amse
école d'économie d'aix-marseille
aix-marseille school of economics

Working Papers / Documents de travail

The Matching Function: A Unified Look into the Black Box

Georgios Angelis
Yann Bramoullé

WP 2023 Nr 15

Aix*Marseille université Socialement engagée
A*Midex Initiative d'excellence Aix-Marseille
FRANCE 2030
Faculté d'Économie et de Gestion Aix*Marseille Université
cnrs
L'ÉCOLE DES HAUTES ÉTUDES EN SCIENCES SOCIALES
Centrale Marseille

The Matching Function:
A Unified Look into the Black Box

Georgios Angelis   Yann Bramoullé

June, 2023*

Abstract

In this paper, we use tools from network theory to trace the properties of the matching function to the structure of granular connections between applicants and firms. We link seemingly disparate parts of the literature and recover existing functional forms as special cases. Our overarching message is that *structure counts*. For rich structures, captured by non-random networks, the matching function depends on whole sets rather than just the sizes of the two sides of the market. For less rich—random network—structures it depends on the sizes of the two sides and a few structural parameters. Structures characterized by greater asymmetries reduce the matching function's efficacy, while denser structures can have ambiguous effects on it. For the special case of the Erdös-Rényi network, we show that the way the network varies with the sizes of the two sides of the market determines if the matching function exhibits constant returns to scale, or even if it is of a specific functional form, such as CES.

*Angelis, georgios.angelis@unibocconi.it: Bocconi University, IGIER; Bramoullé, yann.bramoulle@univ-amu.fr: Aix-Marseille University, CNRS, Aix-Marseille School of Economics. We are grateful to Sanjeev Goyal, Philip Kircher, Pascal Michaillat, Andreas Tryphonides and seminar and conference participants for helpful comments, and especially to Manolis Galenianos for constructive suggestions on an early version of this paper. This work was supported by the French National Research Agency through Grant ANR-17-EURE-0020 and the Excellence Initiative of Aix-Marseille University, A*MIDEX.

1 Introduction

The matching function is the linchpin of most models that depart from the Walrasian equilibrium to capture search frictions in the market (Petrongolo and Pissarides, 2001). The number of contexts in which it has been used highlights its success: most notably the labor market, where unemployed and vacancies coexist in equilibrium, but also credit markets, goods markets, assets trading over the counter, the new monetarist literature aiming to explain the emergence of money, international trade.

The matching function, however, has remained a "black box" for nearly forty years. It is a reduced-form object which economists use for its tractability but no systematic analysis has been done of the frictions implicitly assumed to underlie it, such as information limitations and coordination failures. More specifically, little is known of how the *structure* of the underlying frictions affects the matching function's properties.

Notable contributions that derive a matching function endogenously are Burdett, Shi and Wright (2001) and Albrecht, Gautier and Vroman (2006) in the directed search literature, Calvó-Armengol and Zenou (2005) in the social networks literature, and Stevens (2007) who uses a specific type of queuing system. Each of these contributions derive important implications, but in terms of the matching function, as we will show, each has focused on a particular, highly symmetric structure of the underlying frictions.

In this paper, we propose a network-based model of the matching function, linking the micro to the macro. We consider *any possible* structure of connections between applicants and firms, under a benchmark application-and-offer protocol, to provide the first systematic study of how the structure of the underlying frictions affects the emergent matching function.

The overarching message of our findings is that *structure counts*. Our first insight is that matching improves if there is less inequality in job access among job-seekers: asymmetries unambiguously aggravate miscoordination and hurt the matching process. This result is novel, and may play an important role, empirically, to help explain documented variations in match efficacy. Second, in contexts with search frictions like the labor market, expanding agents' opportunities is not necessarily a good thing: more links can actually worsen matching. More broadly, we show that the matching function is generally a function of whole sets, rather than just the sizes of the two sides of the market as assumed in the literature. In other words, who links where, that is the composition of the market as captured in the network structure, generally matters. We recover existing matching functions as special cases.

In our setup a bipartite network connects applicants (the unemployed) and vacancies. The links can correspond to social ties—as in the social networks literature (e.g. Calvó-Armengol, 2004)—or skills required to apply for that job, or geographic restrictions the applicant has on where to work. In other words a link captures 1-for-1 whether an applicant *knows of* or generally *can* be employed at a vacancy—for whatever reason—and we do not need to take a stance on it for our analysis. The network, characterized by the presence or not of a link between any applicant-vacancy pair, is thus making explicit precisely the frictions the literature has been assuming to implicitly underlie the matching function.

We adopt a simple protocol as to what happens over this network, this collection of links: Each applicant applies to all vacancies they are connected to; each vacancy is offered to an applicant chosen uniformly at random among all applications received—if any application was received. When an applicant receives at least one offer, a "match" or a "meeting" is said to take place. The matching function, which in our setup is an endogenous object, is the expected number of such matches, taking the underlying network structure as given.

We start by analyzing maximally rich structures, where we can differentiate between each applicant and firm. We derive a matching function that depends on whole *sets* rather than just the sizes of the two sides of the market as typically assumed, yet it is given by a compact analytic expression. Our framework is thus consistent with the empirical findings that the *composition* of the market matters for the matching outcome (e.g. Barnichon and Figura, 2015; Hall and Schulhofer-Wohl, 2018) without needing to assume the existence of a matching technology at any level of (dis-)aggregation. In the special case of the complete network, we recover the functional form of the classic balls-in-bins model derived early in the literature (e.g. Butters, 1977) and later as an equilibrium outcome by Burdett, Shi and Wright (2001).

To get closer to the more standard matching functions in the literature more symmetry is required. To this end, we extend our analysis to random networks: the number of links of each applicant is drawn iid from a distribution, and links fall uniformly at random on vacancies on the other side. The resulting structures are less rich than the non-stochastic networks, as both applicants and firms are ex-ante identical. However, they are rich enough: changes in asymmetry are well-defined as mean-preserving spreads in the underlying distribution.

Because all applicants and all firms are now ex-ante identical, the matching function—given again by a compact expression—comes closer to its standard form, depending on the *sizes* of the two sides of the market, rather than whole sets. In the special case when the applicant-degree distribution is degenerate, i.e. all applicants have the same number of links, we recover the matching function of Albrecht, Gautier and Vroman (2006).

Our comparative static results make predictions for *match efficacy*, the residual term of the matching function when the sizes of the two sides of the market are held fixed. Match efficacy is the analog of the production function's total factor productivity and it governs how well the matching process works. It has been shown to fluctuate, and specifically to drop in recessions (e.g. Sedláček, 2014; Mukoyama, Patterson and Şahin, 2018). It is considered a *key concept in understanding turnover in the labor market* (Hall and Schulhofer-Wohl, 2018), yet little is known of the causes of these fluctuations. Our approach provides a natural explanation, based on changes in the network structure connecting applicants to firms. An applicant's degree corresponds to the number of applications they send and hence their search intensity. The network thus captures a structurally rich notion of search intensity among applicants.

Let us first look at the effect of asymmetries. We compare structures where inequality in access to jobs among applicants changes: take a network, hold the firm side fixed and swap a link from it linking firm j with applicant i to it linking firm j with another applicant i'; only the job-finding probabilities of these two applicants are affected. Match efficacy goes down iff applicant i—from whom we take the link away—relies more on it compared to applicant i'. That is, iff the probability that applicant i does not receive an offer from all their other connections excluding j, is higher than the respective probability for applicant i'. In other words, when the "rich" get "richer"—in the sense of access to jobs—match efficacy drops.

Let us now look at the effect of more links, underpinned by an analogous condition. Suppose we add a new link between an applicant i, and a firm j. The job-finding probability of applicant i goes up, while the job-finding probability of all other applicants linking to firm j goes down. The condition for whether the net outcome of this trade-off is positive is whether applicant i, who benefits from the addition, needs that link more compared to the "average" applicant hurt from the addition. Specifically, we show the outcome is positive iff, again, the probability that i does not receive an offer from all the other firms they link to—excluding j—is higher than the average respective probability for all other applicants linking to j.

The result on the effect of asymmetries is novel and we show that it holds very generally, both in arbitrary non-random networks, and in random networks with arbitrary degree distributions. It suggests the informativeness of higher moments of search intensity data, beyond the mean, in explaining changes in match efficacy. For instance, some job-seekers may get excluded from connections to firms during recessions, leading to an increase in inequality of access to jobs and hence to a drop in match efficacy. Validating such network-based explanations could be done by collecting granular microdata on actual applications.

The result on more links is present in the special symmetric structures of Albrecht, Gautier and Vroman (2006) and Calvó-Armengol and Zenou (2005). We show that it holds much more generally for arbitrary non-random networks. Through the lens of such matching functions, the coexistence of higher search intensity during the Great Recession and the observed outward shift of the Beveridge curve does not necessarily stand as a puzzle, as is the case with matching functions in the macro-literature (Elsby, Michaels and Ratner, 2015; Mukoyama, Patterson and Şahin, 2018).

In the final part of the paper, we also make progress in understanding the determinants of the other key dimension of the matching function—*returns to scale.* For the tractable case of the Erdös-Rényi random network, we show that whether the matching function exhibits constant returns to scale or even whether it is of specific functional forms, such as CES, as commonly assumed, depends on how the network "scales," how it varies with the sizes of the two sides of the market. This result echoes Stevens (2007) who uses a queuing system related to the Erdös-Rényi network, to derive the CES functional form.

Taking all our results together, search intensity and a matching technology of certain properties are not two separate things, where the former is "super-imposed" on the latter as in the textbook treatment (e.g. Pissarides, 2000); they are one and the same. Our richer notion of search intensity that takes into account the whole structure of search—the network—*is* the matching function. Directed search and random search are not that different after all.

Network theory allowed us to give a treatment of the problem at significantly greater generality than what has been done before: applicants can be arbitrarily heterogeneous in their number of links, and our analysis applies equally well to small (Burdett, Shi and Wright, 2001) and large (Albrecht, Gautier and Vroman, 2006) economies. Compared with the social networks literature (e.g. Calvó-Armengol, 2004) which takes as primitive the network among applicants, we work with the network between applicants and vacancies. We draw tighter connections to the macro-labor literature precisely because we work with networks of that type. In contrast to the queuing system primitive of Stevens (2007) our network of connections is, in principle at least, fully observable in cross-sectional data.

