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Kazuo NISHIMURA
Harutaka TAKAHASHI
Alain VENDITTI

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A dynamic theory of the Balassa-Samuelson effect: Why has the Japanese economy stagnated for over 30 years?*

Kazuo NISHIMURA[†]

Harutaka TAKAHASHI[‡]

and

Alain VENDITTI[§]

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Abstract: *The Balassa-Samuelson effect ("BS effect") has attracted attention as a theory to explain the stagnation of the Japanese economy over the past 30 years. In particular, it has been used to explain the long-term depreciation of the real effective exchange rate since 1995. Furthermore, macroeconomic data show that the BS effect explains well Japan's long-term economic stagnation. However, the BS effect was originally derived theoretically for small open economies, not for large economies like Japan. In other words, the BS effect cannot be theoretically applied to large economies. This is a serious problem in applying the BS effect empirically. In this paper, we embed Balassa-Samuelson's original argument into the optimal growth theory framework. That is, we set up an optimal growth problem for large countries. It is then shown that there exists a stable optimal steady state and that the BS effect is more directly valid in that optimal steady state. In other words, as a long-run property, the BS effect is applicable to large as well as small countries, although, contrary to the small open economy case, it does not depend on the capital shares of the two sectors.*

Keywords: *Two-sector optimal growth models, optimal steady state, capital intensity, Balassa-Samuelson effect*

Journal of Economic Literature Classification Numbers: C61, C62, E32, F31, O41.

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†RIEB, Kobe University, Kobe, Japan

‡Research fellow, Graduate School of Economics, Kobe University, Kobe, Japan and Professor Emeritus, Meiji Gakuin University, Tokyo, Japan. He would like to thank IMERA-Institute of Advanced Studies of Aix-Marseille University, and Aix-Marseille School of Economics for providing the best research environment for drafting the paper.

§Aix-Marseille Univ., CNRS, AMSE, Marseille, France

1 Introduction

The Balassa-Samuelson effect (BS effect, hereafter) is still an important phenomenon in the theory of economic development, as Balassa [2] states, "As economic development is accompanied by greater inter-country differences in the productivity of tradable goods, differences in wages and service prices increase, and correspondingly so do differences in purchasing power parity and exchange rates." Formally it can be expressed as the following BS effect equation.

$$\frac{\dot{\tilde{p}}_N}{\tilde{p}_N} = \left(\frac{1-\beta}{1-\alpha}\right)\frac{\dot{A}_T}{A_T} - \frac{\dot{A}_N}{A_N} \tag{1}$$

where $\dot{x}$ indicates a time derivative dx/dt, $\tilde{p}_N$ is the relative price of non-tradable goods, where the price of tradable goods is numéraire, α is the capital share of the tradable goods output, β is the capital share of the non-tradable output, and A_i is the TFP of the sector $i = N, T$.[1]

If $\alpha > \beta$ and $\dot{A}_T/A_T > \dot{A}_N/A_N$ hold, then according to (1), it implies $\dot{\tilde{p}}_N/\tilde{p}_N > 0$. In other words, the relative price of non-tradable goods increases. If the perfect purchasing parity (PPP) holds only for tradable goods, it implies that the real exchange rate will appreciate. Valentinyi and Herrendorf [10] report that $\alpha = 0.37$ and $\beta = 0.32$ in the US economy. It implies that $(1-\beta)/(1-\alpha) \approx 1.08 > 1$. Furthermore, p_N grows in such an economy because the TFP growth rate in the tradable goods sector is expected to be greater than that in the non-tradable goods sector.

Another important property is as follows.

Wages are determined entirely by the factor productivity of the tradable goods sector. (2)

Let us call these two properties (1) and (2) collectively the Balassa-Samuelson property. In short, the BS property hereafter.

The Japanese economy has stagnated for the past 30 years, especially the real effective exchange rate index (2010=100), which has been declining since 1995 as shown in Figure 1.

[1] For the standard derivation of the equation (1) and (2), please refer to Asea *et al.* [1] and Couharde *et al.* [4] for comprehensive explanations.

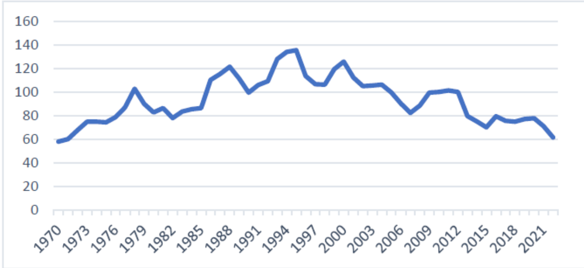

Figure 1: Japan's real effective exchange rate index

Source: Real effective exchange rate index (2010 = 100) – Japan — Data (worldbank.org).

