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Pollution, public debt, and growth: The question of sustainability*

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Abstract

This paper examines an endogenous growth model that allows us to consider the dynamics and sustainability of debt, pollution, and growth. Debt evolves according to the financing adaptation and mitigation efforts and to the damages caused by pollution. Three types of features are important for our analysis: The technology through the negative effect of pollution on TFP; The fiscal policy; The initial level of pollution and debt with respect to capital. Indeed, if the initial level of pollution is too high, the economy is relegated to an endogenous tipping zone where pollution perpetually increases relatively to capital. If the effect of pollution on TFP is too strong, the economy cannot converge to a stable and sustainable long-run balanced growth path. If the income tax rates are high enough, we can converge to a stable balanced growth path with low pollution and high debt relative to capital. This sustainable equilibrium can even be characterized by higher growth and welfare. This last result underlines the role that tax policy can play in reconciling debt and environmental sustainability.

JEL classification: E60; H63; Q54; Q58

Keywords: Environmental damage, Pollution, Fiscal policy, Public debt, Sustainability.

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1 Introduction

In the context of growing public debt and rising costs related to global pollution, it is crucial to have a clear understanding of the interplay between debt and global pollution dynamics. Today, several countries that are particularly vulnerable to the impacts of climate change also find themselves burdened with high levels of debt. The economic repercussions of the COVID-19 pandemic further exacerbated this situation (see e.g Dibley et al., 2021). This issue is widely brought up for developing countries,[1] but is also a major concern for developed countries. Indeed, since 2008, we have observed in all groups of countries an increasing trend in the share of debt in GDP (Figure 1).

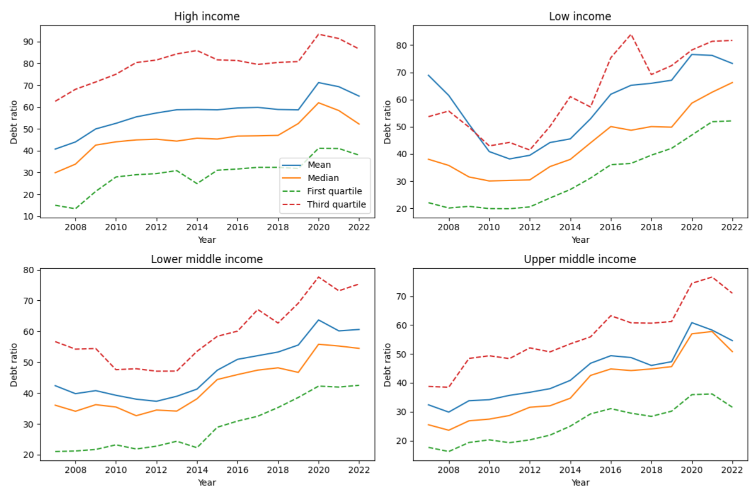

Figure 1: Evolution of the central government debt to GDP ratio by groups of countries (World Bank classification in 2023). IMF Global Debt Database.

Concerning expenses related to global pollution, in addition to the substantial investments linked to the transition to a less polluting economy, the expenses associated with adaptation are expected to grow in all countries (IPCC, 2022). Meanwhile, the connections between public debt and global environmental challenges can be illustrated by the

[1] In December 2023, among 67 low-income countries, the Debt Sustainability Analysis provided by the World Bank Group and the IMF identified 28 countries with a high risk of overall debt distress and 11 countries already in distress. This worrisome observation is coupled with the increase in the frequency and intensity of extreme weather events in these vulnerable countries.

positive and substantial impacts of climate vulnerability on debt (Buhr et al., 2018), suggesting the existence of a detrimental cycle wherein vulnerability perpetuates itself through public debt management. Moreover, the rise in disaster-related losses will result in reduced tax revenues. Zenios (2024) gives an overview of the direct and indirect channels and suggests a possible doom loop between climate change and sovereign debt. We consider in this paper these different dimensions to examine the interplay between debt and pollution dynamics.

As reported by IMF (2023), policymakers face a fundamental trade-off. On the one hand, relying on spending-based measures to achieve emission goals and to adapt could lead to a substantial increase in public debt. On the other hand, limited environmental action exposes the world to adverse consequences from global pollution, increasing the cost of adaptation. The recent proposals from the European Commission regarding new economic governance rules highlight the interconnected nature of questions surrounding fiscal sustainability and environmental concerns. In particular, the Commission considers climate change as a structural trend representing a challenge to the financial stability of Member States' public finances.[2] The Pisani-Ferry and Mahfouz (2023) report is in line with this argument. In the case of France, it recommends using debt to finance the investments needed for the ecological transition. The use of debt should be limited to "green" investments that have a positive impact on the climate and generate long-term economic returns. This policy must be accompanied by more progressive taxation.

This paper contributes to this debate. Within an endogenous growth framework, we study the dynamic path of pollution, debt, and economic growth when public authorities finance mitigation and adaptation to tackle the damages caused by the pollution stock. From a normative perspective, we look at how fiscal and environmental policy instruments can be used to guarantee sustainability and improve welfare.

We develop an overlapping generations (OLG) model where debt, pollution, and growth are endogenous. Households live for two periods and save through two assets, capital, the source of growth, and public debt. The government issues debt securities because taxes on capital and labor incomes do not cover public expenses for pollution mitigation and adaptation, and the service of past debt. The pollution stock evolves

[2] See https://ec.europa.eu/commission/presscorner/detail/en/ip_23_2393.

with production activity and mitigation measures, and is a source of damage by reducing aggregate productivity (TFP). Technology plays an important role: The final good is produced using an AK-type production function to have a simple engine of growth, but considering that the adaptation policy of the government can dampen the negative effect of pollution on TFP.

The long-run equilibria analysis shows that two balanced growth paths (BGPs) may exist, defined by constant ratios of debt and pollution over capital. One is characterized by a low pollution-to-capital ratio and a high debt-to-capital ratio while the reverse is observed for the other. When TFP vulnerability to pollution is not too high, the growth rate is higher at the BGP with low pollution and high debt, which also means higher welfare at this BGP. In that case, the higher long-term growth rate is at the expense of a high level of debt per unit of capital. This suggests a crowding-in effect of debt on growth.

The analysis of dynamics allows us to show that depending on fiscal policy, the TFP vulnerability to pollution, and initial conditions of capital, debt, and pollution stocks, the economy either converges to the BGP with low pollution over capital, collapses, or experiences a perpetual increase in pollution over capital. More precisely, we show that the BGP with a high pollution-to-capital ratio is not sustainable, as the economy cannot converge to this state, while the convergence to the BGP with low pollution-to-capital may be possible for a sufficiently high labor taxation and a reasonable TFP vulnerability to pollution. We identify an extreme case in which sustainability is completely excluded. None of the two BGPs is stable and the economy either collapses or is characterized by a perpetual increase of pollution over capital. This unfavorable situation occurs when the tax rates on labor and capital income are low and TFP vulnerability to pollution is high.

If the BGP with high pollution over capital is the only unstable one, it is a saddle toward which the stock variables cannot converge. Therefore, the stable manifold of this equilibrium defines an endogenous tipping zone (ETZ). If the initial conditions on debt, pollution, and capital are such that the economy is in this zone, the dynamics of pollution relative to capital explode. Interestingly, the higher the debt relative to pollution, the easier the economy can be relegated to the ETZ. It corresponds to a form of debt vulnerability: Higher debt favors unsustainable dynamic paths for pollution.

Finally, in the case of a long-run sustainability, we investigate if the policy can still improve welfare. Therefore, we analyze the effect of the fiscal policy, mitigation and adaptation on the level of the stable BGP. When TFP vulnerability to pollution is not excessive, increasing taxation enhances welfare along the stable BGP, while the effects of environmental policy instruments depend greatly on their efficiency. Insufficiently efficient mitigation in reducing pollution may worsen the pollution-to-capital ratio, whereas adaptation improves welfare only if TFP responds strongly to such expenditure. In general, this analysis shows the crucial role played by adaptation.

Previous contributions studied the macroeconomic implications of the interplay between public debt and environmental factors (Heijdra et al., 2006; Fodha and Seegmuller, 2012, 2014). Nonetheless, debt is often considered as an exogenous instrument, and its dynamic evolution of financing adaptation and mitigation together with the dynamic path of pollution is left aside, meaning that the question of sustainability is not properly addressed. In Baret and Menuet (2024), debt allows financing mitigation expenditure. However, this paper cannot address the question of sustainability by assuming a constant long-term debt-to-output ratio and a stabilizing rule ensuring convergence towards this objective. Moreover, they leave aside adaptation expenditures while they must be significantly integrated into environmental spending. We go one step further by developing a model that incorporates endogenous public debt dynamics and acknowledges the impact of environmental issues on fiscal sustainability. This dimension seems essential to address the economic consequences of global pollution and highlights how it exerts its influence on sovereign debt. Our paper is related to some recent papers that consider endogenous public debt and its dynamics. For example, Boly et al. (2022) examine the relationship between public and environmental debt. We depart from this paper focusing on fiscal sustainability and considering the economic damage entailed by pollution stock and its impact on debt. Catalano et al. (2020) examine the role of fiscal policy in climate change adaptation. They use a calibrated macroeconomic model of an open economy, that does not allow them to explicitly identify the interplay between debt dynamics, growth, and the environment. More generally, and in contrast to the literature that assumes limits on long-term public debt (Baret and Menuet, 2024; Boly et al., 2022; Seghini and Dées, 2024), we do not impose any restrictions on sovereign debt. This is crucial for studying environment-debt interactions and sustainability.

The rest of this paper is organized as follows. Section 2 presents an OLG model in which pollution is proportional to capital stock, and the government issues debt and imposes taxes on capital and labor incomes for financing adaptation and mitigation expenditures. Section 3 defines the equilibrium. Section 4 studies the balanced growth paths and examines the existence and multiplicity of BGPs. Section 5 considers the dynamics and the possibility of an endogenous tipping zone. Section 6 presents some policy implications. The final section provides the conclusion.

2 The model

We consider a dynamic model with pollution and three types of agents, firms, consumers, and a government. Time is discrete, $t = 0, 1, ..., +\infty$, and there is no uncertainty.

2.1 Production

We consider an AK model of economic growth in which TFP decreases with pollution stock.[3] Considering that pollution or climate change is detrimental to production is particularly relevant in addressing debt and environmental issues, as it allows to focus on funding adaptation efforts, extending beyond mere mitigation. The need for adaptation strategies will increase with the intensification of climate change impacts. These adaptations come with associated costs, such as building infrastructure to protect against rising sea levels or creating drought-resistant agriculture. Public action can provide the necessary financial resources to implement these adaptation measures, reducing the vulnerability of countries to environmental shocks. We thus assume that the capacity for adaptation reduces the incremental damage caused by pollution stock. This ability to adapt is ensured by the public authorities, who devote an amount G_{1t}, specifically for

[3] Burke et al. (2015) finds a nonlinear decline in macroeconomic productivity following a change in temperature, across sectors, in 166 rich and poor countries since the 1960s. This finding is confirmed by Kalkuhl and Wenz (2020) which looks at the effects of rising global mean surface temperature on production.

this purpose.[4]

Therefore, the production is given by:

$$Y_t = A\left(\frac{G_{1t}}{P_t}\right) K_t^{\alpha}(\bar{K}_t L_t)^{1-\alpha} \tag{1}$$

with $\bar{K}$ the aggregate level of capital, $\alpha \in (0,1)$ and:

Assumption 1 *$A(X)$ is a strictly increasing function, with $A(0) = A_0 \geqslant 0$, $A(+\infty) = A_1 < +\infty$, and $A'(X)X/A(X) \in (0,1)$.*

This last assumption implies an elasticity of production to the adaptation to pollution ratio lower than one. When the adaptation to pollution ratio goes up it causes a less-than-proportional increase in productivity.

Note that $A(X)$ may capture the fact that climate change destroys a part of aggregate output at each period (Golosov et al., 2014; Dietz and Stern, 2015). It can also represent the health effects of global pollution stock or the impacts of a change in temperature, which results in reduced aggregate productivity (Dasgupta et al., 2021; Burke et al., 2015).