Our setup, naturally, entails some modeling choices. We propose a minimal, novel, and useful setup to study the determinants of the object of interest of this study, the matching function. We abstract for example from potential differences in the quality of applicant-firm matches, which could be introduced adding weights to the edges, or from firms possibly making more than one (or even zero) offers. Opting for generality, we show how *any* network's properties affect the emergent matching function, without taking a stand on where the network is coming from. How these networks look in reality, how they are formed, how they evolve over the business cycle, and how they respond to policy remain to be studied. In this sense our network is *an intermediate object* whose significance this paper highlights.

The rest of the paper is structured as follows. Section 2 lays out the setup. Section 3 derives the matching function for non-random networks and goes over useful special cases. Section 4 gives our comparative statics regarding match efficacy. Section 5 extends our analysis to random networks. Section 6 presents the implications of scaling. Section 7 discusses modeling choices at length. Section 8 discusses avenues for future research and concludes.

2 The setup

We start by introducing the economic environment, some terminology and notation.

Primitives: The economic environment consists of two sets of agents $\mathcal{U}$ and $\mathcal{V}$, of size $U, V \in \mathbb{N}$ respectively, and a (bipartite) graph G linking elements between the two sets.

We take the elements of $\mathcal{U}$ to correspond to *applicants*, i.e. workers searching for a job, and the elements of $\mathcal{V}$ to correspond to *jobs* offered by firms. In other words $\mathcal{U}$ contains the *unemployed*,[1] while $\mathcal{V}$ contains *vacancies* in our setup.

As a convention, we will be indexing the elements of $\mathcal{U}$ by $i = 1, 2, ..., U$ and the elements of $\mathcal{V}$ by $j = 1, 2, ..., V$. Following the search and matching literature we will assume that each firm has a single vacancy to fill, thus we may interchangeably refer to firm j or vacancy j as the counterparty of an applicant i.

The graph G is represented by an adjacency matrix—denote $G = \big(g_{ij}\big)$, where $g_{ij} = 1$ if applicant i is connected to firm j, and $g_{ij} = 0$ otherwise.[2] $G + ij$ (resp. $G - ij$) denotes network G after adding (resp. deleting) a link between applicant i and vacancy j.

We will denote by $d_i = \sum_j g_{ij}$ an applicant's *degree*, that is the number of firms the applicant connects to. Similarly a firm's degree $d_j = \sum_i g_{ij}$ corresponds to the number of applicants the firm connects to. As a matter of accounting it has to hold that the total number of degrees on the two sides are equal, i.e. $\sum_i d_i = \sum_j d_j$.

Finally we refer to an applicant's *neighborhood* as the set of firms the applicant connects to. Specifically, for an applicant i, define $N_i = \{j \in \mathcal{V} : g_{ij} = 1\}$. Similarly we can define the neighborhood of a firm j. It follows that the size of a node's neighborhood equals their degree; for applicant i denote $|N_i| = d_i$.

As in what follows we will be making connections to the search and matching literature, let

[1] In principle the "applicants" can also be people who are not classified as "unemployed" in the data, for example people searching on the job. With slight abuse of terminology we will be using the terms "applicants," "job-seekers" and "unemployed" interchangeably.

[2] Applicants correspond to rows and firms to columns.

us also define the central quantity of that literature, *market tightness*, $\theta = \frac{V}{U}$.

An application and offer protocol: Taking the network of links G as given, we assume applicants apply to all firms they connect to. Each vacancy is offered to an applicant chosen uniformly at random among all applications received, if any application was received.

When an applicant receives (at least) one offer, a "match" (or a "meeting") is said to take place. The key object of interest throughout our analysis is the *matching function* defined as the expected number of matches taking the network structure as given. We denote

$$m(G) = \mathbb{E}[\#\text{matches}|G]$$

An example: Let us consider the following instance[3]

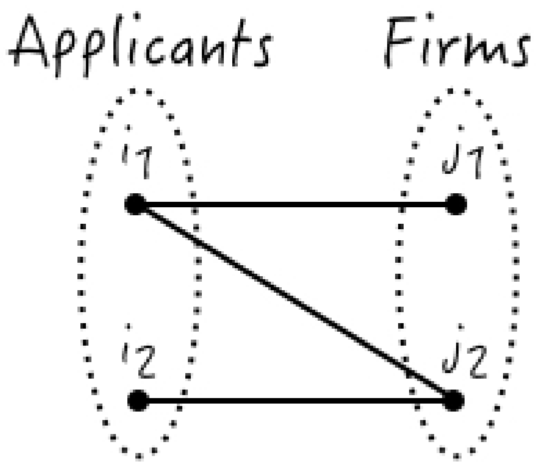


Figure 1: A 2-by-2 example.

Applicant i_1 applies to both firms j_1 and j_2, while applicant i_2 applies only to firm j_2. Accordingly, firm j_1 makes an offer to applicant i_1, and firm j_2 chooses with probability 1/2 to make an offer to i_1 and with probability 1/2 to make an offer to i_2. We take no stand on which offer (if any) both i_1 and i_2 choose.

[3]In matrix form the graph of this example is $G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For exposition, we also note that the degree of applicant i_1 is 2, and of applicant i_2 it is 1. Their corresponding neighborhoods are the sets $\{j_1, j_2\}$, and $\{j_2\}$ respectively.

There are two possible outcomes: (i) j_2 makes an offer to i_2, (ii) j_2 makes an offer to i_1. Outcome (i) is the first best. Outcome (ii) is a state of coexisting vacancy and unemployment, as a result of coordination failure. According to our protocol, each outcome occurs with probability 1/2. Thus, the expected number of matches is given by

$$\begin{aligned} m(G) &= \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 \\ &= 3/2 \end{aligned}$$

A note on interpretation. The links of the graph can correspond to social ties—as in the social networks literature[4] (e.g. Calvó-Armengol, 2004), or skills required to apply for that job, or geographic restrictions the applicant has on where to work. In other words the graph can represent *any* relevant factors restricting the jobs an applicant knows of, or can apply to for whatever reason, and we don't need to take a stance on it for our analysis.

The network structure, that is the presence or not of a link between any applicant-vacancy pair, can thus be taken to make explicit precisely the frictions the search and matching literature has been assuming to implicitly underlie the matching function.[5] The network can be said to capture the underlying *information structure*, broadly conceived, in the economy.

3 Matching in an arbitrary graph

We can extend our computation to any arbitrary graph G, and get a generalized matching function that is a function of *sets* rather than just the sizes of the two sides of the market.

---

[4]The seminal modern contribution here is considered Calvó-Armengol (2004). A series of papers have followed including Calvó-Armengol and Jackson (2004), Calvó-Armengol and Zenou (2005), Ioannides and Soetevent (2006), Galenianos (2014, 2020), Galeotti and Merlino (2014), Espinosa, Kovárík and Ruíz-Palazuelos (2021). Montgomery (1991) is one of the important precursors. An early overview of the use of network tools in economics is provided by Ioannides (1977).

[5]In the words of Petrongolo (VoxEU, 2010) "[Search] frictions derive from several sources, including imperfect information about trading partners, heterogeneous demand and supply, slow mobility, coordination failures and other similar factors."

Proposition 1. *For any given arbitrary graph G connecting job-seekers to vacancies, the matching function defined as the expected number of total matches is given by*

$$m(G) = U - \sum_{i=1}^{U} \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \tag{1}$$

Proof. For any applicant i the probability to receive no offer is $\prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$, and thus their probability of finding a job is

$$\begin{aligned} f_i(G) &\equiv Pr\{i \text{ receives at least one offer}|G\} \\ &= 1 - \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \end{aligned}$$

For each applicant define the indicator random variable showing if they find a job, where

$$Y_i = \begin{cases} 1, & \text{w.p. } f_i(G) \\ 0, & \text{w.p. } 1 - f_i(G) \end{cases}$$

Then the number of matches, taking the graph as given, which by definition is the number of applicants finding a job is also a random variable, and specifically $\#\text{matches}|G = \sum_i Y_i$. The matching function, i.e. the expected number of matches is then

$$\begin{aligned} m(G) &= \mathbb{E}[\#\text{matches}|G] \\ &= \mathbb{E}[\sum_{i=1}^{U} Y_i] \\ &= \sum_{i=1}^{U} \mathbb{E}[Y_i] \\ &= \sum_{i=1}^{U} f_i(G) \\ &= U - \sum_{i=1}^{U} \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \end{aligned}$$

□

Let us pause and appreciate how compact an expression (1) is for how general it is: it gives us the expected number of matches for *any* possible graph G. The matching function is a function of *sets*, as applicants' neighborhoods N_i enter the expression.

We think it is useful to highlight that each applicant i finding a match is a Bernoulli trial with probability of success f_i. The Bernoulli trials are not independent, they all depend on G, but to compute $m(G)$ independence is not required; we only use the linearity of expectation. We also note that the derivation of f_i hinges on each firm deciding independently from all other firms which applicant to make an offer to.

Let us now see how (1) specializes in special types of structures, and specifically how it reduces in being a function only of the sizes U, V as is the case in the literature.

Example 1: The *complete graph* (or family of graphs) is the case where all applicants are connected to all firms. In this case $N_i = \mathcal{V}$, $\forall i$, and $d_j = U$, $\forall j$, thus (1) becomes

$$\begin{aligned} m(G) &= U - \sum_{i=1}^{U} \left(1 - \frac{1}{U}\right)^V \\ &= U\left(1 - \left(1 - \frac{1}{U}\right)^V\right) \end{aligned}$$

It can be seen that the matching function in this case is increasing and concave in its two arguments.[6] We also note the complete graph corresponds to the classic balls-in-bins model. Thus, it is no surprise the above is the same functional form derived early using that model (e.g. Butters, 1977), and later as an equilibrium object by Burdett, Shi and Wright (2001).[7]

The complete graph is an interesting special case as it corresponds to the case when information frictions are eliminated—all applicants know of and can apply to all existing jobs—thus matching is only the outcome of coordination frictions.

It is broadly accepted in the literature (e.g. Petrongolo and Pissarides, 2001; Wright et al.,

[6] Shown in the Online Appendix.

[7] Replace U with m, and V with n to get their eq. 18. We also derive the symmetric of this function, i.e. $m = V\left(1 - \left(1 - \frac{1}{V}\right)^U\right)$, as a special case in our random network treatment; see footnote 16.

2021) that the empirical relevance of this functional form[8] is quite limited.

Example 2: A *double regular graph* (or family of graphs) is the case where every applicant is connected to d_U firms, and each firm is connected to d_V applicants.

This is a doubly-symmetric graph where all applicants and all firms search with the same "intensity," and we show it relates closely to the standard search-and-matching setup. We note that a double regular graph can be thought to correspond to a symmetric equilibrium.