BS property was often applied to explain Japan's real exchange rate depreciation and economic stagnation problems. In other words, as a result of globalization, major Japanese manufacturers in the tradable goods sector moved their main production facilities to China, Thailand, and other countries, and the remaining production facilities in Japan became less efficient. In addition, deregulation implemented in the service sector increased productivity in the non-tradable goods sector. This fact can be confirmed by the Balassa-Samuelson (BS) effect measures reported in the RPROD database.[2] Figure 2 exhibits five different BS effect indicators. Since 1995, three of the five series have sharply declined.

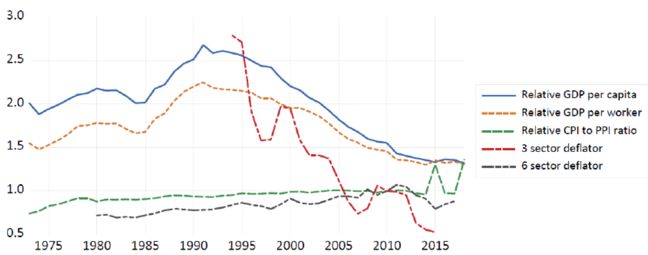

Figure 2: Japan's BS effect measures

Source: CEPII - RPROD.

The rate of TFP growth in the tradable goods sector, as indicated by relative GDP per capita or per worker in Figure 2 above, has declined sharply since 1995, while the relative prices of non-tradable goods, as indicated by the CPI/PPI ratio and the three- and six-sector deflators in Figure 2, especially the three-sector deflator, has declined substantially as expected by the BS property (1) described above. As a result, the real effective exchange rate declined as shown in Figure 1.

Japan's per capita wages have also stagnated due to BS characteristics (2), and as Figure 3 below shows, per capita wages have remained nearly constant during the 1991-2020 reporting

[2]In detail on the RPROD data base, see C. Couharde *et al.* [4].

period.

As a result, the BS property would seem to fully explain the stagnation of the Japanese economy and the depreciation of the real effective exchange rate. However, there is a major theoretical problem with applying the BS property to a large country like Japan. This is because the BS property was originally proven for small developing countries that are given interest rates in the world market. Therefore, it is important to show that the BS property holds for large countries. This issue is addressed in the framework of optimal growth theory.

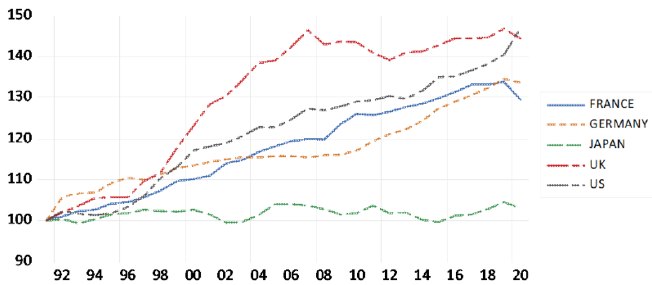

Figure 3: Real per-capita wages (1991:100)

Source: $https://www5.cao.go.jp/j-j/wp/wp-je22/h06_hz020105.html$.

To our knowledge, the BS property has never been formally tested in the framework of two-sector optimal growth theory. Consider two cases. One is the case of a large country that can control interest rates internally, and the other is the case of a small country that cannot set interest rates but can provide access to world capital markets. In a small country, Takahashi and Venditti [9] proved that the exact same equation as the BS effect equation (1) holds. In contrast, this paper proves that the following different BS effect equation (3) holds as a property of the long-run optimal steady state in a large country.

$$\frac{\dot{\tilde{p}}_N}{\tilde{p}_N} = \frac{\dot{A}_T}{A_T} - \frac{\dot{A}_N}{A_N} \tag{3}$$

This is in contrast to the equation for the BS effect for small countries shown in (1). Since (3) does not depend on the capital intensity ratio term as in equation (1), the BS effect shows a more direct relationship with the difference in TFP growth rates. It is important to note that this difference is due to different optimal steady-state conditions, as will be shown below.

The remainder of this paper is organized as follows. Section 2 provides and analyzes the model: Section 2.1 presents a two-sector model comprising tradable and non-tradable goods. Then, in Sections 2.2 and 2.3, we analyze the uniqueness and saddle point stability of the optimal steady state. In Section 3, we discuss the Balassa-Samuelson property in the optimal steady state based on the discussion in Section 2. In Section 4, we provide concluding comments. All the proofs are gathered in a final Appendix.