Example: we can consider the following specifications for $A(X)$:

$$A(X) = \frac{A_1 X}{1+X} \tag{2}$$

with $A(0) = 0$. This function is increasing and concave, with:

$$\frac{A'(X)X}{A(X)} = \frac{1}{1+X} \in (0,1) \tag{3}$$

Let r_t be the interest rate and w_t the wage. At equilibrium, we have $\bar{K}_t = K_t$ and assuming that the population in this economy is equal to one, labor input is $L_t = 1$.

[4]Adaptation is formalized as a flow, that reduces the damage of pollution stock. This implies that adaptation expenses are recurrent during each period. Considering that adaptation is infrastructure, this means that we assume a perfect depreciation of such infrastructure at each period. This appears consistent in our OLG model, where one period represents around 35 years. In this time perspective, we suppose that changing climatic conditions necessitate ongoing adjustments, updates, or improvements.

Therefore, profit maximization gives:

$$r_t = \alpha A \left(\frac{G_{1t}}{P_t}\right) \quad (4)$$

$$w_t = (1-\alpha) A \left(\frac{G_{1t}}{P_t}\right) K_t \quad (5)$$

Returns of factor being a positive function of productivity, they decrease with pollution.

2.2 Pollution

The stock of pollution increases with the emission flow and partly leaves the atmosphere through a natural process in a share $0 < m < 1$. Emission flow is assumed to be proportional to the stock of capital. Mitigation measures G_{2t} are implemented by public authorities to enhance the sinks of emissions. The stock of pollution evolves according to:

$$P_{t+1} = (1-m)P_t - \psi G_{2t} + \mu K_t \quad (6)$$

The parameter $\psi > 0$ captures the efficiency of public abatement and $\mu > 0$ pollution flow resulting from capital stock. This stock of pollution only affects the real side of the economy through its negative effect on the TFP. We will not consider a direct negative effect of pollution on households welfare.

2.3 Government

We consider public actions to tackle environmental issues. Public spending G_t linearly increases with GDP:

$$G_t = gY_t \quad (7)$$

and are divided into public spending which attenuates the effect of pollution on production G_{1t}, i.e. adaptation to climate effect, and mitigation G_{2t}:

$$G_{it} = g_i Y_t \quad (8)$$

for $i = 1, 2$, with $g_1 + g_2 = g$.

Environmental policy instruments consist of public spending on both mitigation and adaptation. While most of the literature has focused on their potential substitutability, these two strategies are now seen as simultaneously needed in the face of climate emergencies. This need is reflected in international ambitions to achieve a balance in climate finance spending between the two strategies (Sadler et al., 2024).

Since we are in an endogenous growth framework, we assume that the government determines its public spending for adaptation and mitigation by fixing their amount per GDP unit. This means that the policy of adaptation will be determined by g_1 and the policy of mitigation by g_2. Of course, g_2 could be considered as not too high, because it is more realistic to assume that a country cannot strongly affect the global stock of pollution through its own policy of mitigation.

To finance these spending, the government collects taxes on labor and capital incomes, τ_L and τ_K, and issues debt B_t. Therefore, its expenditures include repayment of debt and interest payments. The government faces the following budget constraint at each period:

$$R_t^b B_t + G_t = B_{t+1} + \tau_L w_t + \tau_K r_t K_t \tag{9}$$

with R_t^b the interest factor of debt and $B_0 > 0$ the initial stock of debt. We are in an economy with a positive initial stock of public debt. The different policy parameters as well as the interest factor of debt and the income will determine how public debt evolves through time. We will precisely study the interplay between debt accumulation, dynamics of pollution stock, and growth.

2.4 Consumers

Consumers are in overlapping generations. The population size of each generation is constant and normalized to one. Each consumer lives for two periods, consumes in both periods, and saves through two assets, public debt and capital. Capital depreciates at rate $\delta \in (0, 1)$, meaning that return on capital is given by $1 - \delta + (1 - \tau_K) r_t$.

The utility function of the generation born in t is given by:

$$U(c_t, d_{t+1}) = \left(c_t^{\frac{\sigma-1}{\sigma}} + \beta d_{t+1}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{10}$$

with $\beta \in (0, 1)$ and $\sigma > 1$. This last assumption ensures intertemporal substitutability, which will imply a saving rate increasing in the return of assets. The household maximizes his utility under the two budget constraints:

$$c_t + K_{t+1} + B_{t+1} = (1-\tau_L)w_t \quad (11)$$

$$d_{t+1} = [1-\delta+(1-\tau_K)r_{t+1}]K_{t+1} + R^b_{t+1}B_{t+1} \quad (12)$$

Since bonds and capital assets are perfect substitutes, they provide the same return:

$$1-\delta+(1-\tau_K)r_{t+1} = R^b_{t+1} \equiv R_{t+1} \quad (13)$$

Optimal choices give the saving function:

$$K_{t+1} + B_{t+1} = \frac{\beta^\sigma R_{t+1}^{\sigma-1}}{1+\beta^\sigma R_{t+1}^{\sigma-1}}(1-\tau_L)w_t \quad (14)$$

which is increasing in R_{t+1} because $\sigma > 1$, and is in accordance with a saving function increasing in the interest factor of assets.

3 Equilibrium

We define an equilibrium as a function of capital, debt, and pollution stocks. Market clearing is obtained substituting (4), (5) and (13) in (6), (9) and (14). We get the following functions:

$$K_{t+1} + B_{t+1} = \frac{\beta^\sigma R_{t+1}^{\sigma-1}}{1+\beta^\sigma R_{t+1}^{\sigma-1}}(1-\tau_L)(1-\alpha)A\left(\frac{G_{1t}}{P_t}\right)K_t \quad (15)$$

$$B_{t+1} = R_t B_t + gA\left(\frac{G_{1t}}{P_t}\right)K_t - (\tau_L(1-\alpha)+\tau_K\alpha)A\left(\frac{G_{1t}}{P_t}\right)K_t \quad (16)$$

$$P_{t+1} = (1-m)P_t - \psi g_2 A\left(\frac{G_{1t}}{P_t}\right)K_t + \mu K_t \quad (17)$$

We introduce the following variables to conduct our analysis: The growth factor $\gamma_{t+1} \equiv K_{t+1}/K_t$, debt per unit of capital $b_t \equiv B_t/K_t$, and pollution per unit of capital $\pi_t \equiv P_t/K_t$. Using the government budget (8), public adaptation per unit of capital can thus be written as a function of π_t:

$$\frac{G_{1t}}{P_t} = \frac{g_1}{\pi_t}A\left(\frac{G_{1t}}{P_t}\right) \quad (18)$$

Since $A'(x)x/A(x) \in (0,1)$, this equation implicitly defines a decreasing function $G_{1t}/P_t = \varepsilon(\pi_t)$ if $\lim_{x\to 0} A(x)/x > \pi_t/g_1$.[5] Using Assumption 1 and equation (18), we further have

[5]Note that it is always satisfied if $A(0) > 0$. If $A(0) = 0$, the condition is equivalent to $\lim_{x\to 0} A'(x) > \pi_t/g_1$.

$\varepsilon(0) = +\infty$ and $\varepsilon(+\infty) = 0$. As pollution per unit of capital increases, the amount of public spending dedicated to addressing pollution-related issues for each unit of pollution decreases. Because of the productivity cost entailed by pollution, the higher the pollution per unit of capital, the lower the resources that can be allocated to adaptation. This may be counteracted if the government decides to allocate a larger proportion of its budget to climate adaptation efforts, *i.e* if g_1 increases.

The total factor productivity can thus be expressed as a function of pollution per capital. We have $A(G_{1t}/P_t) = A[\varepsilon(\pi_t)] \equiv a(\pi_t)$, with $a'(\pi_t) < 0$, $a(0) = A[\varepsilon(0)] = A(+\infty) = A_1$, and $a(+\infty) = A[\varepsilon(+\infty)] = A(0) = A_0$. Similarly, the interest factor can be written as $R_{t+1} = 1 - \delta + (1 - \tau_K)\alpha a(\pi_{t+1}) \equiv R(\pi_{t+1})$, with $R'(\pi_{t+1}) < 0$.

Example (continued): considering our example given by equation (2), we have $\varepsilon(\pi_t) = g_1 A_1/\pi_t - 1$*, which implies that:*

$$a(\pi_t) = A_1 - \frac{\pi_t}{g_1} \tag{19}$$

which requires that $\pi_t < g_1 A_1$.

Now, we can rewrite the dynamic system (15)-(17) as follows:

$$\gamma_{t+1} + b_{t+1}\gamma_{t+1} = \Sigma(\pi_{t+1})(1-\tau_L)(1-\alpha)a(\pi_t) \tag{20}$$

$$b_{t+1}\gamma_{t+1} = R(\pi_t)b_t + ga(\pi_t) - (\tau_L(1-\alpha) + \tau_K\alpha)a(\pi_t) \tag{21}$$

$$\pi_{t+1}\gamma_{t+1} = (1-m)\pi_t - \psi g_2 a(\pi_t) + \mu \tag{22}$$

with $\Sigma(\pi_{t+1}) \equiv \frac{\beta^\sigma R(\pi_{t+1})^{\sigma-1}}{1+\beta^\sigma R(\pi_{t+1})^{\sigma-1}}$ the saving rate characterized by $\Sigma'(\pi_{t+1}) < 0$ because $\sigma > 1$. Saving falls with π because the interest rate is reduced by pollution per unit of capital.

Rearranging equations (20)-(22), we finally obtain:

$$\gamma_{t+1} = a(\pi_t)\left[\Sigma(\pi_{t+1})(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi_t)}{a(\pi_t)}b_t\right] \tag{23}$$

$$b_{t+1} = \frac{\frac{R(\pi_t)}{a(\pi_t)}b_t + g - (\tau_L(1-\alpha)+\tau_K\alpha)}{\Sigma(\pi_{t+1})(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi_t)}{a(\pi_t)}b_t} \tag{24}$$

$$\pi_{t+1} = \frac{(1-m)\frac{\pi_t}{a(\pi_t)} - \psi g_2 + \frac{\mu}{a(\pi_t)}}{\Sigma(\pi_{t+1})(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi_t)}{a(\pi_t)}b_t} \tag{25}$$

Equations (24) and (25) give the dynamics of (b_t, π_t) for $t \geq 0$, taking into account that both b_t and π_t are predetermined variables. The dynamics of debt per capital is coupled with the dynamics of pollution per unit of capital for two reasons. First, because the depreciation of capital is not complete. This implies that the cost of capital falls with productivity loss but not in the same proportion as growth $\left(\frac{R(\pi_t)}{a(\pi_t)} \text{ still depends on } \pi_t\right)$. Second, because saving rate increases with productivity, a factor that is diminished by pollution per unit of capital.

The value of the growth factor γ_t is deduced from these two variables using equation (23). To conduct our analysis, we focus on relevant situations in which the growth factor, debt per unit of capital, and pollution per unit of capital are all positive. To ensure $\gamma_{t+1} > 0$, $b_{t+1} > 0$ and $\pi_{t+1} > 0$, we assume a primary deficit and the following restrictions:

$$\begin{aligned}
\Sigma(\pi_{t+1})(1-\tau_L)(1-\alpha) &> \frac{R(\pi_t)}{a(\pi_t)} b_t + g - (\tau_L(1-\alpha) + \tau_K \alpha) \\
\mu &> a(\pi_t)\psi g_2
\end{aligned}$$

They are satisfied under the next assumption:

Assumption 2

$$\begin{aligned}
b_t &< \frac{a(+\infty)}{R(+\infty)}[\Sigma(+\infty)(1-\tau_L)(1-\alpha) - g + \tau_L(1-\alpha) + \tau_K \alpha] \\
g &> \tau_L(1-\alpha) + \tau_K \alpha \\
\mu &> a(0)\psi g_2
\end{aligned}$$

The first inequality characterizes an upper bound for public debt, which increases with the amount of savings and decreases with the primary deficit, the second a primary deficit, and the third emission intensity per unit of capital higher than the efficiency of pollution abatement.

