rho > 0, \theta \in ]0, \pi/2[$, so as to avoid boundaries where the derivatives of interest are not defined. Let us show that $\frac{\partial g}{\partial x} \geq 0$ and $\frac{\partial g}{\partial y} \geq 0$. Fixing $\theta = 0$ (respectively, $\theta = \pi/2$) yields $\rho = x$ (respectively, $\rho = y$) and $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial \rho}_{|\theta=0} \geq 0$ (respectively, $\frac{\partial g}{\partial y} = \frac{\partial g}{\partial \rho}_{|\theta=\pi/2} \geq 0$), in this particular direction. Another way to see this result is to notice that $\frac{\partial g}{\partial x}$ is the orthogonal projection of $\frac{\partial g}{\partial \rho}$ along the $y$-axis. The identity of the signs of $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial \rho}$ can be obtained in the same fashion.

Incrementing the derivation order with respect to $\rho$ (i.e., imposing successively $\frac{\partial^2 g}{\partial \rho^2} \leq 0, \frac{\partial^3 g}{\partial \rho^3} \geq 0, \frac{\partial^4 g}{\partial \rho^4} \leq 0$) allows us to obtain the successive and respective non-positive and non-negative partials of order 2, 3 and 4 with respect to $(x, y)$, as the consequence of iterating the previous reasoning by fixing $\theta = 0$ and $\theta = \pi/2$. Therefore, $\frac{\partial^{k_1+k_2} g}{\partial x^{k_1} \partial y^{k_2}}$ is of the sign of $(-1)^{k_1+k_2+1}$, as it is the sign of $\frac{\partial^{k_1+k_2} g}{\partial \rho^{k_1+k_2}}$. Therefore, $[u$ is $4 - idircv$ in $(x, y)] \Longleftrightarrow [u$ is $4 - icv$ in $\rho]$.

Finally, it is easy to obtain the stochastic dominance results of the proposition by applying already known results of one-dimensional stochastic dominance for the $4 - icv$ utility functions and recalled in Theorem 6.

$(\beta)$ comes from Theorem 5, using the change of variable and class of functions shown in $(\alpha)$. $(\gamma)$ is a consequence of $(\alpha)$ and the knowledge of the generators for one-dimensional fourth-order stochastic dominance. QED.

**Proof of Theorem 7**:

Let $P^{k_1, k_2}(x, y; z_1, z_2) = \frac{1}{z_1^{k_1} z_2^{k_2}} \int\int\limits_{[0,z_2][0,z_1]} (z_1 - x)^{k_1-1} (z_2 - y)^{k_2-1} dF(x, y)$. If one inte-