2 The model

This section first describes the production structure and the preferences of our 2-sector model. Then we discuss the intertemporal equilibrium and the steady state. Finally we derive the characteristic polynomial associated with linearization around the steady state.

2.1 The 2-sector economy

We consider an economy producing a non-tradable (N) good $\tilde{y}_N$, and a tradable (T) good $\tilde{y}_T$. Each good is assumed to be produced by using capital k_j and labor l_j, $j = N, T$ in different proportions *via* Cobb-Douglas production functions:

$$\begin{aligned} \tilde{y}_N &= A_N k_N^{\beta} l_N^{1-\beta}, \\ \tilde{y}_T &= A_T k_T^{\alpha} l_T^{1-\alpha}, \end{aligned} \tag{4}$$

where A_i denotes the total factor productivity of sector $i = N, T$. Total labor is given by $1 = l_N + l_T$, and total stock of capital is given by $k = k_N + k_T$. Let us then denote $y_N = \tilde{y}_N / A_N$ and $y_T = \tilde{y}_T / A_T$. We can then rewrite (4) as

$$\begin{aligned} y_N &= k_N^{\beta} l_N^{1-\beta}, \\ y_T &= k_T^{\alpha} l_T^{1-\alpha}. \end{aligned} \tag{5}$$

A firm in each industry maximizes its profit under productivity-normalized output prices p_N and p_T, rental rate of capital r, and wage rate w. Choosing the tradable good as the *numéraire*, i.e. $p_T = 1$, we define from the technologies (5) the following Lagrangian

$$\mathcal{L} = k_T^{\alpha} l_T^{1-\alpha} + p_N \left[k_N^{\beta} l_N^{1-\beta} - y_N \right] + r\left[k - k_N - k_T\right] + w\left[1 - l_N - l_T\right]$$

with p_N, r and w the price of the non-tradable good, the interest rate and the wage rate, all in terms of the price of tradable good. The first-order conditions give

$$\begin{aligned} & r = \alpha k_T^{\alpha-1} l_T^{1-\alpha} = p_N \beta k_N^{\beta-1} l_N^{1-\beta} \\ & w = (1-\alpha) k_T^{\alpha} l_T^{-\alpha} = p_N (1-\beta) k_N^{\beta} l_N^{-\beta} \end{aligned} \tag{6}$$

and we thus derive the following input coefficients:

$$\begin{aligned} a_{00}(w, p_N) = \tfrac{l_N}{y_N} = \tfrac{p_N(1-\beta)}{w}, &\quad a_{10}(r, p_N) = \tfrac{k_N}{y_N} = \tfrac{p_N \beta}{r}, \\ a_{01}(w) = \tfrac{l_T}{y_T} = \tfrac{1-\alpha}{w}, &\quad a_{11}(r) = \tfrac{k_T}{y_T} = \tfrac{\alpha}{r}. \end{aligned} \tag{7}$$

Each coefficient a_{ij} represents the amount of "good" i, that is, labor or intermediate capital good, that it takes to produce one unit of good j - in other words, the non-tradable or tradable good output. Denoting $p = (p_N, 1)'$ and $\omega = (w, r)'$, we can then define the following matrix of input coefficients

$$\mathcal{A}(\omega, p) = \begin{pmatrix} a_{00}(w, p_N) & a_{01}(w) \\ a_{10}(r, p_N) & a_{11}(r) \end{pmatrix}$$

which can basically be obtained from input-output tables available in national accounting data.

Using the results of Benhabib and Nishimura [3], and as stated in Lemma 1 and Lemma 2, the factor-price frontier and the factor market-clearing equations depend on this matrix.

Lemma 1. *$p = \mathcal{A}'(\omega, p)\omega$ and $dp = \mathcal{A}'(\omega, p)d\omega$.*

Lemma 2. *Denote $x = (1, k)'$ and $y = (y_N, y_T)'$. Then $\mathcal{A}(\omega, p)y = x$ and*

$$\mathcal{A}(w, p)dy + \begin{pmatrix} \left(\frac{\partial a_{00}}{\partial w} y_N + \frac{\partial a_{01}}{\partial w} y_T\right) dw + \frac{\partial a_{01}}{\partial p_N} y_N dp_N \\ \left(\frac{\partial a_{10}}{\partial r} y_N + \frac{\partial a_{11}}{\partial r} y_T\right) dr + \frac{\partial a_{11}}{\partial p_N} y_N dp_N \end{pmatrix} = dx.$$

We derive that, at equilibrium, wage rate and rental rate are functions of the non-tradable output price only, that is, $w = w(p_N)$ and $r = r(p_N)$, while the outputs are functions both of the capital stock and the non-tradable output price, $y_j = y_j(k, p_N)$, $j = N, T$.