Lemma 1 *Let us note $\epsilon_a(\pi_t) \equiv a'(\pi_t)\pi_t/a(\pi_t) < 0$. Under Assumption 2, $|\epsilon_a(\pi_t)|$ not infinite and σ close to 1, we get:*

$$\gamma_{t+1} = f_1(b_t, \pi_t) \tag{26}$$

$$b_{t+1} = f_2(b_t, \pi_t) \tag{27}$$

$$\pi_{t+1} = f_3(b_t, \pi_t) \tag{28}$$

where $f_3(b_t, \pi_t)$ and $f_2(b_t, \pi_t)$ are increasing with respect to b_t and π_t, and $f_1(b_t, \pi_t)$ decreasing with respect to b_t and π_t. Equations (27) and (28) determine the dynamics of (b_t, π_t) for all $t \geq 0$, while the growth factor is given by (26).

Proof. See Appendix A. ■

4 Balanced growth paths: multiplicity and main features

We focus here on long-run equilibria. We first show the existence and multiplicity of BGPs. Then, we investigate the main features of these equilibria. We will in particular understand how they are ranked according to the levels of debt per capital, pollution per capital, and growth.

4.1 Existence and multiplicity of BGPs

Along a balanced growth path, capital, debt, and pollution grow at a constant rate $\gamma - 1$. A balanced growth path is thus characterized by $b_t = b_{t+1} = b$ and $\pi_t = \pi_{t+1} = \pi$ solving (27) and (28). Hence, it is a stationary solution (b, π) to:

$$b = \frac{\frac{R(\pi)}{a(\pi)}b + g - (\tau_L(1-\alpha) + \tau_K\alpha)}{\Sigma(\pi)(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha) + \tau_K\alpha) - g - \frac{R(\pi)}{a(\pi)}b} \tag{29}$$

$$\pi = \frac{(1-m)\frac{\pi}{a(\pi)} - \psi g_2 + \frac{\mu}{a(\pi)}}{\Sigma(\pi)(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha) + \tau_K\alpha) - g - \frac{R(\pi)}{a(\pi)}b} \tag{30}$$

Given such a solution, the growth factor corresponds to:

$$\gamma = a(\pi)\left[\Sigma(\pi)(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha) + \tau_K\alpha) - g - \frac{R(\pi)}{a(\pi)}b\right] \tag{31}$$

The ratio of (29) and (30) gives:

$$b = \frac{g - (\tau_L(1-\alpha) + \tau_K\alpha)}{\frac{1-m-R(\pi)}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}} \tag{32}$$

Using (13), it is equivalent to:

$$b = \frac{a(\pi)\pi[g - (\tau_L(1-\alpha) + \tau_K\alpha)]}{\pi(\delta - m) - (1-\tau_K)\alpha a(\pi)\pi - \psi g_2 a(\pi) + \mu} = \frac{g - \tau_L(1-\alpha) - \tau_K\alpha}{X(\pi) - \frac{1-\delta}{a(\pi)} - (1-\tau_K)\alpha} \equiv B_1(\pi) \tag{33}$$

with $X(\pi) \equiv \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$.

Moreover, (30) can be rewritten as:

$$\begin{aligned} b &= \frac{a(\pi)}{R(\pi)} \left[\Sigma(\pi)(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K \alpha - g - X(\pi) \right] \\ &\equiv B_2(\pi) \end{aligned} \tag{34}$$

In the following, we assume:

Assumption 3 *$\delta \geq m$, $\epsilon_a(\pi) < -1$ and finite, σ higher but close to 1.*

The first part of the assumption implies that the rate of pollution absorption is lower than the depreciation rate of capital. This is consistent in our context, as pollution stock can refer to greenhouse gases whose some will remain in the atmosphere for thousands of years. The second part of the assumption implies that total factor productivity is elastic to pollution over capital ratio, illustrating an important vulnerability to climate change (see IPCC, 2022). When pollution per unit of capital increases total factor productivity falls more than proportionally. It implies that $a(\pi)\pi$ is decreasing in π. As a result, $X(\pi)$ is an increasing function of π. The last part of the assumption ensures that Lemma 1 is satisfied.

Example (continued): Note that with the example defined in equation (19), $\epsilon_a(\pi) < -1$ implies that $\pi > g_1 A_1/2$.

This example illustrates the fact that $\epsilon_a(\pi) < -1$ could introduce a lower bound $\underline{\pi} > 0$ defined by $\epsilon_a(\underline{\pi}) = -1$ such that $\epsilon_a(\pi) < -1$ for all $\pi > \underline{\pi}$.

Under Assumptions 2 and 3, the numerator of (33) is positive and the denominator is increasing in π. We thus have $B_1'(\pi) < 0$. In addition, to ensure a positive debt along the balanced growth path, we restrict our attention to cases where:

$$X(\pi) - \frac{1-\delta}{a(\pi)} > (1-\tau_K)\alpha \tag{35}$$

New debt emissions should be higher than the cost of existing debt. Indeed, using (30) and (31), $X(\pi) = \gamma / a(\pi)$. It implies that inequality (35) is equivalent to $\gamma > R(\pi)$. Using (33), we have $b[\gamma - R(\pi)] = a(\pi)[g - \tau_L(1-\alpha) - \tau_K \alpha]$. Since we assume that the government budget is characterized by a primary deficit, a BGP should be characterized by a growth factor larger than the interest factor.

Since the left-hand side of inequality (35) is increasing in π, there exists $\pi_1 > 0$ such that $X(\pi_1) = \frac{1-\delta}{a(\pi_1)} + (1-\tau_K)\alpha$ if there is a value $\widetilde{\pi} > 0$ such that $X(\widetilde{\pi}) < \frac{1-\delta}{a(\widetilde{\pi})} + (1-\tau_K)\alpha$. Then, inequality (35) is satisfied for all $\pi > \pi_1$, with $\lim_{\pi \to \pi_1^+} B_1(\pi) = +\infty$. Note that the existence of $\pi_1 > 0$ can be compatible with Assumption 2.

Lemma 2 *Under Assumptions 1-3, assume that there exists a value $\widetilde{\pi} > 0$ such that $X(\widetilde{\pi}) < \frac{1-\delta}{a(\widetilde{\pi})} + (1-\tau_K)\alpha$. There exists $\pi_1 > 0$ such that $X(\pi) > \frac{1-\delta}{a(\pi)} + (1-\tau_K)\alpha$ for $\pi > \pi_1$.*

Our example illustrates that this lemma is satisfied for a non-empty set of parameters.

Example (continued): Using the example defined in equation (19), we illustrate the existence of the bound π_1. Inequality (35) writes $F(\pi) > (1-\tau_K)\alpha$, with:

$$F(\pi) \equiv \frac{\pi[g_1 A_1(\delta - m) + \psi g_2] + g_1 A_1(\mu - \psi g_2)}{\pi(g_1 A_1 - \pi)} \tag{36}$$

Since $F(\pi)$ is an increasing function and $F(g_1 A_1) = +\infty$, there exists $\pi_1 \in (g_1 A_1/2, g_1 A_1)$ if $F(g_1 A_1/2) < (1-\tau_K)\alpha$. This happens if $\delta - m < A_1(1-\tau_K)\alpha/2$ and

$$g_1 > \frac{4\mu - 2\psi g_2 A_1}{A_1[(1-\tau_K)\alpha A_1 - 2(\delta - m)]}$$

Using (34), a steady state with positive debt per unit of capital should satisfy:

$$\Sigma(\pi)(1-\tau_L)(1-\alpha) > g - \tau_L(1-\alpha) - \tau_K\alpha + X(\pi) \tag{37}$$

Since the left-hand side of this inequality is decreasing and the right-hand side is increasing in π, there exists $\pi_2 > 0$ such that inequality (37) is satisfied for all $\pi < \pi_2$, with $B_2(\pi_2) = 0$. In this case, we also deduce that under Assumption 3, we have $B_2'(\pi) < 0$. For this, we need to have a value $\widehat{\pi} > 0$ such that $\Sigma(\widehat{\pi})(1-\tau_L)(1-\alpha) < g - \tau_L(1-\alpha) - \tau_K\alpha + X(\widehat{\pi})$.

Lemma 3 *Under Assumptions 1-3, assume that there exists a value $\widehat{\pi} > 0$ such that $\Sigma(\widehat{\pi})(1-\tau_L)(1-\alpha) < g - \tau_L(1-\alpha) - \tau_K\alpha + X(\widehat{\pi})$. The interval (π_1, π_2) is non empty if the following inequality is satisfied:*

$$\Sigma(\pi_1)(1-\tau_L)(1-\alpha) > g - \tau_L(1-\alpha) - \tau_K\alpha + \frac{1-\delta}{a(\pi_1)} + (1-\tau_K)\alpha \tag{38}$$

where π_2 *is defined by* $\Sigma(\pi_2)(1-\tau_L)(1-\alpha) = g - \tau_L(1-\alpha) - \tau_K\alpha + X(\pi_2)$.

Example (continued) : In our example, let $\widehat{\pi} = g_1 A_1$ *be such that* $a(\widehat{\pi}) = 0$. *In this case, we have* $X(\widehat{\pi}) = +\infty$, *which ensures the first inequality in the lemma.*

Note that in our example, we have π_1 *is higher but arbitrarily close to* $g_1 A_1/2$ *if* g_1 *tends to* $\frac{4\mu - 2\psi g_2 A_1}{A_1[(1-\tau_K)\alpha A_1 - 2(\delta - m)]}$. *Therefore, inequality (38) is satisfied if* $\Sigma(g_1 A_1/2)(1-\tau_L)(1-\alpha) > g - \tau_L(1-\alpha) - \tau_K\alpha + \frac{2(1-\delta)}{A_1} + (1-\tau_K)\alpha$. *This last inequality is satisfied if* τ_K *and* A_1 *are high enough and the primary deficit is not too important. It proves the existence of* π_2 *and of a non-empty interval* (π_1, π_2).

Since $B_1(\pi_1) = +\infty > B_2(\pi_1)$ and $B_1(\pi_2) > B_2(\pi_2) = 0$, the economy may be characterized by an even number (two) of steady states.

Proposition 1 *Under Assumptions 1-3, and inequality (38), there exists* $\overline{g} > \tau_L(1-\alpha) + \tau_K\alpha$ *such that for* $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \overline{g})$, *there are (at least) two BGPs,* (π_I, b_I) *and* (π_{II}, b_{II}), *with* $\pi_I < \pi_{II}$ *and* $b_I > b_{II}$.

Proof. See Appendix B. ■

A primary deficit ($g > \tau_L(1-\alpha) + \tau_K\alpha$) ensures the possibility of having a positive stationary debt per unit of capital in our context where growth is higher than the interest factor. At the same time, if the environmental expenditure is too high ($g > \overline{g}$), the primary deficit is too significant to observe stationarity in debt per unit of capital. The share of GDP devoted to environmental issues has to be an intermediate to observe stationary solutions. If it is satisfied, we may have two BGP. The BGP characterized by the lowest pollution to capital ratio (π_I) has the highest level of debt over capital (b_I), while the one with the highest pollution-to-capital ratio (π_{II}) is also defined by the lowest debt per unit of capital (b_{II}). We can note that the decreasing relationship we observe between π and b is ensured by the sufficient TFP vulnerability ($\epsilon_a(\pi) < -1$). This property is specific to our analytical framework and is explained in detail in the following section.

4.2 Balanced growth and the role of TFP sensitivity to pollution

We want to clearly understand the links between pollution to capital, debt to capital, and growth that come from the comparison of the two BGPs.

First, we turn our attention to the growth factor γ. From (31), we see that it is a declining function of the debt per capital ratio b, through *a priori* a usual crowding-out effect of debt on investment. Moreover, as pollution generates negative external effects on production and therefore also on the saving rate, the growth factor also depends negatively on the pollution per capital ratio π. In our context of TFP vulnerability ($\epsilon_a(\pi) < -1$), a BGP with a higher level of π is characterized by a lower debt per capital ratio b. Therefore, at the BGPs, the relationship between γ and π (or b) seems ambiguous.