For such graphs (1) gives us the matching function being

$$m(G) = U\left[1 - \left(1 - \frac{1}{d_V}\right)^{d_U}\right]$$

However, by accounting it holds that $Ud_U = Vd_V$, and utilizing this equation we can write

$$m(U, V) = U\left[1 - \left(1 - \frac{1}{d_U}\frac{V}{U}\right)^{d_U}\right]$$

where d_U is taken to be a parameter, and d_V is determined from $Ud_U = Vd_V$. All applicants have the same job-finding probability, which is[9]

$$f(\theta; d_U) = 1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

The matching function in this case can be shown to possess standard properties assumed in the literature: it is constant returns to scale, increasing, and concave in both U, and V. The matching function can also be shown to be approximated at a first-order by a Cobb-Douglas function.[10] We show these results formally in the Online Appendix.

[8]Often people refer to its limiting form: using the result $\lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n = e^x$, the above asymptotically exhibits constant returns to scale, as then the matching function can be taken to be approximately $m(V, U) \approx U\left(1 - e^{-\frac{V}{U}}\right)$.

[9]Naturally, not any choice of a d_U will do; d_U has to be an integer and it has to be such that d_V is also an integer. This points to a limitation of this model (or special case).

[10]We note that the 1st-order approximation result is not specific to this family of graphs: *any* matching function that exhibits CRS to a 1st-order approximation is Cobb-Douglas.

We think the following result is interesting to compare with the more general results that follow on match efficacy.

Proposition 2. *For the double regular network the match efficacy is maximized when* $d_U = 1$.

Proof. See appendix. □

4 Structure and match efficacy

The main result of this section is about aggregate match efficacy. We first provide a related comparative static for the individual level, which has natural empirical analogs and generalizes analogous results of the standard matching function.

Proposition 3. *In terms of their job-finding probability, an applicant (a) invariably benefits by connecting to a new firm, and (b) is hurt if another applicant links to a firm they are connected to.*

Proof. In notation the above comparative statics are respectively

$$f_i(G + ik) > f_i(G), \forall k \notin N_i$$
$$f_i(G + i'k) < f_i(G), \forall k \in N_i, i' \neq i$$

They follow directly from the expression $f_i(G) = 1 - \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$. □

Part (a) illustrates that an applicant's job-finding probability is always improved from higher search intensity. Part (b) is an externality the higher search intensity of one applicant imposes on other applicants. Contrary to the standard matching function where the externality affects all other applicants, in our case it is "local," affecting only the applicants connected to the firm to which the link is added; the rest of the applicants are unaffected.

In other words, an applicant receiving (or losing) a link creates winners and losers, and thus its effect on aggregate match efficacy is a priori ambiguous. The next example illustrates such a trade-off.

Example 3. Consider the following instance where we swap a link from applicant i to i'.

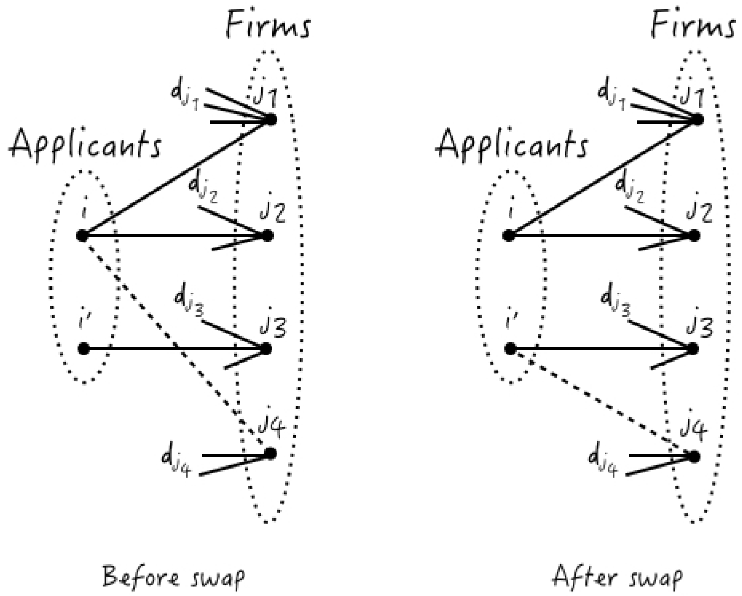


Figure 2: Aggregate efficacy is hurt from the swap iff $\left(1-\frac{1}{d_{j_1}}\right)\left(1-\frac{1}{d_{j_2}}\right) > \left(1-\frac{1}{d_{j_3}}\right)$. In this case applicant i' who is better off even without the extra link, gets a link at the expense of applicant i, raising inequality in access to jobs between them.

Their respective job-finding probabilities before the swap are

$$
\begin{aligned}
f_i^{\text{before}} &= 1 - \left(1 - \frac{1}{d_{j_1}}\right)\left(1 - \frac{1}{d_{j_2}}\right)\left(1 - \frac{1}{d_{j_4}}\right) \\
f_{i'}^{\text{before}} &= 1 - \left(1 - \frac{1}{d_{j_3}}\right)
\end{aligned}
$$

After the swap i loses a link and will necessarily be worse-off, while i' gains a link and will be better-off. Thus a trade-off emerges:

$$
\begin{aligned}
f_i^{\text{after}} &= 1 - \left(1 - \frac{1}{d_{j_1}}\right)\left(1 - \frac{1}{d_{j_2}}\right) < f_i^{\text{before}} \\
f_{i'}^{\text{after}} &= 1 - \left(1 - \frac{1}{d_{j_3}}\right)\left(1 - \frac{1}{d_{j_4}}\right) > f_{i'}^{\text{before}}
\end{aligned}
$$

All other applicants (not shown in the figure) remain unaffected. The outcome of the trade-off depends on how their job-finding probabilities compare *without* the concerned link. More concretely aggregate efficacy is reduced iff $f_i^{\text{before}} + f_{i'}^{\text{before}} < f_i^{\text{after}} + f_{i'}^{\text{after}}$ or

$$\underbrace{\left(1-\frac{1}{d_{j_1}}\right)\left(1-\frac{1}{d_{j_2}}\right)}_{\text{Prob\{i doesn't get offer from connections } \textit{excluding } j_4\text{\}}} \quad > \quad \underbrace{\left(1-\frac{1}{d_{j_3}}\right)}_{\text{Prob\{i' doesn't get offer from connections } \textit{excluding } j_4\text{\}}}$$

This condition states that applicant i is relatively more reliant on the additional link to j_4 than i' to get a job. That is because the probability that i does *not* find a job relying on all their *other* connections, $\left(1-\frac{1}{d_{j_1}}\right)\left(1-\frac{1}{d_{j_2}}\right)$ is greater than the respective probability for i', $\left(1-\frac{1}{d_{j_3}}\right)$. Thus making the swap hurts i more than it benefits i', hence the net outcome is negative.

Theorem 1. *Take an arbitrary network G connecting job-seekers to vacancies.*

(A) Let $\hat{G}$ denote the network resulting from swapping a link $ij \in G$ with link $i'j \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - f_i(\hat{G}) > 1 - f_{i'}(G) \tag{*}$$

(B) Let $\hat{G}$ denote the network resulting from adding link ij, where $ij \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - \bar{f}_{N_j}(G) > 1 - f_i(G) \tag{*'}$$

where $1 - \bar{f}_{N_j}(G) \equiv \frac{1}{d_j}\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right)$.

Proof. See appendix. □

Part (A) of theorem 1 generalizes precisely the instance illustrated in the foregoing example: $1 - f_i(\hat{G})$ is the probability applicant i—the loser of the swap, does *not* receive an offer from all their other connections excluding j, and $1 - f_{i'}(G)$ the respective probability for applicant i'—the winner of the swap. The link swap is harmful iff the former is above the latter, and thus the applicant who is more reliant on the extra connection—i, loses it for the benefit of the applicant who needs it less—i'.

A link swap between applicants is the only operation allowed when holding the firms' degrees fixed and is thus a well-defined comparative static. It corresponds to cases of *redistribution* of links among applicants. Specifically theorem 1 says that any swap, and thus any sequence thereof, that *increases inequality in access to jobs* among applicants will hurt the matching process. In other words, when "the rich get richer"—in terms of access to jobs—match efficacy goes down. Conversely, link swaps that equalize the probabilities of applicants to receive an offer, thus making the network structure *less asymmetric*, improve efficacy.

Part (B) of theorem 1 states that an exactly analogous condition determines whether an additional link, that is *higher* (overall) *search intensity* or equivalently network structures of *higher density*, will improve or hurt overall match efficacy. The right-hand side of the condition is the probability applicant i—the winner of the addition—receives no offer from all their *other* connections, thus determines the reliance of i on the new link; the left-hand side gives the corresponding quantity for the "average" loser of the addition, that is the average probability an applicant connecting to firm j before the addition receives no offer from all their *other* connections excluding firm j.

Theorem 1 formalizes the two main themes of our analysis for match efficacy: structures of higher asymmetry unambiguously hurt overall match efficacy (part A), while structures of higher density can have ambiguous results (part B). We will see different variants of these two themes going forward.

A first variant of the effects of asymmetry expressed only in terms of applicants' degrees can be attained in the special case when all firms have the same degree. In this case: dispersion in applicant's degrees reduces aggregate efficacy.

Proposition 4. *Suppose $d_j = d_V, \forall j$. If (d'_i) is a mean-preserving spread of (d_i), then $m(G') < m(G)$.*

Proof. We have

$$m(G') < m(G) \Leftrightarrow$$

$$U - \sum_{i=1}^{U}\left(1-\frac{1}{d_V}\right)^{d_i'} < U - \sum_{i=1}^{U}\left(1-\frac{1}{d_V}\right)^{d_i} \Leftrightarrow$$
$$\sum_{i=1}^{U}\left(1-\frac{1}{d_V}\right)^{d_i'} > \sum_{i=1}^{U}\left(1-\frac{1}{d_V}\right)^{d_i}$$

Since $\left(1-\frac{1}{d_V}\right)^x$ is a convex function of x, the last inequality holds.[11] □

Corollary 1. *Suppose $d_j = d_V, \forall j$, and let G_R denote the corresponding doubly regular graph, if that exists. Then for any graph G, $m(G) \leq m(G_R)$.*

Thus, when the situation is homogeneous on the firms' side, match efficacy increases with homogeneity on the applicants' side as well. Conversely, any increase in the spread in applicants' degrees will reduce match efficacy.