As can be expected in multi-sector optimal growth models, there is a duality between the *Rybczinski* and *Stolper-Samuelson* effects, i.e.

$$\frac{\partial y_N}{\partial k} = \frac{\partial r}{\partial p_N}. \tag{8}$$

2.2 Intertemporal equilibrium and steady state

The economy is populated by a large number of identical infinitely-lived agents. Without loss of generality, we assume that the total population is constant and normalized to one. At each period, a representative agent inelastically supplies one unit of labor. Furthermore, utility is derived from consuming the non-tradable good $\tilde{c}_N$ and the tradable good $\tilde{c}_T$ according to the following Cobb-Douglas specification:

$$u(c_N, c_T) = c_N^{\theta} c_T^{1-\theta}$$

with $c_N = \tilde{c}_N/A_N$, $c_T = \tilde{c}_T/A_T$ and $\theta \in (0, 1]$. Parameter θ measures the share of the non-tradable good c_N within total utility. The agent's preferences imply properties of interest regarding the (pure) elasticities of intertemporal substitution in consumption goods c_N and

c_T, ε_{00} and ε_{11}, and the (cross-) elasticities of intertemporal substitution between the two goods, ε_{01} and ε_{10}:

$$\begin{array}{l} \varepsilon_{00} = -\frac{u_1}{u_{11}c_N} = \frac{1}{1-\theta},\ \varepsilon_{01} = -\frac{u_1}{u_{12}c_T} = -\frac{1}{1-\theta}, \\ \varepsilon_{10} = -\frac{u_2}{u_{21}c_N} = -\frac{1}{\theta},\ \varepsilon_{11} = -\frac{u_2}{u_{22}c_T} = \frac{1}{\theta}. \end{array} \tag{9}$$

Profit maximization in both sectors described in Section 2.1 yields the demands for capital and labor as functions of the capital stock and the production levels of the non-tradable good, namely $l_j = l_j(k, y_N)$ and $k_j = k_j(k, y_N)$, $j = N, T$. Considering that at the equilibrium $c_N = y_N$, the optimal amount of the non-tradable good is then defined by:

$$y_T = k_T(k, y_N)^{\alpha} l_T(k, y_N)^{1-\alpha} = T(k, c_N).$$

From the envelope theorem, we get: $r = T_k(k, c_N)$ and $p_T = -T_{c_N}(k, c_N)$. The intertemporal optimization problem of the representative agent is then given by:

$$\begin{array}{lll} \max\limits_{\{c_N(t), c_T(t), k(t)\}} & \displaystyle\int_0^{+\infty} c_N(t)^{\theta} c_T(t)^{1-\theta} e^{-\delta t} dt & \\ s.t. & \dot{k}(t) = T(k(t), c_N(t)) - gk(t) - c_T(t) & \\ & k(0)\ given, & \end{array} \tag{10}$$

where $\delta \geq 0$ is the discount rate and $g > 0$ is the depreciation rate of the capital stock. We can write the modified Hamiltonian in current value as:

$$\mathcal{H} = c_N(t)^{\theta} c_T(t)^{1-\theta} + q(t) \left[T(k, c_N(t)) - gk(t) - c_T(t) \right].$$

The necessary conditions, which describe the solution to problem (10), are therefore given by the following equations:

$$q(t) = \frac{\theta c_N(t)^{\theta-1} c_T(t)^{1-\theta}}{p_N(t)} \tag{11}$$

$$q(t) = (1-\theta) c_N(t)^{\theta} c_T(t)^{-\theta} \tag{12}$$

$$\dot{k}(t) = T(k(t), c_N(t)) - gk(t) - c_T(t) \tag{13}$$

$$\dot{q}(t) = (\delta + g - T_k(k(t), c_N(t))) q(t) = (\delta + g - r(t)) q(t) \tag{14}$$

Taking equations (11) to (14), we are now in a position to characterize an equilibrium path $\{k(t), p_N(t),\}_{t \geq 0}$ and to prove the existence of a unique steady state. Indeed, as shown in Section 2.1, we have $r = r(p_N)$ and $c_N = y_N = k_N(k, c_N)^{\beta} l_N(k, c_N)^{1-\beta}$ which gives $c_N = c_N(k)$, and thus $y_T = T(k, c_N(k)) = y_T(k)$. Using (11), (12), we derive:

$$c_T(t) = c_T(k(t), p_N(t)) = c_N(k(t)) \frac{p_N(t)(1-\theta)}{\theta}. \tag{15}$$