To avoid this ambiguity, we exploit the fact that the growth of capital is equal to the growth of the pollution stock. Then, using (30) and (31), the growth factor can be expressed as a function of π:

$$\gamma = 1 - m + \frac{\mu - \psi g_2 a(\pi)}{\pi} \tag{39}$$

Hence, we note that the growth factor is higher than 1, i.e. growth is positive, as soon as m is low enough. The growth factor increases with the pollution flow but decreases with the current pollution stock. Indeed, since mitigation decreases with pollution through its effect on TFP, π has two opposite effects on growth, a positive one through pollution flows and a negative one through the pollution stock.

Therefore, growth is a decreasing function of pollution over capital ratio (and hence an increasing function of b) if and only if the elasticity of TFP with respect to π satisfies:

$$-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)} \tag{40}$$

which may be in accordance with Assumptions 2 and 3. We thus deduce that:

Corollary 1 *Under Assumptions 1-3, inequality (38), and* $\overline{g} > g > \tau_L(1-\alpha) + \tau_K \alpha$, *we have:*

1. $\gamma_I > \gamma_{II}$ *iif* $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$ *for all* $\pi \in (\pi_1, \pi_2)$*;*

2. $\gamma_I < \gamma_{II}$ *iif* $-\epsilon_a(\pi) > \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$ *for all* $\pi \in (\pi_1, \pi_2)$*.*

In case 1, the growth rate decreases with π, so it is lower in the state with low debt and high pollution (π_{II}, b_{II}). This configuration is characterized by a not excessive TFP vulnerability to pollution. When the production is not too sensitive to pollution through the TFP, a higher level of π means a lower pollution flow over pollution stock, which implies lower growth.

In case 2, the high TFP vulnerability explains that growth is higher in the state (π_{II}, b_{II}). Indeed, a higher level of pollution over capital implies a strong increase in the pollution flow because of the decrease in public mitigation. Then, the pollution flow over the pollution stock increases, which implies higher growth.

This result is important as it reveals that as long as we consider a not excessive TFP vulnerability to pollution (case 1 of Corollary 1), all other things being equal, a BGP characterized by a lower pollution level per unit of capital is associated with higher capital growth. Recall that this BGP also has a higher level of debt over capital. A direct implication of Corollary 1 is that when TFP vulnerability is not excessive, a BGP with higher growth means a BGP with higher debt over capital. In contrast, when TFP vulnerability is very high, a BGP with lower growth means a BGP with higher debt over capital. Therefore, there is a crowding-in effect of debt on growth in the first case and a crowding-out effect in the second one.

Using (29) and (31), the intertemporal budget constraint evaluated at a BGP can be written:

$$b[\gamma - R(\pi)] = a(\pi)[g - \tau_L(1-\alpha) - \tau_K\alpha] \tag{41}$$

When pollution over capital is low, the TFP and, therefore, the interest factor are high. This means that both the primary deficit and debt services are high. Debt over capital is high, even if growth is higher than at the steady state with a higher ratio of pollution over capital (case 1 of Corollary 1). This explains that higher debt can be compatible with higher growth. This is an interesting result regarding the macroeconomic literature that mainly finds that public debt usually has a crowding-out effect on growth (see the seminal contribution by Diamond, 1965), except in the presence of some financial imperfections (see Woodford, 1990). Using (41), if the TFP is constant and, therefore, the interest factor too, we immediately deduce that public debt over capital and growth are inversely related. Public debt always has a crowding-out effect on growth.

Now, we want to understand precisely why a high TFP vulnerability to pollution is the source of a negative link between π and b when we compare both BGPs. We start by examining the extreme case in which TFP is not sensitive to pollution damage (i.e. $\epsilon_a(\pi) = 0$). It implies that $a(\pi) = a$, $R(\pi) = R$ and $\Sigma(\pi) = \Sigma$ are constant. Equations (33) and (34) thus become:

$$B_1(\pi) = \frac{a[g - (\tau_L(1-\alpha) + \tau_K\alpha)]}{\delta - m - (1-\tau_K)\alpha a + \frac{\mu - \psi g_2 a}{\pi}} \tag{42}$$

$$B_2(\pi) = \frac{a}{R}\left[\Sigma(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - g - \frac{1-m}{a} - \frac{1}{\pi}\left(\frac{\mu}{a} - \psi g_2\right)\right] \tag{43}$$

with $B_1'(\pi) > 0$ and $B_2'(\pi) > 0$. In that case, if there still exist several steady states, the one with the highest level of pollution over capital will be also characterized by the highest level of debt over capital. Comparing equations (42) and (43) with equations (33) and (34) provides insights into the differences that arise.

Equation (42) represents debt over capital as the ratio of the primary deficit over the new debt emission, which increases with growth, minus the cost of debt services measured by the interest factor. As previously mentioned, the growth of capital is equal to the growth of pollution at a BGP, which explains that it is decreasing in the pollution stock over capital and hence that there is a positive relationship between π and b. When productivity is negatively affected by pollution over capital, several adding effects may imply a reversal of this link. These effects can be perceived by using equation (33). Production being affected negatively by π, a higher π implies a lower primary deficit $a(\pi)[g - \tau_L(1-\alpha) - \tau_K\alpha]$. In addition, the cost of debt reimbursement goes down with productivity loss. These adding effects, due to endogenous productivity, are important when vulnerability to pollution is high enough. In that case, we have a negative relationship between debt over capital and pollution over capital. Equation (43) comes from the equilibrium on the asset market taking into account that capital growth is equal to pollution growth and considering the government budget constraint. Debt per unit of capital is equal to the difference, discounted by the interest factor, between savings and the sum of the primary deficit and the increase of capital (which is here equal to pollution growth). When productivity is constant, the only effect of π on b is positive and is due to its negative effect on pollution growth, as previously mentioned. When the TFP decreases with pollution over capital, several adding effects may overturn the link between debt over capital and pollution over capital. These effects can be perceived by using equation (34). First, the interest factor decreases with pollution over capital, which implies that the saving rate decreases too. Second, a lower TFP decreases income discounted by the interest factor.[6]

To summarize, TFP vulnerability to pollution implies that higher pollution to capital reduces product so that the debt to capital ratio reduces too because less deficit has to be financed and less saving is available to buy public debt. Turning to the analysis of the dynamics is now essential to determine toward which equilibria the economy will converge.

5 Dynamics, endogenous tipping zone and sustainability

The first main question we ask is whether the economy might converge to a BGP. Only in such a case, the economy will be sustainable in the long run. Otherwise, either the economy will collapse, or pollution over capital will follow an explosive dynamic path. We will especially identify a zone in terms of initial conditions, that we call the endogenous tipping zone, such that the economy will not be sustainable.

5.1 Stability of the BGPs

Proposition 1 states that two BGPs with positive debt, capital, and pollution may coexist. We analyze the dynamics in this interesting case. The question is to know toward which BGP the economy will converge. We aim to identify the conditions for a sustainable or explosive and unsustainable dynamic path. The objective is to highlight the respective roles of fiscal instruments, TFP vulnerability to pollution, and initial conditions on

[6] We can notice that the negative relationship between b and π may be observed because the depreciation of capital is not complete ($\delta \neq 1$) and/or because the saving rate increases with the productivity ($\sigma > 1$). Otherwise, when $\delta = 1$ and $\sigma = 1$, $R(\pi)/a(\pi)$ and $\Sigma(\pi)$ are constant. From equation (29), the debt ratio b does no longer evolve with π. The reduction in the primary deficit and the cost of debt induced by the increase in π are proportional to the decrease in savings so that the debt-to-capital ratio does not depend on productivity and thus not on π.

pollution debt and capital.

The dynamics are driven by equations (24) and (25). They can alternatively be driven by a combination of (24) and (25), and equation (25).

Let $\Phi_t \equiv \frac{b_t}{\pi_t} = \frac{B_t}{P_t}$. Using (24) and (25), we obtain:

$$\Phi_{t+1} = \frac{\frac{R(\pi_t)}{a(\pi_t)}\Phi_t + \frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi_t}}{X(\pi_t)} \tag{44}$$

Then, $\Phi_{t+1} \geqslant \Phi_t$ is equivalent to:

$$\Phi_t \leqslant \frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi_t[X(\pi_t) - \frac{1-\delta}{a(\pi_t)} - (1-\tau_K)\alpha]} \equiv B_3(\pi_t) \tag{45}$$

with $B_3(\pi_t) = B_1(\pi_t)/\pi_t$, $B_3(\pi_1) = \infty$, $B_3(\pi_2) > 0$ and $B_3'(\pi_t) < 0$.

Using Lemma 1, $\pi_{t+1} \geqslant \pi_t$ is equivalent to $g_3(b_t, \pi_t) \geqslant \pi_t$.[7] Using (25) and (34), this inequality rewrites $b_t \geqslant B_2(\pi_t)$. This is equivalent to $\Phi_t \geqslant B_2(\pi_t)/\pi_t \equiv B_4(\pi_t)$, given by:

$$B_4(\pi_t) = \frac{a(\pi_t)}{R(\pi_t)\pi_t}\left[\Sigma(\pi_t)(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - g - X(\pi_t)\right] \tag{46}$$

with $B_4'(\pi_t) < 0$, $B_4(\pi_1) > 0$ and $B_4(\pi_2) = 0$.

We can draw a phase diagram using these different ingredients and the results of the previous section. The stationary values of debt over pollution are $\Phi_I = \frac{b_I}{\pi_I}$ and $\Phi_{II} = \frac{b_{II}}{\pi_{II}}$. Since $B_3(\pi_t)$ and $B_4(\pi_t)$ are both decreasing and $\pi_I < \pi_{II}$, we deduce that $\Phi_I > \Phi_{II}$. We further note that $B_3'(\pi_I) < B_4'(\pi_I)$, while $B_3'(\pi_{II}) > B_4'(\pi_{II})$.

The qualitative picture of the dynamics is represented in Figure 2. We conjecture that the steady state (π_I, Φ_I) is stable, whereas the steady state (π_{II}, Φ_{II}) is a saddle. Since the two dynamic variables π_t and Φ_t are predetermined, a saddle is generically unstable. We will now confirm this conjecture by the analysis of local dynamics.

Proposition 2 *Under Assumptions 1-3, inequality (38), and $g \in (\tau_L(1-\alpha)+\tau_K\alpha, \overline{g})$, we have the following:*

1. *The steady state $(\pi_{II}, b_{II}, \Phi_{II})$ is a saddle;*

[7] Using the proof of Lemma 1, $g_3(b_t, \pi_t) = \pi_t$ is equivalent to $G_3(\pi_t, b_t, \pi_t) = \pi_t$ and we have $g_3(b_t, \pi_t) \geqslant \pi_t$ is equivalent to $G_3(\pi_t, b_t, \pi_t) \geqslant \pi_t$ because one function is an increasing transformation of the other one.