5 Matching in random graphs

In the foregoing part of our analysis we showed that matching generally depends on whole sets. In such a setup, therefore, to get closer to the standard matching function that only depends on sizes of the two sides, one needs to impose extreme symmetry as that in the double regular or the complete graphs, thus losing all structural richness. We now introduce the family of structures described as (bipartite) random graphs which feature symmetry, as all applicants are ex-ante identical, yet retain enough structural richness to be of interest.

Random graph characterization: Take applicant degrees to be i.i.d. draws from a given distribution $\vec{p} = (p_0, p_1, ...p_V)$, where $p_k \equiv Pr\{d_i = k\}$. For each applicant, conditional on a given draw from that distribution, the links are assumed to fall at random on an equal number of distinct firms among the V.[12]

The applicant-degree distribution $\vec{p}$ can be any arbitrary distribution over the non-negative integers. The way the random graph is characterized induces a distribution of degrees on

[11]The Online Appendix provides some background material on mean-preserving spreads over arbitrary vectors which are not necessarily a probability distribution.

[12]More formally that is random sampling of d_i elements from a population of V without replacement.

the firm side, which we will show is a binomial distribution.

We highlight that even though all applicants are ex-ante identical in these structures, we can still meaningfully talk about changes in the asymmetry of the structure in the form of mean-preserving spreads in the underlying distribution $\vec{p}$.

We will denote the firm-degree distribution by $\vec{z} = (z_0, z_1, ...z_U)$, where $z_k \equiv Pr\{d_j = k\}$. We will also denote the mean degree on the applicant and firm sides by $\bar{d}_U, \bar{d}_V$ respectively, i.e. $\bar{d}_U = \sum_k kp_k$ and $\bar{d}_V = \sum_k kz_k$.

Lemma 1. *Conditional on an applicant-degree distribution $\vec{p}$, the degrees on the firm side follow a binomial distribution, denote $d_j \sim Bin(\lambda, U)$, where $\lambda = \frac{\bar{d}_U}{V}$.*

Proof. We have

$$z_k = Pr\left\{\sum_{i=1}^{U} X_{ij} = k\right\}$$

where X_{ij} is an indicator, being 1 if applicant i links to (has applied to) firm j.

Since all i are ex-ante i.i.d, X_{ij} are also i.i.d with probability[13]

$$\begin{aligned} Pr\{X_{ij} = 1\} &= \sum_{k=1}^{V} Pr\{X_{ij} = 1 | d_i = k\} p_k \\ &= \sum_{k=1}^{V} \frac{\binom{V-1}{k-1}}{\binom{V}{k}} p_k \end{aligned}$$

Define $\lambda \equiv \sum_{k=1}^{V} \frac{\binom{V-1}{k-1}}{\binom{V}{k}} p_k$. Now, by noticing that $\frac{\binom{V-1}{k-1}}{\binom{V}{k}} = \frac{k}{V}$, it follows that

$$\lambda = \frac{\bar{d}_U}{V}$$

Then X_{ij} are Bernoulli with probability of success λ, and thus $d_j \sim Bin(\lambda, U)$. □

We note that λ is a function of $\vec{p}, V$ but for notational simplicity we are not denoting this explicitly.

[13]The second line follows from a standard combinatorial argument: we want to find how many choices include a particular element i, among all the $\binom{V}{k}$ possible choices. We fix element i, and are free to choose the remaining $k-1$ elements from the remaining $V-1$ elements of the pool: these are precisely $\binom{V-1}{k-1}$.

Remark: Since $d_j \sim Bin(\lambda, U)$, it follows that $\bar{d}_V = \lambda U$, and thus[14]

$$\frac{\bar{d}_V}{U} = \frac{\bar{d}_U}{V}$$

This is a useful relationship we will invoke again in our analysis later.

Corollary 2. *A mean-preserving spread in the distribution of d_i's leaves the distribution of d_j's unchanged.*

Proof. This follows from d_j's following a binomial distribution, and its parameter λ depending only on $\bar{d}_U, V$, which stay constant with a mean-preserving spread. □

We will return to this result when we do comparative statics.

5.1 Moving to matching

We now derive the matching function in the stochastic network case. The matching function now is a *double* expectation, over who makes an offer to whom (as before), but also over the realized networks G. In other words, the matching function now is

$$m = \mathbb{E}_G[m(G)]$$

We first prove a lemma we will need regarding the *excess degree* of a firm an applicant connects to. The excess degree, denote by $\tilde{d}$, refers to the number of edges leaving the firm *other than* the edge of the said applicant.[15]

Lemma 2. *The excess degrees of all firms an applicant connects to (a) are i.i.d, and (b) it holds that $Pr\{\tilde{d} = k\} = \frac{(1+k)z_{1+k}}{\bar{d}_v}$.*

[14] In fact it can be shown this is an accounting identity that has to hold for any bipartite random graph.

[15] We note that the result $Pr\{\tilde{d} = k\} = \frac{(1+k)z_{1+k}}{\bar{d}_v}$ is a special case of a more general result known for the configuration model (e.g. Newman, 2003; Jackson, 2010). The result is exact in our case, while in the configuration model it is approximate and holds asymptotically for a large number of nodes.

Proof. The degrees of firms are i.i.d following $Bin(\lambda, U)$. Thus the excess degrees of a firm an applicant connects to are also i.i.d and $\tilde{d} \sim Bin(\lambda, U-1)$, since $U-1$ only of the firm's degree Bernoulli trials remain to be determined. It follows that

$$\begin{aligned}
Pr\{\tilde{d} = k\} &= \binom{U-1}{k} \lambda^k (1-\lambda)^{U-1-k} \\
&= \frac{(U-1)!}{k!(U-1-k)!} \lambda^k (1-\lambda)^{U-1-k} \\
&= \frac{1+k}{\lambda U} \frac{U!}{(1+k)!(U-1-k)!} \lambda^{1+k} (1-\lambda)^{U-1-k} \\
&= \frac{(1+k) z_{1+k}}{\bar{d}_V}
\end{aligned}$$

□

Theorem 2. *The matching function in our stochastic network model, defined as $m = \mathbb{E}_G[m(G)]$, is given by*

$$m = U \left(1 - \sum_{d_U=0}^{V} p_{d_U} (1-\phi)^{d_U} \right) \tag{2}$$

where $\phi = \frac{1-z_0}{\bar{d}_V}$. $z_0 = (1-\lambda)^U$ is the probability a firm receives no applications.

Proof. See appendix. □

Remark: Suppose $\vec{p}$ is degenerate, say $\Pr\{d_i = d\} = 1$. Then all applicants have the same number of connections and thus send the same number of applications, and (2) becomes[16]

$$\begin{aligned}
m(U, V; \vec{p}) &= U \left(1 - \sum_{d_U=0}^{V} p_{d_U} (1-\phi)^{d_U} \right) \\
&= U \left(1 - \left(1 - \frac{1-z_0}{\bar{d}_V} \right)^d \right)
\end{aligned}$$

If also $U, V \to \infty$ holding V/U constant, then $z_0 \to e^{-\bar{d}_V}$. Using $Ud = V\bar{d}_V$,

$$m(U, V; d) = U \left(1 - \left(1 - \frac{1 - e^{-dU/V}}{dU/V} \right)^d \right)$$

We notice this is the matching function derived by Albrecht, Gautier and Vroman (2006).

[16] Taking the special case when $d = 1$, we get $m(U, V) = V \left(1 - (1 - \frac{1}{V})^U \right)$, which is the symmetric function of the balls-in-bins model we got in section 3. See also footnote 7.

5.2 The special case of Erdös-Rényi

As it is an important benchmark in the literature of random graphs, we derive the matching properties of the Erdös-Rényi network, which we show is a special case of our model.

Lemma 3. *When $d_i \sim Bin(\mu, V)$, our stochastic network becomes the Erdös-Rényi network, that is it can be created drawing each link with the same probability μ.*

Proof. We have already shown the firm degree distribution is binomial with parameter $\lambda = \frac{\bar{d}_U}{V}$. But since $d_i \sim Bin(\mu, V)$, $\bar{d}_U = \mu V$. Thus $\lambda = \mu$, and $d_j \sim Bin(\mu, U)$.

It follows that in this case the network can be constructed drawing each link with probability μ as this process amounts to precisely V Bernoulli trials for each applicant, and U Bernoulli trials for each vacancy all with probability of success μ. □

Corollary 3. *In the case of the Erdös-Rényi model the matching function is given by*

$$m = U\left(1 - \left[1 - \frac{1-(1-\mu)^U}{U}\right]^V\right)$$

Proof. See appendix. □

Two polar cases are readily verifiable: As we would expect, for $\mu = 0$, we have the empty graph, and $m = 0$; For $\mu = 1$, we have the complete graph, and $m = U\left(1 - \left[1 - \frac{1}{U}\right]^V\right)$.

Corollary 4. *The matching function in the Erdös-Rényi model is increasing in μ, and thus it is maximized when $\mu = 1$ (the complete graph).*

Proof. It follows directly from the expression for m. □

Contrasting this with the result on double regular graphs, it indicates that a higher search intensity has generally an ambiguous effect on match efficacy. We will see another variant of this finding in the more general analysis that follows.

5.3 The effects of asymmetry and density; again

From (2), the applicants' job-finding probability is given generally from the expression

$$f = \frac{m}{U} = 1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U}$$

where $\phi = \frac{1-z_0}{\bar{d}_V}$.