grates by parts $k_1-1$ times with respect to $x$, then one obtains: $\frac{1}{(k_1-1)!} \int_{[0,z_2]} F^{k_1}(z_1;y) \ (z_2-y)^{k_2-1} \ dy.$
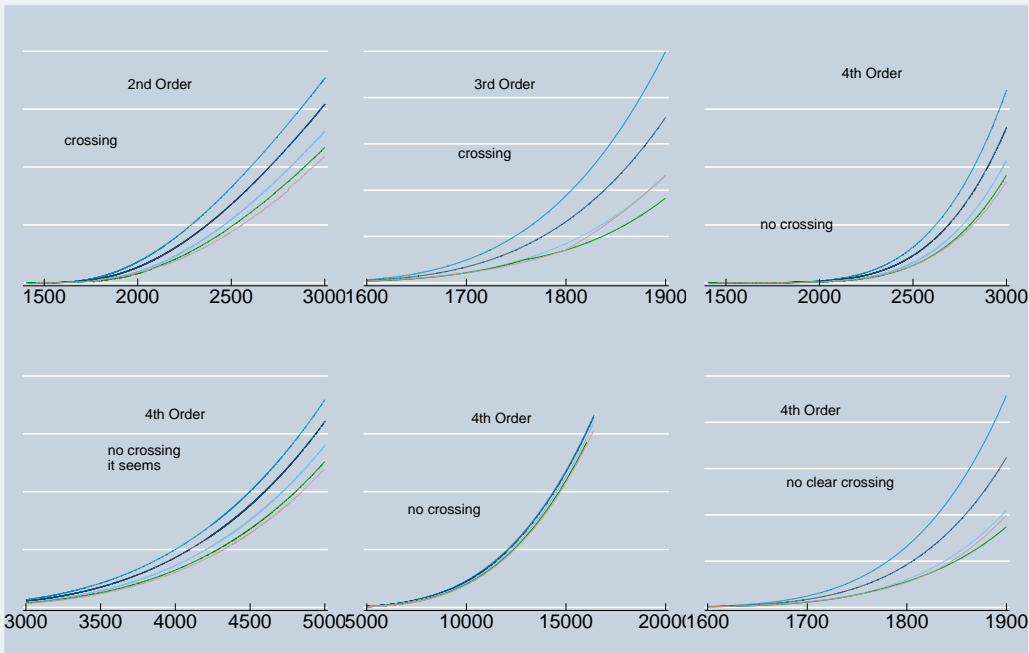
Then, $k_2 - 1$ additional successive integration by parts with respect to $y$ gives:

$\frac{1}{(k_1-1)!(k_2-1)!} \ \frac{1}{z_1^{k_1} z_2^{k_2}} F^{k_1,k_2}(z_1, z_2)$, where $F^{k_1,k_2}$ is the joint density function $f(x,y)$ integrated $k_1$ times with respect to $x$ and $k_2$ times with respect to $y$. This implies the equivalence of the two considered orderings. QED.
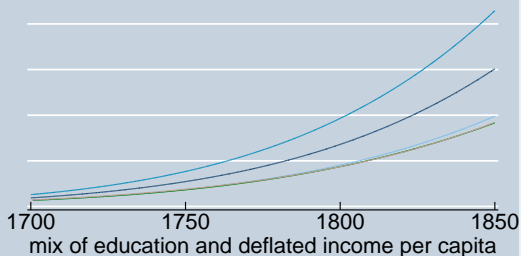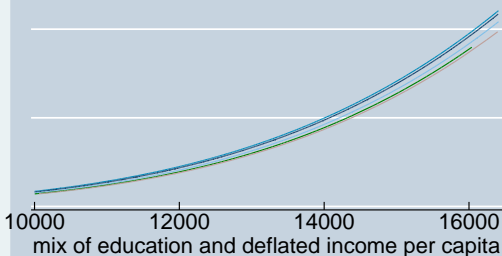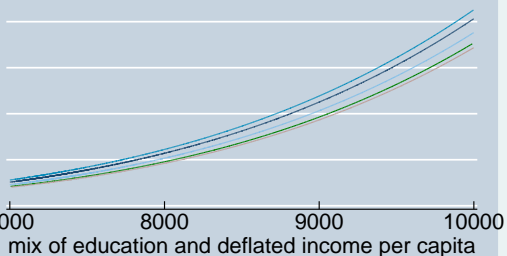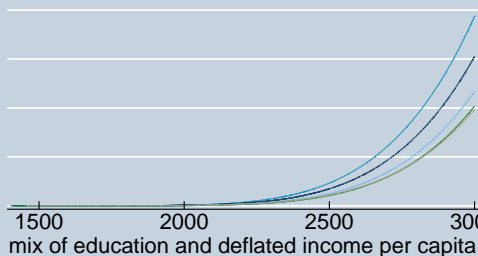
**Insert Figures 1 and 2**

# Dominance Tests
## Details of domains

# Dominance Tests 5th Order
## Details of domains

mix of education and deflated income per capita

# Dominance Tests 6th Order

## Details of domains



mix of education and deflated income per capita

mix of education and deflated income per capita

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