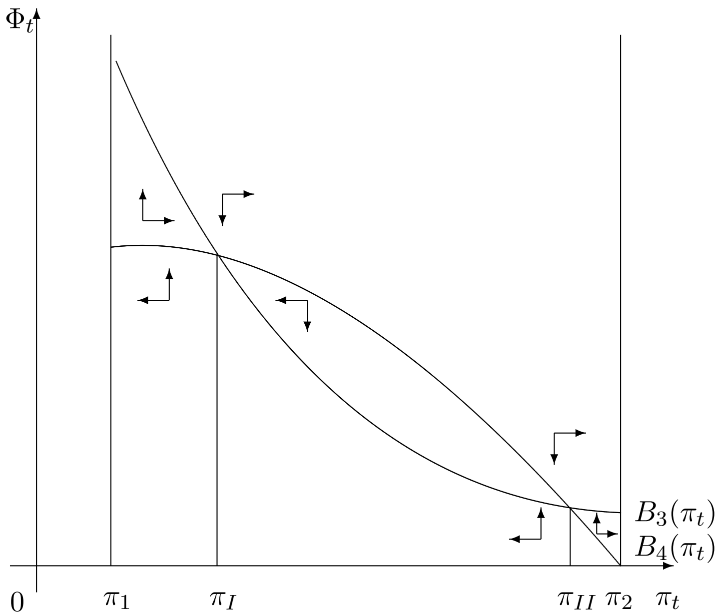

Figure 2: Dynamics

2. *The steady state* (π_I, b_I, Φ_I) *is locally stable if* τ_L *is high enough and* $\epsilon_a(\pi)$ *is not too negative.*

Proof. See Appendix C. ■

This proposition shows that the economy never converges to the steady state with a high level of pollution over capital. As pollution, debt, and capital are all predetermined, the saddle (π_{II}, b_{II}) is never achieved and therefore delimits a pollution trap, as we will discuss later. Indeed, when initial conditions are characterized by too high levels of pollution and debt with respect to capital, the economy cannot converge to a long-run BGP. Both b_t and π_t increase across time. The too high level of pollution over capital implies a too low TFP and GDP growth to be compatible with the convergence to a stable and sustainable BGP. When pollution over capital is not too high, the economy would converge to the BGP characterized by the low level of pollution over capital and the highest level of debt over capital or pollution (π_I, b_I). Such a dynamic path could experience oscillations converging to the steady state. Depending on the level of pollution over the capital, the convergence to this steady state does not require a so low debt relative to capital. The stability of this BGP requires a not too strong TFP vulnerability to pollution and a high tax rate on labor income. If the first condition is not fulfilled, any increase in pollution over capital implies a strong decrease in GDP and, therefore, investment in future capital due to the high TFP vulnerability to pollution. Future pollution over capital increases even more, generating an explosive dynamic path (see for instance equation (25)). Under the last condition, savings are low enough to prevent an unsustainable build-up of debt and capital.

5.2 Endogenous tipping zone (ETZ)

Since the steady-state $(\pi_{II}, b_{II}, \Phi_{II})$ is a saddle, it has one stable and one unstable manifold. By inspection of Figure 2, we observe that the stable manifold of this steady-state delineates a zone such that when the economy is on the right side of this manifold, pollution will be explosive. We call this zone an Endogenous Tipping Zone (ETZ), because it is endogenously defined by the dynamic behavior of the economy. In the following proposition, we show that:

Proposition 3 *Under Assumptions 1-3, inequality (38), and* $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \overline{g})$*, the stable manifold of the steady state* $(\pi_{II}, b_{II}, \Phi_{II})$ *has a negative slope at least in the neighborhood of the steady state, while the unstable manifold has a negative slope but higher than the stable one.*

Proof. See Appendix E. ■

The stable manifold (SM) of the steady state $(\pi_{II}, b_{II}, \Phi_{II})$ is clearly represented in Figure 3. Since the two dynamic variables are predetermined, the economy is generically on the left or the right side of this decreasing curve. If the initial conditions are such that the economy is on the left side of (SM), the dynamics could be characterized by convergence to the stable steady state with an increase in the long run of debt over pollution and capital and a decrease in pollution per unit of capital.

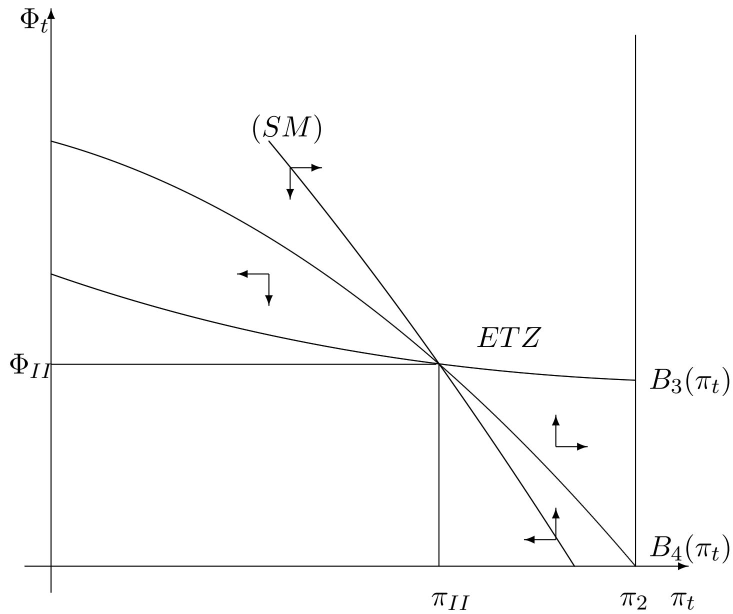

Figure 3: Endogenous Tipping Zone (ETZ)

If the initial conditions are such that we are on the right side of (SM), pollution per unit of capital will increase *a priori* indefinitely. Since the unstable manifold is negatively sloped, pollution will also increase with respect to debt. The economy is in the ETZ if the stock of pollution is sufficiently high with respect to capital. Interestingly, at least around the BGP, the ETZ is delimited by a negatively slopped relationship between Φ_t and π_t. This means that the economy will experience explosive paths for lower levels of pollution over capital when the debt relative to pollution is higher. When an economy is already burdened with high levels of debt, it may struggle to allocate additional resources toward pollution reduction efforts and adaptation. A vicious cycle is triggered, where a high level of debt does not translate into significant spending on pollution control. Instead, it illustrates an inability to adapt and mitigate adequately, thereby increasing damages and making it more difficult to stabilize debt and pollution per unit of capital. In contrast, the lower the debt over pollution, the higher the level of pollution over capital to have an explosive path. This means that lower public debt and higher capital allow to reach a sustained growing economy more easily.

Considering the damage of pollution stock on production, we provide a theoretical mechanism explaining why, for a given pollution-to-capital ratio, the likelihood of observing explosive paths for pollution is higher in countries with higher debt. Such a detrimental situation is pointed out in the literature, but as we can remark in Zenios (2024), the papers that attempt to formalize it focus mainly on the impact of climate risks on public spending; He highlights that these risks increase the cost of public debt, making public finances even more vulnerable. We obtain a similar conclusion of debt vulnerability considering a purely deterministic context.

The importance of our result is all the greater as public debt-to-GDP ratios have risen for decades and have reached record levels in a significant number of both developed and developing countries (see Figure 1 and WorldBank, 2023). This trend, combined with the heightened vulnerability to environmental issues, emphasizes the need to propose adapted policy tools to avoid a vicious circle of pollution over debt.

5.3 Unsustainability

As it is suggested in Proposition 2, the system can also be completely unsustainable. This happens if no steady state is stable, i.e. if the equilibrium (π_I, b_I, Φ_I) is unstable. In such a situation, pollution and debt over capital will follow an explosive dynamic paths. As we will see, it can occur if the TFP vulnerability to pollution is high and the tax rates are low. The next proposition provides sufficient conditions for such an undesirable configuration:

Proposition 4 *Under Assumptions 1-3, inequality (38), and* $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \overline{g})$*, the steady state* (π_I, b_I, Φ_I) *is unstable if* τ_L *is low enough,* $\epsilon_a(\pi) < -2$*,* $1 + m > 2\delta$ *and* $1 - \tau_K > \mu/[2\alpha a(\pi_2)\pi_2]$.

Proof. See Appendix D. ■

This proposition gives sufficient conditions to have all steady states saddle or unstable. It particularly requires sufficiently low tax rates on capital and labor incomes and a strong TFP vulnerability to pollution. When the tax rates are low, the primary deficit is important, as is the cost of debt. Both effects directly deteriorate public finance. The positive impact of low tax rates on aggregate savings is not sufficient to compensate. Low taxation creates conditions promoting unsustainable debt levels and hence hinders the government's ability to tackle pollution issues. Moreover, when TFP is highly vulnerable to pollution damage, the economy can never converge to a long-run BGP. As we have already seen, a small increase in pollution over capital implies a strong decrease in production and capital, which implies a larger future increase in pollution over capital. Either pollution and debt will go to infinity, characterizing a vicious circle of debt and pollution or the economy will collapse.

To summarize this section, high levels of tax rates are key ingredients to rule out an explosive accumulation of pollution and debt. However, the economy may be unsustainable for technological reasons, i.e. a high TFP vulnerability to pollution. Finally, high initial levels of debt and pollution with respect to capital promote instability of the dynamic path. Note that a high initial debt over capital is not *a priori* a source for unsustainability since it may reinforce the possibility of converging to the stable steady state with low pollution over capital. It will depend on the level of pollution over capital.

6 Policy implications

We are now interested in a configuration of possible sustainability. This means that we consider the case in which the BGP (π_I, b_I, Φ_I) is locally stable.[8] In this encouraging situation, we examine which public policy can improve welfare. For this aim, we precisely study the effect of policy variables that allow the management of environmental adaptation and mitigation, g_1 and g_2, and the fiscal revenue, τ_L and τ_K, on the welfare at a BGP.

We start by evaluating the welfare at a BGP $i = \{I, II\}$. The consumptions are given

[8] Based on Proposition 2, this implies a TFP vulnerability to pollution not too important. We also consider in this section the case in which growth is decreasing with pollution ratio π, i.e. $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$.

by:

$$\begin{aligned} c_{it} &= (1-\Sigma(\pi_i))(1-\tau_L)w_{it} \\ d_{it+1} &= R(\pi_i)\Sigma(\pi_i)(1-\tau_L)w_{it} \end{aligned}$$

with $w_{it} = (1-\alpha)a(\pi_i)\gamma_i^t K_0$. Substituting these two consumptions into the utility function (10), we get:

$$U(c_{it}, d_{it+1}) = \left[(1-\Sigma(\pi_i))^{\frac{\sigma-1}{\sigma}} + \beta(R(\pi_i)\Sigma(\pi_i))^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} (1-\tau_L)(1-\alpha)a(\pi_i)\gamma_i^t K_0$$

Therefore, the main factor determining the utility evaluated at a BGP is the growth factor γ_i. Based on (39), we see that policy variables affect γ_i through their impact on π_i. In addition, we note that the policy variable g_2 has also a direct negative effect on the growth rate.

One necessary step to determine how policy variables modify welfare is to examine their effects on (π_i, b_i). For this aim, we redefine all relevant equations as functions of (π_i, b_i) and policy instruments $(\tau_L, \tau_K, g_1, g_2)$. Productivity can thus be defined as a function of g_1 and π_i:

$$A(G_1/P) = A[\theta(\pi_i, g_1)] \equiv \Theta(\pi_i, g_1)$$

where $\theta(\pi_i, g_1)$ is defined by (18), and is increasing in g_1 and decreasing in π_i. We deduce the sign of the two following derivatives: $\Theta_{g_1}(\pi_i, g_1) > 0$, $\Theta_{\pi_i}(\pi_i, g_1) < 0$. The function $\Theta_{g_1}(\pi_i, g_1)$ captures how productivity responds to a variation in mitigation.

We present the influence of fiscal policy, via τ_L and τ_K, and environmental policy, through g_1 and g_2, focusing on the potentially stable and sustainable steady state (b_I, π_I) (see Proposition 2).[9]

Lemma 4 *Under Assumptions 1-3 and inequality (38), we have the following properties:*

- *An increase in τ_L and/or τ_K increases b_I and reduces π_I;*
- *An increase in g_1 reduces π_I if $\Theta_{g_1}(\pi, g_1)$ is high enough, τ_L is high and $g - \tau_L(1-\alpha) - \tau_K\alpha$ is not too small;*
- *If $\psi < \pi$, an increase in g_2 reduces b_I and increases π_I.*

[9] The effects of comparative statics on the other steady state (b_{II}, π_{II}) are the opposite.

Proof. See Appendix F. ■

Lemma 4 emphasizes that at the low pollution BGP (b_I, π_I), more stringent taxation reduces pollution over capital. This is because more fiscal revenue reduces the primary deficit and hence promotes capital investment. However, even though the government earns more revenue from taxation, the reduction in pollution to capital ratio goes with an increase in the debt ratio. Higher fiscal pressure leading to a higher debt over capital can appear counter-intuitive. It is driven by two key assumptions: Productivity reacts to the fall in pollution per unit of capital and there is a primary deficit $(g > \tau_L(1-\alpha) + \tau_K\alpha)$. In this context, the increase in the debt ratio is a positive side effect of productivity gains, which is facilitated by the reduction in pollution damage. Note that even if the qualitative effect of tax on capital and labor is similar (in particular, both reduce saving) it masks different mechanisms. Labor tax reduces labor income while capital taxation reduces the cost of capital and debt and hence the saving rate.