We have already established that a mean-preserving spread in d_i's leaves the distribution of d_j's unchanged (corollary 3), and thus ϕ is left unchanged as well. We can further show the following proposition

Proposition 5. *A mean-preserving spread*[17] *in the distribution of d_i's reduces the applicants' job-finding probability f.*

Proof. Denote by $\vec{p}\,'$ a mean-preserving spread of $\vec{p}$. Equivalently, the two distributions have the same mean, and $\vec{p}$ second-order stochastically dominates (SOSD) $\vec{p}\,'$ (Mas-Colell, Whinston and Green, 1995, proposition 6.D.2). The definition of SOSD holds that for every non-decreasing concave functions $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ it holds that

$$\sum_{d_U} p'_{d_U} u(d_U) \leq \sum_{d_U} p_{d_U} u(d_U)$$

Now, $-(1-\phi)^{d_U}$ is an increasing and (strictly) concave function, and then from the definition of SOSD we have

$$\sum_{d_U} p'_{d_U}[-(1-\phi)^{d_U}] \leq \sum_{d_U} p_{d_U}[-(1-\phi)^{d_U}] \Rightarrow$$
$$1 - \sum_{d_U} p'_{d_U}(1-\phi)^{d_U} \leq 1 - \sum_{d_U} p_{d_U}(1-\phi)^{d_U} \Rightarrow$$

[17] A mean-preserving spread is defined as a compound lottery, say $d'_U = d_U + Y$, where $\mathbb{E}[Y|d_U] = 0$ (Rothschild and Stiglitz, 1970). For example define $Y = 0$, if $d_U = 0$, and $Y = \begin{cases} +1, \text{w.p. } 1/2 \\ -1, \text{w.p. } 1/2 \end{cases}$ if $d_U \geq 1$.

$$f' \leq f$$

with equality holding iff $\phi = 0$ or $\phi = 1$. $\square$

Corollary 5. *Given $\bar{d}_U$, match efficacy is maximized when everyone sends the same number of applications, $\bar{d}_U$.*

Proof. We have

$$1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U} < 1 - (1-\phi)^{\bar{d}_U}$$

following from Jensen's inequality. $\square$

We note that proposition 5 and its corollary echo the results on the impact of heterogeneity on match efficacy of section 4.

We now move to study the effect of uniformly increasing search intensity across applicants. As illustrated in figure 3, when all applicants send the same number of applications, i.e. $\Pr\{d_i = d\} = 1$, the matching function exhibits an inverted-U shape as a function of d. Thus uniformly increasing search intensity has an ambiguous effect on match efficacy.

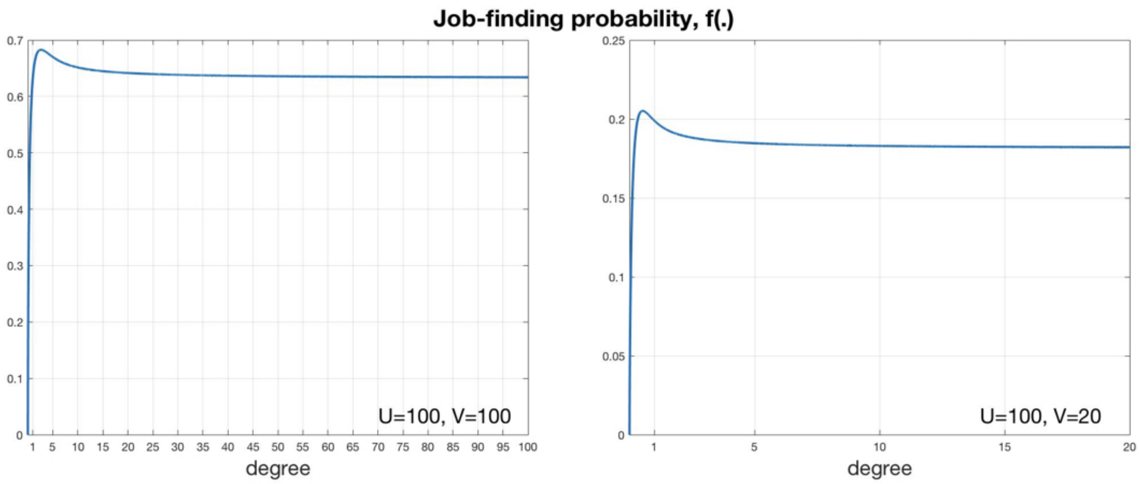


Figure 3: Job-finding probability $f(.)$ as a function of $d \in [0, V]$, when all applicants have the same degree d. U, V are held fixed. $f(.)$ (and thus $m(.)$) has a characteristic inverted-U shape as a function of degree d.

We notice the possibility that the peak of the inverted-U can be to the left of $d = 1$ in the case of a slack market of relatively low V (figure on the right, $U = 100, V = 20$). That means that in this case, having everyone send the same number of applications, maximum efficacy is achieved when everyone sends a single application. Furthermore, quantitatively, even when the peak is at a higher degree, it is early, or in other words the matching function has a "long" right tail. This means that even when there are benefits from multiple applications, congestion effects outweigh these benefits quite fast. The peak in the left figure, a perfectly balanced market with $U = V = 100$ occurs at $d = 3$.

Finally, we compare the efficacy of three networks: (i) the double regular network, (ii) the Erdös-Rényi network, and (iii) the 1-side regular network where all applicants have the same degree (Albrecht, Gautier and Vroman, 2006). To do so, in the following figure we hold U, V fixed, and vary the applicants' degree d. In cases (i) and (iii) all applicants have exactly the same degree, d. In case (ii) there is (ex-post) heterogeneity in applicants' degrees, but they all have the same (ex-ante) expected degree, d; in other words for the three networks to be comparable, we vary μ in the Erdös-Rényi model by varying d, where $\mu = \frac{d}{V}$.

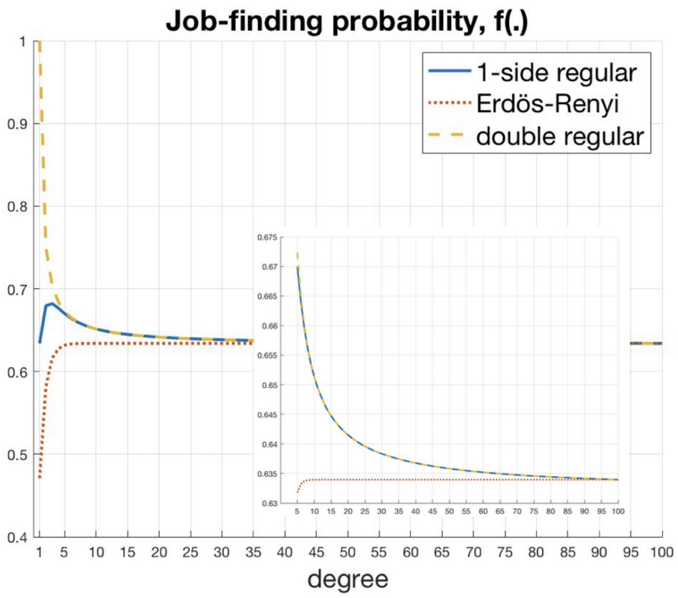


Figure 4: Comparing the efficacy of (i) the double regular network, (ii) the Erdös-Rényi network, and (iii) the 1-side regular network where all applicants have the same degree, holding fixed $U = V = 100$. The detail in the graph zooms in the range of $d = 5..100$ for visual clarity.

There are a few comments to make on the plot. First, we confirm that the job-finding probability is decreasing for the double regular graph, increasing for Erdös-Rényi, and of the inverted-U shape in the case of the 1-side regular network. We notice that for the double regular network the probability is exactly 1 for a degree of 1, as this is the case where every applicant links to a single firm and thus there are no search or coordination frictions.

Second, the fact that efficacy in Erdös-Rényi is lower everywhere reflects our results that applicant-side heterogeneity is harmful for efficacy: in Erdös-Rényi there is applicant-side heterogeneity in degrees, while in both other networks there isn't. The fact that efficacy is lower in the case of the 1-side regular network compared to the double regular network suggests that heterogeneity is harmful on the firm side as well: neither network has heterogeneity on the applicant side, but the double regular doesn't have heterogeneity on the firm side either, while the 1-side regular has the degrees on the firm side following a binomial distribution.

Lastly, we see the 1-side regular and double regular networks quickly converging to each other as degrees increase. This is because the matching functions of the two differ only by the probability of a firm to receive no applications, z_0, and this probability goes to 0 as d increases.[18] All three converge to the same limit at $d = V$, which corresponds to the matching function of the complete network, i.e. the classic balls-in-bins case.

6 The importance of network scaling

The complete graph is the special case when all applicants know of all vacancies. If the number of vacancies changes, for the graph to remain complete, each applicant has to scale their degree up or down accordingly. Thus working with the complete graph structure (the balls-in-bins model) implicitly assumes some type of *scaling* of the degree of all applicants as the size of the graph changes. The same applies to the double regular network.

Upon reflection, the question of scaling applies to all graphs and relates to the question of

[18] Our formulas for the two cases are $f(\theta; d) = 1 - \left(1 - \frac{1}{d}\theta\right)^d$ for the double regular network, and $f(\theta; d) = 1 - \left(1 - \frac{1-z_0}{d}\theta\right)^d$ for the 1-side regular network.

how does applicant degrees change, when market conditions—the sizes of the two sides of the market—change. This question requires an economic model of network generation and thus lies outside the scope of this paper. We can, however, illustrate its significance following the classic analysis of the the Erdös-Rényi network, where the key parameter of the distribution is parameterized as a function of the network size, to study the change in properties of the graph (e.g. Jackson, 2010, p. 89).

In the Erdös-Rényi network it holds that

$$f = 1 - \left[1 - \frac{1 - z_0}{U}\right]^V, \text{ where } z_0 = (1-\mu)^U, \ \mu = \frac{\bar{d}_U}{V}$$

We take the limit where $U, V \to +\infty$ holding V/U fixed, to get

$$f \to 1 - e^{-(1-z_0)V/U}, \text{ where } z_0 \to e^{-U\bar{d}_U/V}$$

Now, scaling refers to how $\bar{d}_U$ changes when U, V change.[19] For example, if $\bar{d}_U$ is constant, the matching function will exhibit constant returns to scale, while, if $\bar{d}_U$ scales linearly in V (i.e. μ stays constant), the matching function will exhibit increasing returns to scale (figure 5). Finally, if $\bar{d}_U$ scales according to the following expression,

$$\bar{d}_U = -\frac{V}{U} \ln\left(1 + \frac{U}{V} \ln\left(1 - \frac{(U^{-\gamma} + V^{-\gamma})^{-\frac{1}{\gamma}}}{U}\right)\right), \ \gamma > 0$$

the matching function will be of the CES form, as shown in proposition 6 that follows. It can be seen that in this case $\bar{d}_U$ is homogeneous of degree 0 in U, V, and as illustrated in figure 6 it is increasing and concave in market tightness $\frac{V}{U}$.

[19] We notice this job-finding probability corresponds to another matching function assumed in the literature, where a fraction of applications z_0 gets "lost" (e.g. Petrongolo and Pissarides, 2001). In our case no application is "lost." z_0 is the fraction of firms receiving no applications and it is an endogenous quantity.