Straightforward computations then yield:

$$\frac{\partial c_T}{\partial k} = \frac{p_N(1-\theta)}{\theta} \frac{\partial c_N}{\partial k} \text{ and } \frac{\partial c_T}{\partial p_N} = \frac{p_N(1-\theta)}{\theta} \frac{\partial c_N}{\partial p_N} - \frac{c_N}{p_N}. \tag{16}$$

Considering (11)-(14) and (16), the motion equations write:

$$\begin{array}{rcl} \dot{k} & = & y_T(k) - gk - c_T(k, p_N) \\ \dot{p}_N & = & \frac{p_N(t)}{\theta}\Big[\delta + g - r(p_N)\Big]. \end{array} \tag{17}$$

Any solution $\{k(t), p_N(t)\}_{t\geq 0}$ that also satisfies the transversality condition:[3]

$$\lim_{t\to+\infty} e^{-\delta t} q(t)k(t) = 0$$

with $q(t)$ as given by (11), is called an equilibrium path. A steady state is defined by a vector (c_N^*, k^*, p_N^*) solution of

$$\begin{array}{rcl} y_T(k) & = & gk + c_T = gk + c_N(k)\frac{p_N(1-\theta)}{\theta} \\ r(p_N) & = & \delta + g. \end{array} \tag{18}$$

We get the following result:

Proposition 1. *There exists a unique steady state $(c_N^*, k^*, p_N^*,) > 0$ solution of the system of nonlinear equations (18) with $c_N^* = c_N(k^*)$ and $c_T^* = c_N(k^*)\frac{p_N^*(1-\theta)}{\theta}$.*

Proof. See Appendix 5.1

2.3 Characteristic polynomial

Linearizing the dynamical system around (c_N^*, k^*, p_N^*) gives a 2×2 Jacobian matrix $\mathcal{J}$ which is provided in Appendix 5.2. Let us denote $\mathcal{T}$ the trace and $\mathcal{D}^\theta$ the determinant of $\mathcal{J}$. Proposition 2 displays some properties of the eigenvalues of $\mathcal{J}$ and the expression of the characteristic polynomial.

Proposition 2. *If λ is an eigenvalue of the Jacobian matrix $\mathcal{J}$, then $\delta - \lambda$ is also an eigenvalue and thus $\mathcal{T} = \delta$. The degree-2 characteristic polynomial is given by:*

$$\mathcal{P}^\theta(\lambda) = \lambda^2 - \lambda\delta + \mathcal{D}^\theta \tag{19}$$

where

$$\mathcal{D}^\theta = \frac{\left(\frac{\partial y_T}{\partial k} - g - \frac{\partial c_T}{\partial k}\right)\frac{\partial r}{\partial p_N}}{\theta} \tag{20}$$

Moreover, the two roots are real and distinct.

Proof. See Appendix 5.2.

The results on the structure of the characteristic roots are in line with the conclusions of Kurz [6] and Levhari and Liviatan [7]. Based on Proposition 2, we can further prove the saddle-point stability of the stationary steady state as exhibited in Proposition 1.

Proposition 3. *For any $\alpha, \beta \in (0, 1)$ and any $\delta \geq 0$, the unique steady state (c_N^*, k^*, p_N^*) is saddle-point stable*

Proof. See Appendix 5.3.

[3]See Michel [8] and Kamihigashi [5] for some proof of the necessity of the transversality condition.

3 The Balassa-Samuelson effect

We focus in this Section on the Balassa-Samuelson effect. The first question to answer is to check whether such a property is satisfied along the optimal steady state. Indeed, we have solved the model through a stationary version of the dynamical equations based on the considerations of the variables $y_N = \tilde{y}_N/A_N$, $y_T = \tilde{y}_T/A_T$, $c_N = \tilde{c}_N/A_N$ and $c_T = \tilde{c}_T/A_T$. We need now to consider the real variables that are affected by the growth rates of productivities A_N and A_T. More precisely we need to consider the price of the non-tradable good $\tilde{p}_N$ which is linked to the stationary price p_N as follows: $p_N = \tilde{p}_N A_N/A_T$. The following Proposition establish that in the optimal growth framework, the BS property is modified as formulated by equation (3).

Proposition 4. *At the unique steady state, the Balassa-Samuelson property holds, i.e.*
$\frac{\dot{\tilde{p}}_N^*}{\tilde{p}_N^*} = \frac{\dot{A}_T}{A_T} - \frac{\dot{A}_N}{A_N}$

Proof. See Appendix 5.4.

Building on Proposition 2 showing the saddle-point property of the steady state, we can also conclude that the Balassa-Samuelson property holds not only at the optimal steady state but also along the optimal path.