Concerning environmental policy variables, an increase in the share of the public budget allocated to adaptation leads to the following consequences: The rise in g_1 generates competing effects on productivity and hence on pollution to capital ratio. First, it puts pressure on the primary deficit, leading to a reduction in capital accumulation due to a crowding-out effect. This increases pollution per unit of capital (π) and reduces productivity. Second, the increase in g_1 has a direct positive effect on productivity, as a larger share of public spending is allocated to adaptation. When productivity is sufficiently sensitive to g_1 ($\Theta_{g_1}(\pi, g_1)$ high enough), the negative effect driven by the crowding-out is surpassed by the direct positive effect of g_1. Pollution per unit of capital (π) goes down while productivity goes up.

Concerning the effect of mitigation, the efficiency of public spending to reduce pollution flow (captured by ψ) is crucial, because it determines how pollution and debt respond to an increase in g_2 at the BGP. More precisely, we can observe a backfire effect of mitigation policy in the sense that it can increase pollution per unit of capital and reduce debt. Indeed, as for g_1, the rise in g_2 puts pressure on the primary deficit, leading to a reduction in capital accumulation due to a crowding-out effect. This, in turn, leads to an increase in pollution per unit of capital (π), resulting in decreased productivity $a(\pi)$. At the same time, the increase in g_2 has a direct negative effect on pollution, whose importance depends on ψ, as a larger share of public spending is allocated to mitigation. However, when mitigation is not sufficiently efficient to reduce pollution flow, i.e. $\psi < \pi$, this direct effect of an increase in g_2 on pollution is less than proportional to $a(\pi)$. The negative feedback effects of an increase in mitigation on capital accumulation, and hence on productivity, surpasses the initial effect of the policy.

Now, we can identify welfare-improving policy scenarios along the BGP $i = I$, i.e. policy scenarios that increase the growth factor γ_I. Using the relationship between π_I and policy instruments presented in Lemma 4, Corollary 1, that gives the link between π_I and the growth rate, and equation (39), we have the following result:

Proposition 5 *Under Assumptions 1-3, inequality (38) and* $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$*, the welfare along the BGP* $i = I$

- *increases with taxation* τ_L *and* τ_K
- *increases with the share of budget allocated to adaptation* g_1 *if* $\Theta_{g_1}(\pi, g_1)$ *is high enough,* τ_L *is high and* $g - \tau_L(1-\alpha) - \tau_K\alpha$ *is not too small*
- *decreases with the share of budget allocated to mitigation* g_2 *if* ψ *is sufficiently low*

Proof. See Appendix G. ■

The results in terms of welfare highly depend on $-\epsilon_a(\pi)$. This condition comes from Corollary 1. As $-\epsilon_a(\pi)$ is lower than $\frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$, the growth rate (and hence welfare) evaluated at a BGP decreases with pollution over capital.

When the TFP vulnerability to pollution is still not too high, a more stringent fiscal policy is a means to increase welfare as it allows for a reduction in the pollution-to-capital ratio. For environmental instruments, this is not so evident because their financing costs can heavily burden the primary budget. This creates a crowding-out effect that outweighs the benefits when the effectiveness of these policies in achieving their main objectives (reducing pollution or increasing adaptation) is limited. However, when TFP reacts sufficiently to adaptation expenditures, this spending can improve growth and welfare despite the fact the primary deficit enlarges.

To illustrate in particular that the condition $\Theta_{g_1}(\pi, g_1)$ high can be in accordance with an elasticity of productivity with respect to pollution ratio satisfying $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$, let us come back on our example.

Example (continued): In our example, $-\epsilon_a(\pi) = \frac{\pi/g_1}{A_1 - \pi/g_1} = \frac{A_1}{a(\pi)} - 1$*. Thus, condition (40) is equivalent to* $\mu > A_1 \psi g_2$*, which holds under Assumption 2. This means that* $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$*. Moreover, in our example we have* $\Theta_{g_1}(\pi, g_1) = \pi/g_1^2$ *which is high if* g_1 *is low. In such a case, an increase of* g_1 *can improves welfare, as shown in Proposition 5.*

Finally, according to Assumption **??**, mitigation is not highly efficient. Consequently, relying solely on this instrument is not a viable option for increasing welfare, as it does not directly dampen the fall in productivity caused by global pollution.

7 Conclusion

This paper examines the complex interplay between public debt, environmental quality, and economic growth, particularly pertinent in the context of growing public debt and global pollution concerns. We address these issues within an endogenous growth framework that incorporates public debt dynamics, adaptation and mitigation spending, and feedback effects of pollution on productivity. We identify two balanced growth paths characterized by varying levels of pollution and debt relative to capital. Depending on fiscal policy, initial conditions, and the responsiveness of productivity to pollution and adaptation, the economy either converges to a sustainable BGP, collapses, or experiences perpetual increases in debt and pollution. Unsustainable debt and pollution arise particularly when pollution-induced damage severely impacts productivity, underscoring the importance of policy interventions for environmental and fiscal stability.

Our results suggest that technological efficiency is fundamental: Improving technology to adapt to the impacts of climate change is important for welfare at a sustainable path. Government should start tackling global emissions (mitigation and adaptation) before a pollution threshold is reached; crossing this threshold, on the other hand, pushes the economy into a tipping zone of unsustainability. Finally, since environmental policies can be financed through both public debt and income tax revenues, we find conditions under which fiscal policy is a key element in reconciling debt sustainability and environmental sustainability.

Our findings suggest that increasing taxation, particularly in contexts where productivity is highly sensitive to pollution damage, can enhance welfare along stable growth paths. This emphasizes the crucial role of efficient environmental policy instruments in promoting adaptation.

Appendix

A Proof of Lemma 1

We examine the properties of equations (23)-(25). We have $R(\pi_t)/a(\pi_t) = (1-\delta)/a(\pi_t) + (1-\tau_K)\alpha$ which is increasing in π_t as $a(\pi_t)$ is a decreasing function. Moreover, as $\Sigma(\pi_t)$ is also decreasing, we can summarize the properties of equations (23)-(25) as follows:

$$\gamma_{t+1} = G_1(b_t, \pi_t, \pi_{t+1}), \text{ with } G_{1,b_t} < 0, G_{1,\pi_t} < 0, G_{1,\pi_{t+1}} < 0 \quad \text{(A. 1)}$$

$$b_{t+1} = G_2(b_t, \pi_t, \pi_{t+1}), \text{ with } G_{2,b_t} > 0, G_{2,\pi_t} > 0, G_{2,\pi_{t+1}} > 0 \quad \text{(A. 2)}$$

$$\pi_{t+1} = G_3(b_t, \pi_t, \pi_{t+1}), \text{ with } G_{3,b_t} > 0, G_{3,\pi_t} > 0, G_{3,\pi_{t+1}} > 0 \quad \text{(A. 3)}$$

By definition of the function $a(\pi)$, we have:

$$\epsilon_a(\pi) = \frac{A'(G_1/P)G_1/P}{A(G_1/P)} \frac{\epsilon'(\pi)\pi}{\epsilon(\pi)} = -\frac{\frac{A'(G_1/P)G_1/P}{A(G_1/P)}}{1 - \frac{A'(G_1/P)G_1/P}{A(G_1/P)}}$$

which can take any value in the interval $(-\infty, 0)$, depending on the technological features. We also have:

$$\frac{R'(\pi)\pi}{R(\pi)} = \frac{(1-\tau_K)\alpha a(\pi)}{R(\pi)} \epsilon_a(\pi)$$

and therefore:

$$\frac{\Sigma'(\pi)\pi}{\Sigma(\pi)} = \frac{\sigma - 1}{1 + \beta^\sigma R(\pi)^{\sigma-1}} \frac{(1-\tau_K)\alpha a(\pi)}{R(\pi)} \epsilon_a(\pi)$$

We deduce that:

$$\frac{G_{3,\pi_{t+1}}\pi_{t+1}}{G_3} = -\frac{a(\pi_{t+1})\Sigma(\pi_{t+1})(1-\tau_L)(1-\alpha)}{\gamma_{t+1}}\frac{(1-\tau_K)\alpha a(\pi_{t+1})}{R(\pi_{t+1})}\frac{\sigma-1}{1+\beta^\sigma R(\pi_{t+1})^{\sigma-1}}\epsilon_a(\pi_{t+1})$$

Since the first two ratios on the right-hand side are smaller than 1, $\frac{G_{3,\pi_{t+1}}\pi_{t+1}}{G_3} < 1$ if σ is close to 1 and $|\epsilon_a(\pi_{t+1})|$ is not infinite. In this case, equation (A. 3) implicitly defines π_{t+1} as an increasing function of b_t and π_t. Substituting this equation in (A. 2), we get $b_{t+1} = G_2(b_t, \pi_t, g_3(b_t, \pi_t)) \equiv g_2(b_t, \pi_t)$, which is increasing with respect to b_t and π_t. These two equations determine the dynamics of (b_t, π_t) for all $t \geq 0$. Then the growth factor is given by $\gamma_{t+1} = G_1(b_t, \pi_t, g_3(b_t, \pi_t)) \equiv g_1(b_t, \pi_t)$, which is decreasing with respect to b_t and π_t.

B Proof of Proposition 1

There exist (at least) two BGPs if there are two solutions $\pi \in (\pi_1, \pi_2)$ to the equation $B_1(\pi) = B_2(\pi)$. Since $B_1(\pi_1) > B_2(\pi_1)$ and $B_1(\pi_2) > B_2(\pi_2)$, it requires the existence of at least one value of $\pi \in (\pi_1, \pi_2)$ such that $B_1(\pi) < B_2(\pi)$.

Using equations (33) and (34), the inequality $B_1(\pi) < B_2(\pi)$ is equivalent to:

$$\Omega(\pi) > g - \tau_L(1-\alpha) - \tau_K\alpha \tag{B. 4}$$

with

$$\begin{aligned}\Omega(\pi) \quad \equiv \quad & \frac{a(\pi)}{R(\pi)}\left[X(\pi) - \frac{1-\delta}{a(\pi)} - (1-\tau_K)\alpha\right] \\ & [\Sigma(\pi)(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - g - X(\pi)]\end{aligned} \tag{B. 5}$$

By construction, we have $\Omega(\pi_1) = \Omega(\pi_2) = 0$ and $\Omega(\pi) > 0$ for all $\pi \in (\pi_1, \pi_2)$. Let us consider $\widetilde{\pi} = \lambda\pi_1$, with $\lambda \in (1, \pi_2/\pi_1)$ a constant independent of g. Then, $\widetilde{\pi}$ does not depend on g and $\Omega(\widetilde{\pi}) > 0$. When g tends to $\tau_L(1-\alpha) + \tau_K\alpha$, inequality (B. 4) evaluated at $\pi = \widetilde{\pi}$ is satisfied. By continuity, there exists $\overline{g} > \tau_L(1-\alpha) + \tau_K\alpha$ such that $\Omega(\widetilde{\pi}) > g - \tau_L(1-\alpha) - \tau_K\alpha$ for all $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \overline{g})$. This proves the existence of two solutions π_I and π_{II}, with $\pi_1 < \pi_I < \pi_{II} < \pi_2$. Since $B_1(\pi)$ and $B_2(\pi)$ are decreasing functions, the associated stationary values b_I and b_{II} are ranked in the following way: $b_I > b_{II}$.

C Proof of Proposition 2

The dynamic system we consider is given by equation (44) and equation (25) which rewrites:

$$\pi_{t+1} = \frac{X(\pi_t)\pi_t}{\Sigma(\pi_{t+1})(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi_t)}{a(\pi_t)}\pi_t\Phi_t} \quad \text{(C. 6)}$$

Let us note $R(\pi)/a(\pi) = (1-\delta)/a(\pi) + (1-\tau_K)\alpha \equiv \widetilde{R}(\pi)$. We have $X'(\pi) > \widetilde{R}'(\pi)$ and, using (35), $X(\pi) > \widetilde{R}(\pi)$.