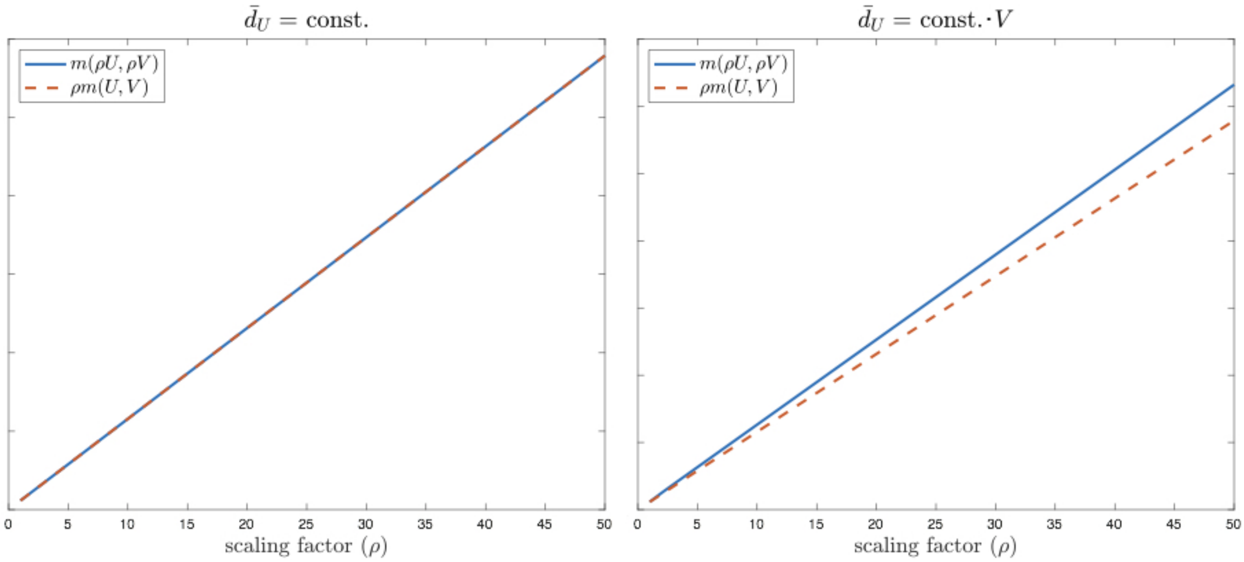


Figure 5: Returns to scale

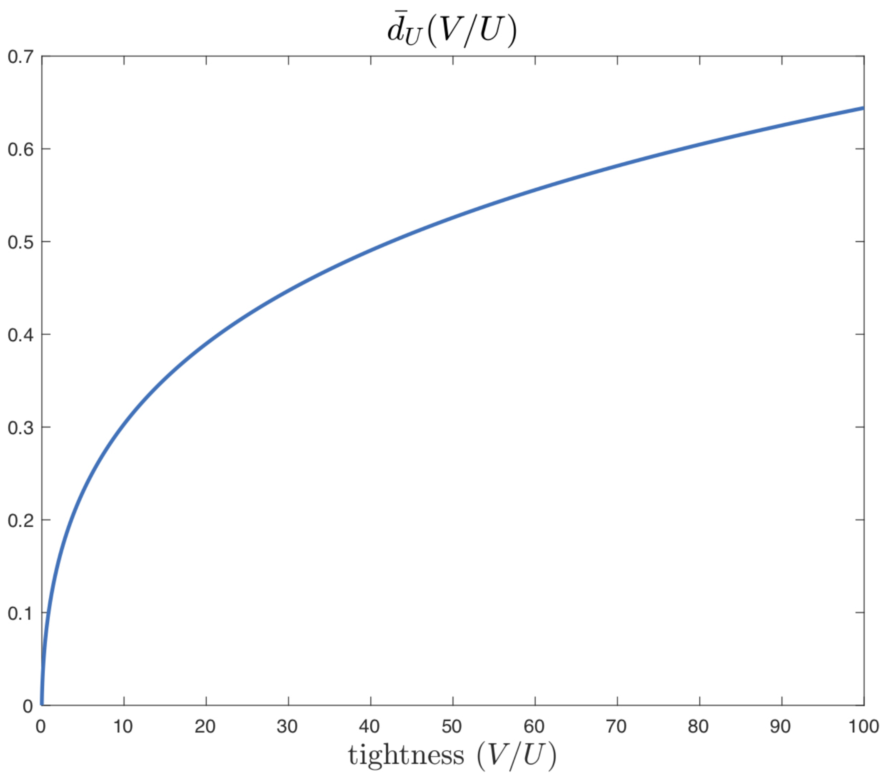


Figure 6: $\bar{d}_U$ yielding CES

Proposition 6. *Take a function $\tilde{m}(U,V)$ such that $\tilde{m}(U,V) < U,V$. Then, in the Erdös-Rényi network, if*

$$\bar{d}_U = -\frac{V}{U}\ln\left(1 + \frac{U}{V}\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right)\right) \tag{**}$$

it will hold that $f = \frac{\tilde{m}(U,V)}{U}$, and $\bar{d}_U > 0$.

Proof. See appendix. □

It follows that if $\tilde{m}(U,V) = m_0\big(U^{-\gamma}+V^{-\gamma}\big)^{-\frac{1}{\gamma}}$, $\gamma > 0, m_0 \in (0,1]$,[20] the Erdös-Rényi network gives rise to the CES matching function. The Leontief can be derived as the limit case of the CES when $\gamma \to \infty$. The Cobb-Douglas can also be derived as a limit case of the more general CES function $m_0\big((1-\eta)U^{-\gamma} + \eta V^{-\gamma}\big)^{-\frac{1}{\gamma}}$, $\eta \in (0,1)$, when $\gamma \to 0$.[21]

Our result on being able to generate specific matching functions can be taken to illustrate "how much" or rather "what type" of a knife-edge case the Cobb-Douglas, the CES, or in fact any specification of the aggregate matching function are. Our analysis suggests that, assuming one of these specifications amounts to assuming a particular type of *scaling* of the applicant-degree distribution: For any pair of U, V, $\bar{d}_U$ has to scale "appropriately"—as given by (**), for the matching function to be of the respective functional form, or more generally to satisfy constant returns to scale.

The question as to *how* the network scales with market conditions is ultimately an empirical one, and our analysis suggests what type of data are needed to address it.

Connection to Stevens (2007). In the large-economy limit of the Erdös-Rényi network we are working with in this section, the applicant-degree distribution is Poisson with parameter $\bar{d}_U$. Stevens (2007) describes the underlying network of connections by a queuing system. Even though the queuing system cannot be mapped exactly to the Erdös-Rényi network because its characterization is inherently dynamic, we argue the two are closely related.

[20] For this functional form it can be checked that $\tilde{m}(U,V) < U,V$.

[21] For the Cobb-Douglas we have to restrict U,V in the regions where $m < U,V$. The fact that Cobb-Douglas is less tractable than the CES in the discrete case is known in the literature. Cobb-Douglas can also be derived as the 1st-order approximation to the CES as shown in the appendix of section 3.

Stevens (2007) describes the underlying network of connections and offer protocol as a ("telephone-line") queuing system (Cox and Miller, 1965 section 4.4; Ross, 2010 section 6.3). Applications arrive at a Poisson rate of $u\alpha$. These can equivalently be thought as independent arrivals from u applicants each at a Poisson rate α. This means that at any time interval τ the number of applications ("customer arrivals") from each applicant is a Poisson distributed random variable with parameter $\alpha\tau$. Thus on the applicant side the model is virtually identical to our Erdös-Rényi network.

Things change slightly from our setup regarding how applications "fall" on vacancies, and how offers (matches) are made: the firm is modeled as a "server" processing applications at a Poisson rate of $v\gamma$. As soon as an application arrives, if it finds the server empty, it starts being processed; if the server is busy, the application is lost. Once processing finishes, an offer (match) is made and the server returns to being empty and ready to process another application. The (stationary) probability the server is empty is known to be $\frac{v\gamma}{v\gamma+u\alpha}$, and thus the matching function is

$$u\alpha\frac{v\gamma}{v\gamma+u\alpha}$$

that is the rate of arrival times the fraction of time it finds the server empty.

This is a useful comparison for two reasons: (a) it links to an alternative characterization of the network of connections and offers protocol which are specified as arrival processes and thus for comparison we have to see them over some time interval τ; (b) it relates to scaling, as one of the main insights of Stevens (2007) is that the matching function is of the CES form precisely when average application intensity—α, and recruitment intensity—γ, are endogenously chosen in a way that depends on market conditions, that is, when intensities "scale" appropriately (Stevens, 2007, proposition 3).

7 Discussion of assumptions

This section discusses some elements of our framework, and modeling choices we made.

Applicant links falling uniformly at random on vacancies. We note that essentially our random network model assumes firms have no limit on how many applications they want to screen; they accept as many applications as applicants send. One way of relaxing this assumption is by assuming that each firm has a randomly drawn degree, and "meeting" amounts to connecting the stubs of applicant and firm degrees. That model is the bipartite configuration model analyzed by Newman, Strogatz and Watts (2001); asymptotically it will have the same behavior as ours, with the only difference being that z_k will not be necessarily binomial, but an arbitrary distribution, and a primitive. That setup will still have (asymptotically) a matching function given by (2) but z_0 will be a primitive.

Burdett, Shi and Wright (2001) and Albrecht, Gautier and Vroman (2006) are examples of how to derive their respective networks, in equilibrium, when links fall randomly on firms.

A note on applicant-degree distributions. Having applicants with zero degree in the set of "unemployed" may appear somewhat atypical. Most of the analysis will not be affected by limiting attention to distributions with $Pr\{d_i = 0\} = 0$. Naturally this would preclude the Erdös-Rényi network which does have a positive fraction of applicants with 0 degree.

In other words, our random graph approach treats in a unified way the decision to enter the labor market (extensive margin of search—send 0 vs sending ≥ 1 applications) and the search effort an applicant puts (intensive margin of search—number of applications sent). People with 0 links (and hence 0 applications) will be the "voluntarily" unemployed, while people who send ≥ 1 applications and don't find a job, the "involuntarily" unemployed. The quotation marks are there to highlight that the "voluntarily" unemployed may just be shut out of the network of job search, not having for example the relevant skills at the moment, or the right connections, or they may not know how to search.[22]

Vertical heterogeneity and uniform at random offers. Our setup abstracts from potential differences in applicants' quality in their abilities. Such a dimension could be introduced, for example, by adding weights to the edges to reflect differences in the applicant-job fit. The probability to receive an offer could then be proportional to that weight. The extent to which "good" applicants would tend to be good fits for multiple positions, and

[22]Some firms will also have zero degree, but this is perfectly normal: a firm can find no match either because nobody applied to its vacancy, or because it made an offer to someone who takes another offer.

thus receive more offers, could exacerbate the coordination frictions in the matching process.