The BS effect equation derived in Proposition 4 indicates a sharp contrast to that of the BS effect equation (1) presented in the Introduction. Note that this equation does not rely on a capital intensity term as indicated in the equation (1). That is, it shows that the rate of change in non-tradable relative prices is exclusively related to the difference in TFP growth rates between the two sectors.

It is interesting to consider why the formulas are different for large and small countries. In the small countries, the interest rate is given in the world market and the wage is determined only in the tradable goods sector. Thus, the allocation of capital goods between sectors is determined by the capital intensity of each sector. Under the standard assumption that the capital intensity of the non-tradable goods sector is lower than that of the tradable goods sector, an increase in capital will increase the output of the tradable goods sector and decrease that of the non-tradable goods sector due to the Rybczynski theorem. This change in output affects relative output prices, which appear as the capital intensity ratio in the BS formula, as shown in (1). In contrast, in the case of large countries, since the interest rate is determined domestically based on the productivity of the sectors, the capital intensities of the two sectors do not work directly to determine output prices. Thus, the relative intensity term is removed from the formula in the large country case, as shown in Proposition 4.

Finally, even for the large countries, Property (2) still holds from the following first-order conditions on the wage rate at the optimal steady state,

$$\tilde{w}^* = (1-\alpha) A_T k_T^{\alpha} l_T^{-\alpha} \tag{21}$$

This relation clearly indicates that only labor productivity in the tradable goods sector determines the wage rate.

4 Concluding comments

The BS effect in large countries provided in Proposition 4 is shown to be in contrast to the equation for the BS effect in small countries as indicated in (1). For large countries such as Japan and the United States, the BS effect was found not to depend on the capital intensity ratio of the non-tradable and tradable sectors, but more directly only on the difference in TFP growth rates of the sectors. Thus, the results more directly support the stagnation theory of Japan based on the properties of the BS effect described in the introduction. An open question remains to study whether our conclusions are related to the fact that we consider specific Cobb-Douglas utility and production functions. The next step will then be to extend our model to general utility and production functions in order to provide a general formulation of the Balassa-Samuelson effect.

5 Appendix

5.1 Proof of Proposition 1

From the first-order conditions (6) and using the steady state equation $r = \delta + g$ we get:

$$\begin{aligned} \omega_{10} &= \tfrac{r}{w} = \tfrac{\beta}{1-\beta}\tfrac{l_N}{k_N} = \tfrac{\alpha}{1-\alpha}\tfrac{l_T}{k_T} \\ r &= \beta p_N \left(\tfrac{l_N}{k_N}\right)^{1-\beta} = \beta p_N \left(\tfrac{1-\beta}{\beta}\omega_{10}\right)^{1-\beta} = \delta + g \end{aligned} \tag{22}$$

We then derive

$$\omega_{10} = \tfrac{\beta}{1-\beta}\left(\tfrac{\delta+g}{\beta p_N}\right)^{\frac{1}{1-\beta}} \tag{23}$$

Consider again the first-order conditions (6), we get

$$\alpha\left(\tfrac{l_T}{k_T}\right)^{1-\alpha} = \beta p_N \left(\tfrac{l_N}{k_N}\right)^{1-\beta} \quad \Leftrightarrow \quad \alpha\left(\tfrac{1-\alpha}{\alpha}\omega_{10}\right)^{1-\alpha} = \beta p_N\left(\tfrac{1-\beta}{\beta}\omega_{10}\right)^{1-\beta} \tag{24}$$

Solving for p_N using (23) then yields

$$p_N^* = \tfrac{\delta+g}{\beta}\left(\tfrac{\alpha}{\delta+g}\right)^{\frac{1-\beta}{1-\alpha}}\left[\tfrac{\beta(1-\alpha)}{\alpha(1-\beta)}\right]^{1-\beta} \tag{25}$$

Consider now Lemma 2, solving $\mathcal{A}(\omega, p)y = x$ with respect to k using $y_T = gk + c_T$ with $c_T = (1-\theta)p_N c_N/\theta$ gives

$$k = \frac{a_{10} + \frac{(1-\theta)p_N c_N}{\theta}(a_{11}a_{00} - a_{10}a_{01})}{a_{00}(1-ga_{11}) + ga_{10}a_{01}} \tag{26}$$

with

$$\begin{aligned} a_{00} &= \tfrac{(1-\beta)p_N\omega_{10}}{\delta+g}, \quad a_{10} = \tfrac{p_N\beta}{\delta+g}, \\ a_{01} &= \tfrac{(1-\alpha)\omega_{10}}{\delta+g}, \quad a_{11} = \tfrac{\alpha}{\delta+g}. \end{aligned} \tag{27}$$