Differentiating equation (44), we get:

$$\frac{d\Phi_{t+1}}{\Phi} = \frac{\widetilde{R}(\pi)}{X(\pi)}\frac{d\Phi_t}{\Phi} + \left[\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\frac{\widetilde{R}(\pi)}{X(\pi)} - \frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi\Phi X(\pi)} - \frac{X'(\pi)\pi}{X(\pi)}\right]\frac{d\pi_t}{\pi} \quad \text{(C. 7)}$$

Differentiating (C. 6), we get:

$$\left[1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)a(\pi)}{\gamma}\right]\frac{d\pi_{t+1}}{\pi} = \frac{R(\pi)\pi\Phi}{\gamma}\frac{d\Phi_t}{\Phi} + \left[1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)\right]\frac{d\pi_t}{\pi} \quad \text{(C. 8)}$$

where $1 + \Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)a(\pi)/\gamma > 0$ under Lemma 1.

The trace T and the determinant D of the associated Jacobian matrix are given by:

$$T = \frac{\widetilde{R}(\pi)}{X(\pi)} + \frac{1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)}{1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)a(\pi)}{\gamma}} > 1 \quad \text{(C. 9)}$$

$$D = \frac{\frac{\widetilde{R}(\pi)}{X(\pi)}\left(1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\right) + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi\Phi X(\pi)} + \frac{X'(\pi)\pi}{X(\pi)}\right)}{1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)a(\pi)}{\gamma}} \quad \text{(C. 10)}$$

and define the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$. We have $P(0) = D > 0$. Since $P(-\infty) = +\infty$, $P(+\infty) = +\infty$, and $T > 1$, the two roots are positive or complex conjugates.

Using $\widetilde{R}(\pi) = R(\pi)/a(\pi)$, $\gamma = a(\pi)X(\pi)$ and equation (44) at a steady state, T and

D rewrite:

$$T = \frac{\widetilde{R}(\pi)}{X(\pi)} + \frac{1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)}{1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{X(\pi)}} > 1 \quad \text{(C. 11)}$$

$$D = \frac{\frac{\widetilde{R}(\pi)}{X(\pi)}\left(1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\right) + \frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\left(1 - \frac{\widetilde{R}(\pi)}{X(\pi)} + \frac{X'(\pi)\pi}{X(\pi)}\right)}{1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{X(\pi)}}$$

$$= \frac{\frac{\widetilde{R}(\pi)}{X(\pi)}\left(1 + \frac{X'(\pi)\pi}{X(\pi)}\right)(1 + \pi\Phi)}{1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{X(\pi)}} \quad \text{(C. 12)}$$

We deduce that:

$$P(1) = 1 - T + D$$

$$= \frac{\left(1 - \frac{\widetilde{R}(\pi)}{X(\pi)}\right)\left(\frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{X(\pi)} - \frac{X'(\pi)\pi}{X(\pi)}\right) + \frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\left(\frac{X'(\pi)\pi}{X(\pi)} - \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\right)}{1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{X(\pi)}} \quad \text{(C. 13)}$$

Using (45) and (46), we rewrite $B_3(\pi)$ and $B_4(\pi)$ as follows:

$$B_3(\pi) = \frac{g - \tau_L(1-\alpha) - \tau_K\alpha}{\pi[X(\pi) - \widetilde{R}(\pi)]} \quad \text{(C. 14)}$$

$$B_4(\pi) = \frac{1}{\widetilde{R}(\pi)\pi}\left[\Sigma(\pi)(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - g - X(\pi)\right] \quad \text{(C. 15)}$$

We deduce that:

$$\frac{B_3'(\pi)\pi}{B_3(\pi)} = -1 - \frac{X'(\pi) - \widetilde{R}'(\pi)}{X(\pi) - \widetilde{R}(\pi)}\pi \quad \text{(C. 16)}$$

$$\frac{B_4'(\pi)\pi}{B_4(\pi)} = -1 - \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{\Phi\widetilde{R}(\pi)\pi} - \frac{X(\pi)}{\Phi\widetilde{R}(\pi)\pi}\frac{X'(\pi)\pi}{X(\pi)} \quad \text{(C. 17)}$$

After some computations, we can show that $\frac{B_3'(\pi)\pi}{B_3(\pi)} < \frac{B_4'(\pi)\pi}{B_4(\pi)}$ is equivalent to:

$$\left(1 - \frac{\widetilde{R}(\pi)}{X(\pi)}\right)\left(\frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{X(\pi)} - \frac{X'(\pi)\pi}{X(\pi)}\right) + \frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\left(\frac{X'(\pi)\pi}{X(\pi)} - \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\right) > 0 \quad \text{(C. 18)}$$

We recall that $B_3'(\pi_I) < B_4'(\pi_I)$ and $B_3'(\pi_{II}) > B_4'(\pi_{II})$. By inspection of equations (C. 13) and (C. 18), we deduce that $P(1) > 0$ at the steady state (π_I, Φ_I) and $P(1) < 0$ at the steady state (π_{II}, Φ_{II}). Therefore, the steady-state (π_{II}, Φ_{II}) is a saddle, with one eigenvalue between 0 and 1 and one higher than 1. The steady-state (π_I, Φ_I) is stable if $D < 1$ and unstable if $D > 1$.

Using (C. 12) and $\pi\Phi = b$, $D < 1$ is equivalent to:

$$\widetilde{R}(\pi)(1+b) < \frac{X(\pi) + \Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{1 + \frac{X'(\pi)\pi}{X(\pi)}} \tag{C. 19}$$

Using the expression of $\widetilde{R}(\pi)$ and equation (34), the left-hand side of inequality (C. 19) is given by:

$$\widetilde{R}(\pi)(1+b) = \Sigma(\pi)(1-\tau_L)(1-\alpha) - [g - \tau_L(1-\alpha) - \tau_K\alpha] - [X(\pi) - \widetilde{R}(\pi)] < \Sigma(\pi)(1-\tau_L)(1-\alpha) \tag{C. 20}$$

which is arbitrarily low if the tax rate τ_L is high. Using $X(\pi) = \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$, we also have:

$$\frac{X'(\pi)\pi}{X(\pi)} = \frac{-\frac{1-m}{a(\pi)}\epsilon_a(\pi) + \frac{\psi g_2}{\pi} - \frac{\mu}{a(\pi)\pi}(1+\epsilon_a(\pi))}{\frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}} \tag{C. 21}$$

Therefore, $X'(\pi)\pi/X(\pi)$ is not too high if $\epsilon_a(\pi)$ is not too negative.

We deduce that we have $D < 1$, which ensures the stability of the steady state (π_I, Φ_I), if τ_L is sufficiently close to high and close to 1 and $\epsilon_a(\pi)$ is not too negative. Note that at least when D is close to 1, the eigenvalues are complex conjugates, meaning that the dynamic path converges with oscillations around the steady state.

D Proof of Proposition 4

Using the proof of Proposition 2, we know that the steady state (π_I, Φ_I) is unstable if $D > 1$, i.e.

$$\widetilde{R}(\pi)(1+b) > \frac{X(\pi) + \Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)}{1 + \frac{X'(\pi)\pi}{X(\pi)}} \tag{D. 22}$$

which requires τ_L sufficiently low and $\epsilon_a(\pi)$ sufficiently negative.

Using $X(\pi) = \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$ and (C. 21), the inequality $X'(\pi)\pi/X(\pi) > 1$ can be written:

$$-\frac{1-m}{a(\pi)}\epsilon_a(\pi) + \frac{\psi g_2}{\pi} - \frac{\mu}{a(\pi)\pi}(1+\epsilon_a(\pi)) > \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$$

which is equivalent to:

$$-\frac{1-m}{a(\pi)}(1+\epsilon_a(\pi)) + 2\frac{\psi g_2}{\pi} - \frac{\mu}{a(\pi)\pi}(2+\epsilon_a(\pi)) > 0$$

This inequality is satisfied for $\epsilon_a(\pi) < -2$. Using the facts that $X'(\pi)\pi/X(\pi) > 1$ and $\Sigma'(\pi) < 0$, inequality (D. 22) is satisfied if:

$$\widetilde{R}(\pi)b > \frac{X(\pi)}{2} - \widetilde{R}(\pi) \tag{D. 23}$$

The left-hand side of this inequality is positive, whereas the right-hand side is strictly negative if $X(\pi) < 2\widetilde{R}(\pi)$, i.e.

$$-\frac{\psi g_2}{\pi} < \frac{1-2\delta+m}{a(\pi)} + 2(1-\tau_K)\alpha - \frac{\mu}{a(\pi)\pi} \tag{D. 24}$$

Since $a(\pi)\pi$ is decreasing, this last inequality is satisfied if $1+m > 2\delta$ and $1-\tau_K > \mu/[2\alpha a(\pi_2)\pi_2]$. It gives a sufficient condition to have $D > 1$.

E Proof of Proposition 3

The Jacobian matrix of the linearized dynamic system is given by:

$$J = \begin{pmatrix} \frac{\widetilde{R}(\pi)}{X(\pi)} & \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\frac{\widetilde{R}(\pi)}{X(\pi)} - \frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi\Phi X(\pi)} - \frac{X'(\pi)\pi}{X(\pi)} \\ \frac{R(\pi)\pi\Phi}{\gamma den} & \frac{1}{den}\left[1+\frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}+1\right)\right] \end{pmatrix}$$

with:

$$den \equiv 1 + \frac{\Sigma'(\pi)\pi(1-\tau_L)(1-\alpha)a(\pi)}{\gamma} \in (0,1)$$

Let $E_s = (1, e_s)$ be the eigenvector associated to the stable eigenvalue $\lambda_s \in (0,1)$. We have $JE_s = \lambda_s E_s$. Using the second equation of this system, we deduce that:

$$\frac{R(\pi)\pi\Phi}{\gamma den} = e_s\left[\lambda_s - \frac{1}{den}\left[1+\frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}+1\right)\right]\right]$$

Since we have:

$$\lambda_s < 1 < \frac{1}{den}\left[1+\frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}+1\right)\right]$$

we deduce that $e_s < 0$. This means that the eigenvector associated with the stable eigenvalue has a negative slope. On the stable manifold, we have $\frac{d\Phi_t}{\Phi}/\frac{d\pi_t}{\pi} = 1/e_s < 0$ in the neighborhood of the steady state.

Let $E_u = (1, e_u)$ be the eigenvector associated to the unstable eigenvalue $\lambda_u > 1$. $det(J - \lambda_u I) = 0$ is equivalent to:

$$\left(\frac{\widetilde{R}(\pi)}{X(\pi)} - \lambda_u\right)\left\{\frac{1}{den}\left[1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)\right] - \lambda_u\right\}$$
$$= -\frac{R(\pi)\pi\Phi}{\gamma den}\left[\frac{g - \tau_L(1-\alpha) - \tau_K\alpha}{\pi\Phi X(\pi)} + \frac{X'(\pi)\pi}{X(\pi)} - \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\frac{\widetilde{R}(\pi)}{X(\pi)}\right]$$

Since $X'(\pi) > \widetilde{R}'(\pi)$, $X(\pi) > \widetilde{R}(\pi)$ and $\lambda_u > 1$, the right-hand side of this equation is strictly negative, which implies that:

$$\lambda_u < \frac{1}{den}\left[1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)\right]$$

Using

$$\frac{R(\pi)\pi\Phi}{\gamma den} = e_u\left[\lambda_u - \frac{1}{den}\left[1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)\right]\right]$$

we deduce that $e_u < 0$, which means that the eigenvector associated to the unstable eigenvalue has a negative slope. On the unstable manifold, we have $\frac{d\Phi_t}{\Phi}/\frac{d\pi_t}{\pi} = 1/e_u < 0$ in the neighborhood of the steady state. Since $\lambda_u > \lambda_s$, we even have $1/e_u > 1/e_s$, which means that around the steady state, the negative slope of the unstable manifold is greater than the one of the stable manifold.