In the absence of differences in quality the uniform-at-random offer from the firm side seems the most natural choice. In fact, we argue, there are many contexts where the choice among applicants is so multidimensional, that, which applicant really "clicks" for a position they look fit for "in paper," is unpredictable and thus can appear as uniform at random.[23]

Multiple applications and single offers. In our application-and-offer protocol, the two sides of the market are asymmetric: applicants apply to *all* vacancies they are connected to, while firms make *a single* offer. This asymmetry stems from the nature of the problem: an offer, once made, is assumed to be binding; this is not the case for applications. Since firms are assumed to have a single vacancy to fill, they necessarily make a single offer.

We think extending the analysis to cases where firms make multiple offers would be naturally done in a context where (large) firms have multiple vacancies to fill. Such a setup may give rise to matching functions where the two sides of the market enter more symmetrically compared to the matching functions we have derived here.[24]

Relatedly, we do not consider cases where a firm makes no offers, or cases where firms make multiple rounds of offers. To make such extensions meaningful would require to introduce notions of quality and preferences. Such decisions are also inherently dynamic: for a firm to decide whether it does not like any candidate[25], or whether it wants to make an offer to its second-best candidate if its top choice turned their offer down, is a function of what types of candidates the firm expects to "meet" next period (e.g. Martellini and Menzio, 2020). We abstract from all these dimensions in the current analysis as our emphasis is on the

[23]It is common that unpredictable factors, such as "cultural" fit, enter the decision process. Similarly, in the academic job market, internal politics among groups and which one will manage to make the strongest case to hire their preferred candidate affect the outcome.

[24]In such setups, we conjecture that a critical role will be played by correlations between the number of applications sent at firms and the number of vacancies at these firms.

[25]A firm might also make no offer because it is a "phantom" (Cheron and Decreuse, 2017). This however is a different issue relating to the measurement of vacancies and if an observed vacancy is "really" a vacancy. Incidentally, this issue has gained some attention recently in the policy debate (e.g. Michaillat, 2023).

implications of the network structure, by definition a cross-sectional object.[26]

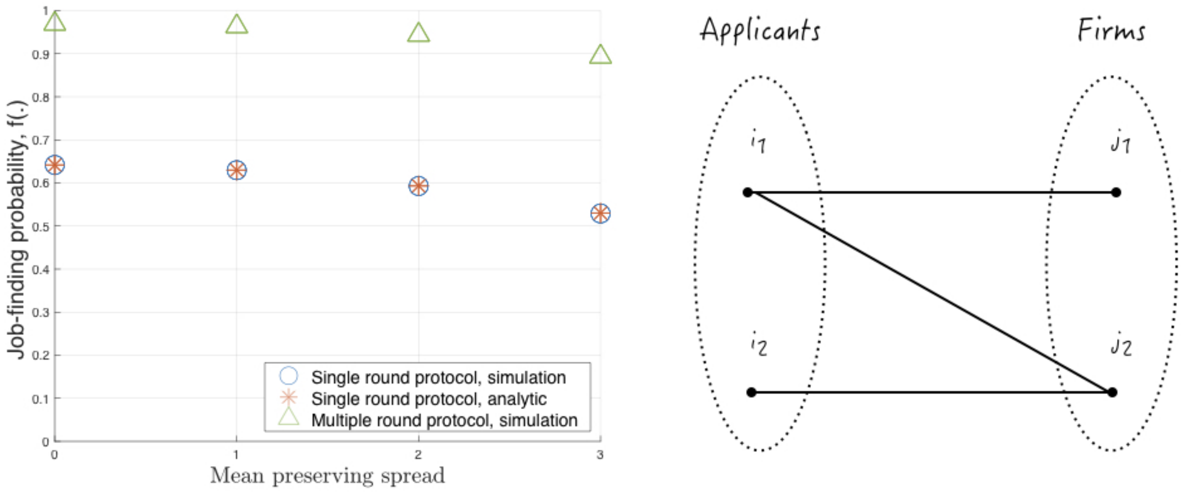
We do think however some further commenting is useful. The number of rounds determines the extent of coordination frictions. Since our interest is on the role of the network structure, we have kept the application-and-offer protocol fixed throughout, while we vary the network structures. To get a sense of how the two interact let us modify our protocol to the polar case of allowing arbitrarily many rounds of offers to the same pool of applicants: assume each applicant chooses randomly one of the offers they receive; this applicant-vacancy pair exits the market, while unmatched firms choose randomly another applicant in their applications list to make an offer to; the process is repeated until all possible matches are exhausted.[27]

With such a protocol, the result that increasing the number of links across applicants can have a negative impact on match efficacy due to congestion effects will go away. This can be seen in the limit: in the complete graph, allowing for multiple rounds necessarily means the matching function reaches its efficacy limit, that is $m(U, V) = \min\{U, V\}$, i.e. the maximum number of possible matches will ultimately be formed.

However, the result that a mean-preserving spread in the applicant degree distribution hurts match efficacy is maintained, even though, naturally, the level of the job-finding probability changes. We show this through simulations. In the left figure below, the y-axis plots the applicant job-finding probability and as we move to the right, the figure plots the outcome of four applicant-degree distributions each being a MPS of the one to its left. The reason the result carries through can be seen going back to our figure 1, reproduced below to the right: more well-connected applicants (i_1 in the figure) can receive and accept an offer (offer j_2 in the figure) that "blocks" less connected applicants from being able to receive an offer even if multiple rounds are allowed for.

[26]The single-round assumption is commonly held among related papers that look at the allocation of jobs to applicants in the cross-section (e.g. Shimer, 2005; Albrecht, Gautier and Vroman, 2006; Galenianos and Kircher, 2009; as well as Calvó-Armengol, 2004; Calvó-Armengol and Zenou, 2005).

[27]This type of analysis is inspired from Kircher (2009) and Gautier and Holzner (2017) who, however, contrary to us, impose some amount of coordination of how applicants are recalled over rounds.



(A) Protocol comparison

(B) A "blocking" pattern

Figure 7: Figure A plots the applicant job-finding probability and as we move to the right of the x-axis each applicant-degree distributions is a mean-preserving spread of the one to its left. More specifically, the sizes of the two sides of the market are held constant, $U = V = 100$; $\bar{d}_U = 20$; and for $x = \{0, 1, 2, 3\}$ half of the applicants are randomly assigned a high degree of $d^h = \bar{d}_U + x \cdot 5$, and the rest a low degree of $d^l = \bar{d}_U - x \cdot 5$. We run 3000 simulations of network creation and matching and take their average for each data point. For the single round protocol we have the analytic expression of the matching function as well and show it is identical to the expectation computed through simulation. Figure B illustrates that in the presence of such patterns in the network, when applicant i_1 receives and accepts an offer from firm j_2, applicant i_2 is cut out of the network even if we allow for multiple rounds thereafter.

8 Concluding remarks

The key message of this paper is that *structure counts* for the properties of the emergent matching function, to the point we can claim that the underlying structure *is* the matching function. A natural next question is, of course, how do economic forces determine that structure and how does that structure change over time.

It is worth highlighting one sense of structure we have not covered in our analysis. For any plausible interpretation of a link, one might expect correlation patterns to emerge. Suppose applicant i connects to two jobs—say j, j'. Conditional on applicant i' connecting to job

j, they may have a higher than average probability to also connect to job j'. This relates to the fundamental notion of *clustering* (or transitivity) in the networks literature, and it is absent from our random network structures, for which the probabilities of each link are independent. We think this is of first-order interest to be refined in future theoretical work and be tested empirically.

Apart from its realism, clustering is expected to matter quantitatively, as we would expect coordination failures be exacerbated in its presence: to put it simply, clustering implies that the same people compete for the same jobs. A description of the underlying network of connections featuring clustering can be seen as a relaxation of what is commonly done, to assume fully segmented *submarkets* (or "locations"),[28] an approach that comes with the known empirical challenge of choosing the "boundaries" of these submarkets.

Networks can also guide refined welfare types of exercises,[29] where the planner takes into account the whole network of connections, thus finding the constrained optimum by eliminating only coordination frictions. The constrained optimum is a known and easy to solve problem in the networks literature, it is the max-flow problem.

Starting from the job-finding probability for the random network, given in (2) and taking 1st-order Taylor approximation around $\bar{d}_u$, we get

$$f \overset{\text{1st}}{\approx} 1 - (1-\phi)^{\bar{d}_u}$$

which is the job-finding probability in the 1-side regular network (Albrecht, Gautier and Vroman, 2006) giving that network an additional special role. Taking the 2nd-order Taylor approximation yields

$$f \overset{\text{2nd}}{\approx} 1 - (1-\phi)^{\bar{d}_u}\left[1 + \frac{1}{2}[\ln(1-\phi)]^2 Var(d_u)\right]$$

giving yet another variant of our theme that asymmetries, as these are captured here by the variance of the distribution of connections, hurt match efficacy.

[28]E.g. Sahin et al., 2014, Herz and van Rens, 2020 in the empirical, and Menzio and Shi, 2011 in the theoretical literature. Under clustering, there will be subsets of highly connected applicants and firms, who however are not fully isolated from the rest of the market. Such subsets would form a "submarket."

[29]Such type of exercises have been the focus of large part of the literature, e.g. Moen, 1997; Shimer, 2005; Albrecht, Gautier and Vroman, 2006; Kircher, 2009; Galenianos and Kircher, 2009; Sahin et al., 2014.

In terms of empirical work, all parts of our analysis suggest that to further unpack the implications of the structure that underlies the emergent matching function, we need granular, applicant-vacancy level data on links. What are the stylized facts of these networks, and how they evolve over the business cycle are open questions.

In terms of theoretical work what is needed is to further understand how these networks are formed: as the outcome of choices of people to acquire information of new jobs and firms to advertise them, the choices of people to move to jobs of similar skills to theirs, or to acquire new skills to expand the set of jobs they can apply to, or to relocate. Models of network creation and models of directed search can usefully be blended in that direction.

Bibliography

Albrecht, James, Pieter A. Gautier, and Susan Vroman. 2006. "Equilibrium Directed Search with Multiple Applications." *The Review of Economic Studies*, 73(4): 869–891.

Barnichon, Regis, and Andrew Figura. 2015. "Labor Market Heterogeneity and the Aggregate Matching Function." *American Economic Journal: Macroeconomics*, 7(4): 222–49.