Solving $\mathcal{A}(\omega, p)y = x$ with respect to y_N using $c_N = y_N$ gives:

$$c_N = \frac{1 - a_{01}\left(gk + \frac{(1-\theta)p_N c_N}{\theta}\right)}{a_{00}} \tag{28}$$

Using (26) into (28) we find

$$c_N^* = \frac{1-ga_{11}}{a_{00}(1-ga_{11})+ga_{10}a_{01}+\frac{(1-\theta)p_N^* a_{01}}{\theta}}$$

with p_N^* as given by (25). We then derive c_T^* and k^*. $\square$

5.2 Proof of Proposition 2

Linearizing the dynamical system around (c_N^*, k^*, p_N^*) gives the Jacobian matrix $\mathcal{J}$:

$$\mathcal{J} = \begin{pmatrix} \frac{\partial y_T}{\partial k} - g - \frac{\partial c_T}{\partial k} & \frac{\partial y_T}{\partial p_N} - \frac{\partial c_T}{\partial p_N} \\ 0 & \frac{1}{\theta} p_N \frac{\partial r}{\partial p_N} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ \mathcal{J}_3 & \mathcal{J}_4 \end{pmatrix} \tag{29}$$

with $\frac{\partial y_N}{\partial k} = \frac{\partial r}{\partial p_N}$.

Since the optimization program (10) has an Hamiltonian structure, and as initially proved by Kurz [6] and Levhari and Liviatan [7], if λ is a characteristic root then $\delta - \lambda$ is also a characteristic root. This is confirmed by showing that $\mathcal{T} = \delta$.

Note that from Lemma 1 we can also derive the sectoral demands for capital and labor as functions of the capital stock and the production of the non-tradable good, namely $l_j = l_j(k, y_N)$, $k_j = k_j(k, y_N)$, $j = N, T$, with $c_N = y_N$. The optimal amount of the tradable good can be also expressed as:

$$y_T = T(k, y_N) = k_T(k, y_N)^\alpha l_T(k, y_N)^{1-\alpha}$$

and from the envelope theorem, we get: $r = T_k(k, y_N)$ and $p_N = -T_{y_N}(k, y_N)$. From Lemmas 1 and 2, we obtain the following derivatives:

$$\frac{\partial r}{\partial p_N} = \frac{\partial y_N}{\partial k} = \frac{\partial c_N}{\partial k} = -\frac{a_{01}}{a_{11}a_{00}-a_{10}a_{01}}, \quad \frac{\partial y_T}{\partial k} = \frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}}$$

Using the input coefficients (7) then yields at the steady state

$$\frac{\partial r}{\partial p_N} = \frac{\partial y_N}{\partial k} = \frac{\partial c_N}{\partial k} = \frac{(\delta+g)(1-\alpha)}{p_N(\beta-\alpha)}, \quad \frac{\partial y_T}{\partial k} = -\frac{(\delta+g)(1-\beta)}{\beta-\alpha} \tag{30}$$

Morever, using $c_T = (1-\theta)p_N c_N/\theta$ we have

$$\frac{\partial c_T}{\partial k} = \frac{1-\theta}{\theta} p_N \frac{\partial c_N}{\partial k} \tag{31}$$

From the Jacobian matrix (29), we then derive

$$\mathcal{T} = \delta + \frac{\partial y_T}{\partial k} - (\delta + g) + p_N \frac{\partial c_N}{\partial k}$$

and we conclude from (30)

$$\frac{\partial y_T}{\partial k} - (\delta + g) + p_N \frac{\partial c_N}{\partial k} = 0$$

It follows therefore that $\mathcal{T} = \delta$. We finally conclude that, because of the triangular structure of the Jacobian matrix, the two characteristic roots are real and distincts. $\square$

5.3 Proof of Proposition 3

We have already proved that the Trace of the Jacobian matrix satisfies $\mathcal{T} = \delta$. Let us now compute the Determinant. Using (30) we get

$$\begin{array}{rcl} \frac{1}{\theta} p_N \frac{\partial r}{\partial p_N} & = & \frac{(\delta+g)(1-\alpha)}{\beta-\alpha} \\ \frac{\partial y_T}{\partial k} - g - \frac{\partial c_T}{\partial k} & = & \frac{1}{\theta}\frac{(\delta+g)(1-\alpha)}{\alpha-\beta} + \delta \end{array} \tag{32}$$

We then derive

$$\begin{array}{rcl} \mathcal{D}^\theta & = & \frac{1}{\theta}\frac{(\delta+g)(1-\alpha)}{\beta-\alpha}\left[\frac{1}{\theta}\frac{(\delta+g)(1-\alpha)}{\alpha-\beta} + \delta\right] \\ & = & -\frac{1}{\theta^2}\frac{(\delta+g)(1-\alpha)}{(\beta-\alpha)^2}\left[(\delta+g)(1-\alpha) + \theta\delta(\alpha-\beta)\right] \end{array} \tag{33}$$