F Proof of Lemma 4

Using (29)-(31), the system that defines the stationary solutions can be rewritten as:

$$b_i = \frac{g_1 + g_2 - (\tau_L(1-\alpha) + \tau_K\alpha)}{z(\pi_i, g_1, g_2) - y(\pi_i, g_1, \tau_K)} \equiv \mathcal{B}_1(\pi_i, g_1, g_2, \tau_K, \tau_L) \quad \text{(F. 25)}$$

$$\begin{aligned} b_i &= \frac{[x(\pi_i, g_1, \tau_K)(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - (g_1+g_2) - z(\pi_i, g_1, g_2)]}{y(\pi_i, g_1, \tau_K)} \\ &\equiv \mathcal{B}_2(\pi_i, g_1, g_2, \tau_K, \tau_L) \end{aligned} \quad \text{(F. 26)}$$

with

$$x(\pi_i, g_1, \tau_K) = \frac{\beta^\sigma(1-\delta+\alpha(1-\tau_K)\Theta(\pi_i, g_1))^{\sigma-1}}{1 + \beta^\sigma(1-\delta+\alpha(1-\tau_K)\Theta(\pi_i, g_1))^{\sigma-1}}.$$

The sign of partial derivatives are: $x_{g_1} > 0$; $x_{\pi_i} < 0$; $x_{\tau_K} < 0$.

$$y(\pi_i, g_1, \tau_K) = (1-\delta)/\Theta(\pi_i, g_1) + \alpha(1-\tau_K).$$

The sign of partial derivatives are: $y_{\pi_i} > 0$; $y_{g_1} < 0$; $y_{\tau_K} < 0$;

and

$$z(\pi_i, g_1, g_2) = \frac{1-m}{\Theta(\pi_i, g_1)} - \frac{\psi g_2}{\pi_i} + \frac{\mu}{\Theta(\pi_i, g_1)\pi_i}.$$

The sign of partial derivatives are: $z_{\pi_i} > 0$; $z_{g_1} < 0$; $z_{g_2} < 0$.

Total differentiation of Equations (F. 25) and (F. 26) gives:

$$\mathcal{C} \times \begin{bmatrix} \mathrm{d}b \\ \mathrm{d}\pi \end{bmatrix} = \mathcal{D} \times \begin{bmatrix} \mathrm{d}g_1 \\ \mathrm{d}g_2 \\ \mathrm{d}\tau_L \\ \mathrm{d}\tau_K \end{bmatrix}$$

where

$$\mathcal{C} = \begin{bmatrix} 1 & -B_1'(\pi) \\ 1 & -B_2'(\pi) \end{bmatrix}$$

with $B_1'(\pi) = \frac{\partial \mathcal{B}_1(\pi, g_1, g_2, \tau_K, \tau_L)}{\partial \pi}$ and $B_2'(\pi) = \frac{\partial \mathcal{B}_2(\pi, g_1, g_2, \tau_K, \tau_L)}{\partial \pi}$ and where

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 & \mathcal{D}_4 \\ \mathcal{D}_5 & \mathcal{D}_6 & \mathcal{D}_7 & \mathcal{D}_8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\frac{b\Theta_{g_1}}{(\Theta)^2}(\delta-m+\mu/\pi)}{z-y} & \frac{\pi+\psi b}{\pi(z-y)} & \frac{\alpha-1}{z-y} & \frac{-\alpha(1+b)}{z-y} \\ \frac{x_{g_1}(1-\tau_L)(1-\alpha)-1+\frac{\Theta_{g_1}}{(\Theta)^2}(b(1-\delta)+\mu/\pi+1-m)}{y} & \frac{\psi-\pi}{\pi y} & \frac{(1-\alpha)(1-x)}{y} & \alpha\left(\frac{(1-\sigma)x(1-\tau_L)(1-\alpha)}{y^2(1+\beta^\sigma(y\Theta)^{1-\sigma})} + \frac{1+b}{y}\right) \end{bmatrix} \quad \text{(F. 27)}$$

The determinant of matrix $\mathcal{C}$ is $\det \mathcal{C}(\pi) = B_1'(\pi) - B_2'(\pi)$. We recall that $B_1'(\pi_I) < B_2'(\pi_I)$ and $B_1'(\pi_{II}) > B_2'(\pi_{II})$. We thus have $\det \mathcal{C}(\pi_I) < 0$ and $\det \mathcal{C}(\pi_{II}) > 0$. Moreover, under Assumptions 1-3, we have $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_7, \mathcal{D}_8 > 0$[10], $\mathcal{D}_3, \mathcal{D}_4 < 0$. The sign of $\mathcal{D}_5$ and $\mathcal{D}_6$, referring to the impact of environmental policy variables, g_1 and g_2, depends on policy and model parameters and is discussed later.

We obtain the effects of policy variables on stationary variables examining:

$$\begin{bmatrix} \mathrm{d}b \\ \mathrm{d}\pi \end{bmatrix} = \frac{1}{\det \mathcal{C}(\pi)} \times \begin{bmatrix} -B_2'(\pi) & B_1'(\pi) \\ -1 & 1 \end{bmatrix} \times \mathcal{D} \times \begin{bmatrix} \mathrm{d}g_1 \\ \mathrm{d}g_2 \\ \mathrm{d}\tau_L \\ \mathrm{d}\tau_K \end{bmatrix}$$

[10] As σ is higher but close to one under Assumption 3.

- Effect of g_1

$$\frac{\mathrm{d}b}{\mathrm{d}g_1} = \frac{-B_2'(\pi)\mathcal{D}_1 + B_1'(\pi)\mathcal{D}_5}{\det \mathcal{C}(\pi)}$$

$$\frac{\mathrm{d}\pi}{\mathrm{d}g_1} = \frac{-\mathcal{D}_1 + \mathcal{D}_5}{\det \mathcal{C}(\pi)}$$

Using the expression for $\mathcal{D}_1$ and $\mathcal{D}_5$ given in (F. 27), we have:

$$\begin{aligned}\frac{\mathrm{d}\pi}{\mathrm{d}g_1} \quad = \quad & \Bigg(\frac{x_{g_1}(1-\tau_L)(1-\alpha) - 1 + \frac{\Theta_{g_1}}{\Theta^2}\left(b(1-\delta) + \frac{\mu}{\pi} + 1 - m\right)}{y} \\ & -\frac{1 + \frac{b\Theta_{g_1}}{\Theta^2}\left(\delta - m + \frac{\mu}{\pi}\right)}{z-y}\Bigg) \times \frac{1}{\det \mathcal{C}(\pi)}\end{aligned}$$

with

$$x_{g1} = (\sigma-1)\alpha\Theta_{g_1}\frac{\beta^\sigma(1-\delta+\alpha(1-\tau_K)\Theta(\pi,g_1))^{\sigma-2}}{(1+\beta^\sigma(1-\delta+\alpha(1-\tau_K)\Theta(\pi,g_1))^{\sigma-1})^2} > 0$$

which is increasing in Θ_{g_1}. Rewriting the previous equation, we obtain:

$$\begin{aligned}\frac{\mathrm{d}\pi}{\mathrm{d}g_1} \quad = \quad & \left(\frac{-z}{y(z-y)} + \frac{x_{g_1}(1-\tau_L)(1-\alpha)}{y} + \frac{\Theta_{g_1}}{y\Theta^2}\Big[b(1-\delta) + \frac{\mu}{\pi} + 1 - m\right. \\ & \left. -b\frac{y}{z-y}\left(\frac{\mu}{\pi} + \delta - m\right)\Big]\right) \times \frac{1}{\det \mathcal{C}(\pi)} \qquad \text{(F. 28)}\end{aligned}$$

Using (F. 25) and (F. 26), we have:

$$\frac{y}{z-y} < \frac{x(1-\tau_L)(1-\alpha)}{g - \tau_L(1-\alpha) - \tau_K\alpha}$$

The right-hand side of this inequality is small if τ_L is high and $g - \tau_L(1-\alpha) - \tau_K\alpha$ is not too small. In this case, the term into brackets in equation (F. 28) is positive such that for Θ_{g_1} high enough, we have $\text{sgn}\left(\frac{\mathrm{d}\pi_I}{\mathrm{d}g_1}\right) < 0$ and $\text{sgn}\left(\frac{\mathrm{d}\pi_{II}}{\mathrm{d}g_1}\right) > 0$. Given the expression for $\frac{\mathrm{d}b}{\mathrm{d}g_1}$, we do not conclude concerning the impact of g_1 on b in this configuration.

We can note that as long as Θ_{g_1} is low enough, $\mathcal{D}_5 < 0$. This implies $\frac{\mathrm{d}b}{\mathrm{d}g_1} < 0$ and $\frac{\mathrm{d}\pi}{\mathrm{d}g_1} > 0$ for the BGP $i = I$ and the reverse for the BGP $i = II$.

- Effect of g_2.

$$\frac{\mathrm{d}b}{\mathrm{d}g_2} = \frac{-B_2'(\pi)\mathcal{D}_2 + B_1'(\pi)\mathcal{D}_6}{\det \mathcal{C}(\pi)}$$

$$\frac{\mathrm{d}\pi}{\mathrm{d}g_2} = \frac{-\mathcal{D}_2 + \mathcal{D}_6}{\det \mathcal{C}(\pi)} = \frac{1}{\pi}\left(\frac{\psi - \pi}{y} - \frac{\pi + \psi b}{z - y}\right)$$

When $\psi < \pi$ we have $\mathcal{D}_6 < 0$. In that case, an increase in g_2 means that π_I and b_{II} increases while π_{II} and b_I decrease.

- Effect of τ_L

$$\frac{\mathrm{d}b}{\mathrm{d}\tau_L} = \frac{-B_2'(\pi)\mathcal{D}_3 + B_1'(\pi)\mathcal{D}_7}{\det \mathcal{C}(\pi)}$$

$$\frac{\mathrm{d}\pi}{\mathrm{d}\tau_L} = \frac{-\mathcal{D}_3 + \mathcal{D}_7}{\det \mathcal{C}(\pi)}$$

We have $\frac{\mathrm{d}\pi}{\mathrm{d}\tau_L} < 0$ and $\frac{\mathrm{d}b}{\mathrm{d}\tau_L} > 0$ for the BGP $i = I$ and the reverse, $\frac{\mathrm{d}\pi}{\mathrm{d}\tau_L} > 0$ and $\frac{\mathrm{d}b}{\mathrm{d}\tau_L} < 0$, for the BGP $i = II$.

- Effect of τ_K

$$\frac{\mathrm{d}b}{\mathrm{d}\tau_K} = \frac{-B_2'(\pi)\mathcal{D}_4 + B_1'(\pi)\mathcal{D}_8}{\det \mathcal{C}(\pi)}$$

$$\frac{\mathrm{d}\pi}{\mathrm{d}\tau_K} = \frac{-\mathcal{D}_4 + \mathcal{D}_8}{\det \mathcal{C}(\pi)}$$

We have $\frac{\mathrm{d}\pi}{\mathrm{d}\tau_K} < 0$ and $\frac{\mathrm{d}b}{\mathrm{d}\tau_K} > 0$ for the BGP $i = I$ and the reverse, $\frac{\mathrm{d}\pi}{\mathrm{d}\tau_K} > 0$ and $\frac{\mathrm{d}b}{\mathrm{d}\tau_K} < 0$, for the BGP $i = II$.

G Proof of Proposition 5

We examine how policy variables affect γ_I to deduce the welfare effect along the stable BGP $i = I$. Using (39), we have:

$$\gamma_I = 1 - m + \frac{\mu - \psi g_2 a(\pi_I)}{\pi_I} \tag{G. 29}$$

The proof of Lemma 4 reveals that $\frac{\mathrm{d}\pi_I}{\mathrm{d}\tau_L} < 0$ and $\frac{\mathrm{d}\pi_I}{\mathrm{d}\tau_K} < 0$ and gives the conditions to have $\frac{\mathrm{d}\pi_I}{\mathrm{d}g_1} < 0$ or > 0. In the configuration where the fall in π_I is good for growth $\left(-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}\right)$, we directly have the policy scenarios for fiscal and adaptation policies that increase welfare.

As regards mitigation g_2, it exerts a direct negative effect on growth, in addition to its effect through π. Lemma 4 gives a condition such that $\frac{\mathrm{d}\pi_I}{\mathrm{d}g_2} > 0$, meaning that in the configuration where the fall in π_I is good for growth, the increase in g_2 has a double negative effect on welfare.

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