We then derive that if $\alpha > \beta$ then $\mathcal{D}^\theta < 0$ for any δ. When $\beta > \alpha$, $\mathcal{D}^\theta < 0$ if and only if

$$(\delta+g)(1-\alpha) + \theta\delta(\alpha-\beta) \quad = \quad \delta h(\theta) + g(1-\alpha) > 0 \text{ with } h(\theta) = 1 - \alpha(1-\theta) - \theta\beta \tag{34}$$

Straightforward computations then show that if $\beta > \alpha$, $g(0) = 1-\alpha > g(1) = 1-\beta > 0$ with $g'(\theta) = \alpha - \beta < 0$ for all $\theta \in [0,1]$. It follows that $g(\theta) > 0$ for all $\theta \in [0,1]$ and $\mathcal{D}^\theta < 0$. The steady state is therefore a saddle-point for any capital-intensity difference between the tradable and non-tradable sectors and for any $\delta \geq 0$.

□

5.4 Proof of Proposition 4

Choosing again the tradable good as the *numéraire*, i.e. $\tilde{p}_T = 1$, we define from the technologies (4) the following Lagrangian

$$\tilde{\mathcal{L}} = A_T k_T^\alpha l_T^{1-\alpha} + \tilde{p}_N\left[A_N k_N^\beta l_N^{1-\beta} - \tilde{y}_N\right] + \tilde{r}\left[k - k_N - k_T\right] + \tilde{w}\left[1 - l_N - l_T\right]$$

with $\tilde{p}_N$, $\tilde{r}$ and $\tilde{w}$ the price of the tradable good, the interest rate and the wage rate, all in terms of the price of tradable good. The first-order conditions give

$$\begin{array}{l} \tilde{r} = \beta\tilde{p}_N A_N k_N^{\beta-1} l_N^{1-\beta} = \alpha A_T k_T^{\alpha-1} l_T^{1-\alpha} \\ \tilde{w} = (1-\beta)\tilde{p}_N A_N k_N^\beta l_N^{-\beta} = (1-\alpha) A_T k_T^\alpha l_T^{-\alpha} \end{array} \tag{35}$$

Compared to the expression of the Lagrangian $\mathcal{L}$ we clearly have the following relationships:

$$p_N = \frac{\tilde{p}_N A_N}{A_T}, \; r = \frac{\tilde{r}}{A_T} \text{ and } w = \frac{\tilde{w}}{A_T}$$

with $\mathcal{L} = \tilde{\mathcal{L}}/A_T$. Proceeding as in the proof of Proposition 1, we get:

$$\begin{array}{rcl} \tilde{\omega}_{10} & = & \frac{\tilde{r}}{\tilde{w}} = \frac{\beta}{1-\beta}\frac{l_N}{k_N} = \frac{\alpha}{1-\alpha}\frac{l_T}{k_T} \\ \tilde{r} & = & \beta\tilde{p}_N A_N\left(\frac{1-\beta}{\beta}\tilde{\omega}_{10}\right)^{1-\beta} = rA_T = (\delta+g)A_T \end{array} \tag{36}$$

which gives

$$\tilde{\omega}_{10} = \frac{\beta}{1-\beta}\left(\frac{A_T(\delta+g)}{\beta\tilde{p}_N A_N}\right)^{\frac{1}{1-\beta}} \tag{37}$$

Considering the first-order conditions (35), we get

$$\beta\tilde{p}_N A_N\left(\frac{l_N}{k_N}\right)^{1-\beta} = A_T\alpha\left(\frac{l_T}{k_T}\right)^{1-\alpha} \quad \Leftrightarrow \quad \beta\tilde{p}_N A_N\left(\frac{1-\beta}{\beta}\tilde{\omega}_{10}\right)^{1-\beta} = A_T\alpha\left(\frac{1-\alpha}{\alpha}\tilde{\omega}_{10}\right)^{1-\alpha} \tag{38}$$

Solving for $\tilde{p}_N$ using (37) then yields

$$\tilde{p}_N^* = \frac{A_T(\delta+g)}{\beta A_N}\left[\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\left(\frac{\alpha}{\delta+g}\right)^{\frac{1}{1-\alpha}}\right]^{1-\beta}$$

We then derive

$$\frac{\dot{\tilde{p}}_T^*}{\tilde{p}_T^*} = \frac{\dot{A}_N}{A_N} - \frac{\dot{A}_T}{A_T}$$

□

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