Burdett, Kenneth, Shouyong Shi, and Randall Wright. 2001. "Pricing and Matching with Frictions." *Journal of Political Economy*, 109(5): 1060–1085.

Butters, Gerard R. 1977. "Equilibrium Distributions of Sales and Advertising Prices." *Review of Economic Studies*, 44(3): 465–491.

Calvó-Armengol, Antoni. 2004. "Job contact networks." *Journal of Economic Theory*, 115(1): 191–206.

Calvó-Armengol, Antoni, and Matthew O. Jackson. 2004. "The Effects of Social Networks on Employment and Inequality." *American Economic Review*, 94(3): 426–454.

Calvó-Armengol, Antoni, and Yves Zenou. 2005. "Job matching, social network and word-of-mouth communication." *Journal of Urban Economics*, 57(3): 500–522.

Cheron, Arnaud, and Bruno Decreuse. 2017. "Matching with Phantoms." *Review of Economic Studies*, 84(3): 1041–1070.

Cox, D.R., and H.D. Miller. 1965. *The Theory of Stochastic Processes. Methuen's mono-*

graphs on applied probability and statistics, Methuen.

Elsby, Michael W. L., Ryan Michaels, and David Ratner. 2015. “The Beveridge Curve: A Survey.” *Journal of Economic Literature*, 53(3): 571–630.

Espinosa, María Paz, Jaromír Kovárík, and Sofía Ruíz-Palazuelos. 2021. “Are close-knit networks good for employment?” Universidad Pablo de Olavide, Department of Economics Working Papers 21.06.

Faberman, R. Jason, and Marianna Kudlyak. 2019. “The Intensity of Job Search and Search Duration.” *American Economic Journal: Macroeconomics*, 11(3): 327–57.

Galenianos, Manolis. 2014. “Hiring through referrals.” *Journal of Economic Theory*, 152(C): 304–323.

Galenianos, Manolis. 2020. “Referral Networks and Inequality.” *The Economic Journal*, 131(633): 271–301.

Galenianos, Manolis, and Philipp Kircher. 2009. “Directed search with multiple job applications.” *Journal of Economic Theory*, 144(2): 445–471.

Galeotti, Andrea, and Luca Paolo Merlino. 2014. “Endogenous Job Contact Networks.” *International Economic Review*, 55(4): 1201–1226.

Gautier, Pieter A., and Christian L. Holzner. 2017. “Simultaneous Search and Efficiency of Entry and Search Intensity.” *American Economic Journal: Microeconomics*, 9(3): 245–82.

Hall, Robert E., and Sam Schulhofer-Wohl. 2018. “Measuring Job-Finding Rates and Matching Efficiency with Heterogeneous Job-Seekers.” *American Economic Journal: Macroeconomics*, 10(1): 1–32.

Herz, Benedikt, and Thijs van Rens. 2020. “Accounting for Mismatch Unemployment.” *Journal of the European Economic Association*, 18(4): 1619–1654.

Ioannides, Yannis M. 1977. “Evolution of Trading Structures.” In *The Economy as an Evolving Complex System II.* , ed. Brian W. Arthur, Steven N. Durlauf and David A. Lane, 129–67. Reading: Addison-Wesley.

Ioannides, Yannis M., and Adriaan R. Soetevent. 2006. “Wages and Employment in a Random Social Network with Arbitrary Degree Distribution.” *American Economic Review (Papers & Proceedings)*, 96(2): 270–274.

Jackson, Matthew O. 2010. *Social and Economic Networks.* Princeton University Press.

Kircher, Philipp. 2009. "Efficiency of Simultaneous Search." *Journal of Political Economy*, 117(5): 861–913.

Lange, Fabian, and Theodore Papageorgiou. 2020. "Beyond Cobb-Douglas: Flexibly Estimating Matching Functions with Unobserved Matching Efficiency." National Bureau of Economic Research, Inc NBER Working Papers 26972.

Martellini, Paolo, and Guido Menzio. 2020. "Declining Search Frictions, Unemployment, and Growth." *Journal of Political Economy*, 128(12): 4387–4437.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic Theory. OUP Catalogue*, Oxford University Press.

Menzio, Guido, and Shouyong Shi. 2011. "Efficient Search on the Job and the Business Cycle." *Journal of Political Economy*, 119(3): 468–510.

Michaillat, Pascal. 2023. "The Fake Problem of Fake Vacancies." *Pascal's Newsletter.*

Moen, Espen R. 1997. "Competitive Search Equilibrium." *Journal of Political Economy*, 105(2): 385–411.

Montgomery, James D. 1991. "Social Networks and Labor-Market Outcomes: Toward an Economic Analysis." *American Economic Review*, 81(5): 1407–18.

Mukoyama, Toshihiko, Christina Patterson, and Ayşegül Şahin. 2018. "Job Search Behavior over the Business Cycle." *American Economic Journal: Macroeconomics*, 10(1): 190–215.

Newman, M. E. J. 2003. "The Structure and Function of Complex Networks." *SIAM Review*, 45(2): 167–256.

Newman, M. E. J., S. H. Strogatz, and D. J. Watts. 2001. "Random graphs with arbitrary degree distributions and their applications." *Physical Review E*, 64(2).

Petrongolo, Barbara. 2010. "The Diamond, Mortensen and Pissarides Nobel: Search and market frictions." *VoxEU.*

Petrongolo, Barbara, and Christopher A. Pissarides. 2001. "Looking into the Black Box: A Survey of the Matching Function." *Journal of Economic Literature*, 39(2): 390–431.

Pissarides, Christopher A. 2000. *Equilibrium Unemployment Theory, 2nd Edition.* Vol. 1

of *MIT Press Books*, The MIT Press.

Ross, Sheldon M. 2010. *Introduction to Probability Models.* Academic Press.

Rothschild, Michael, and Joseph Stiglitz. 1970. "Increasing risk: I. A definition." *Journal of Economic Theory*, 2(3): 225–243.

Sahin, Ayşegül, Joseph Song, Giorgio Topa, and Giovanni L. Violante. 2014. "Mismatch Unemployment." *American Economic Review*, 104(11): 3529–64.

Sedláček, Petr. 2014. "Match efficiency and firms' hiring standards." *Journal of Monetary Economics*, 62(C): 123–133.

Shimer, Robert. 2005. "The Assignment of Workers to Jobs in an Economy with Coordination Frictions." *Journal of Political Economy*, 113(5): 996–1025.

Stevens, Margaret. 2007. "New Microfoundations for the Aggregate Matching Function." *International Economic Review*, 48(3): 847–868.

Wright, Randall, Philipp Kircher, Benoît Julien, and Veronica Guerrieri. 2021. "Directed Search and Competitive Search Equilibrium: A Guided Tour." *Journal of Economic Literature*, 59(1): 90–148.

Appendices

A Section 3, Matching in an arbitrary graph

Proposition. *For the double regular network matching efficacy is maximized when $d_U = 1$.*

Proof. The job-finding probability in the double regular graph case is

$$f = 1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

To study its monotonicity holding θ fixed and varying d_U, let us define and study the monotonicity of the auxiliary function

$$h(d_U) = \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

where naturally $d_U \geq \theta$ for the function to be well-defined.[30] From now on we drop the subscript U to simplify notation.

Define

$$\begin{aligned}\tilde{h}(d) &= \ln(h(d)) \\ &= d \ln\left(1 - \frac{1}{d}\theta\right)\end{aligned}$$

Now,

$$\begin{aligned}\tilde{h}'(d) &= \ln\left(1 - \frac{1}{d}\theta\right) + \frac{d\frac{1}{d^2}\theta}{1 - \frac{1}{d}\theta} \\ &= \ln\left(1 - \frac{1}{d}\theta\right) + \frac{\frac{1}{d}\theta}{1 - \frac{1}{d}\theta}\end{aligned}$$

We can show this is always > 0. Exponentiate both sides to get

$$e^{\tilde{h}'} = \left(1 - \frac{1}{d}\theta\right) e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}}$$

But we know $e^x \geq 1 + x, \forall x \geq 0$, thus

$$\begin{aligned}e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq 1 + \frac{\frac{1}{d}\theta}{1 - \frac{1}{d}\theta} \Rightarrow \\ e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq \frac{1}{1 - \frac{1}{d}\theta} \Rightarrow \\ \left(1 - \frac{1}{d}\theta\right) e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq 1\end{aligned}$$

Since $e^{\tilde{h}'} \geq 1$, it follows that $\tilde{h}' \geq 0$, thus $\tilde{h}$ is increasing in d, thus h is increasing in d, and hence f is decreasing in d. □

B Section 4, Structure and match efficacy

Theorem. *Take an arbitrary network G connecting job-seekers to vacancies.*

[30]This constraint is imposed in our model from $Ud_U = Vd_V$, and $d_V \geq 1$.

(A) Let $\hat{G}$ denote the network resulting from swapping a link $ij \in G$ with link $i'j \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - f_i(\hat{G}) > 1 - f_{i'}(G)$$

(B) Let $\hat{G}$ denote the network resulting from adding link ij, where $ij \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - \bar{f}_{N_j}(G) > 1 - f_i(G)$$

where $1 - \bar{f}_{N_j}(G) \equiv \frac{1}{d_j} \sum_{k \in N_j} \prod_{j' \in N_k/\{j\}} \left(1 - \frac{1}{d_{j'}}\right)$.

Proof. Part (A): Since nothing changes for any other applicant other than i, i', only these two matter, thus

$$\begin{gathered}
m(\hat{G}) < m(G) \Leftrightarrow \\
f_i(\hat{G}) + f_{i'}(\hat{G}) < f_i(G) + f_{i'}(G) \Leftrightarrow \\
-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) - \prod_{k \in N_{i'} \bigcup \{j\}} \left(1 - \frac{1}{d_k}\right) < -\prod_{k \in N_i} \left(1 - \frac{1}{d_k}\right) - \prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) \Leftrightarrow \\
-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_i} \left(1 - \frac{1}{d_k}\right) < -\prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_{i'} \bigcup \{j\}} \left(1 - \frac{1}{d_k}\right) \Leftrightarrow \\
-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right)\left(1 - \frac{1}{d_j}\right) < -\prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right)\left(1 - \frac{1}{d_j}\right) \Leftrightarrow \\
-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right)\left[1 - \left(1 - \frac{1}{d_j}\right)\right] < -\prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right)\left[1 - \left(1 - \frac{1}{d_j}\right)\right] \Leftrightarrow \\
\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) > \prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) \Leftrightarrow \\
1 - f_i(\hat{G}) > 1 - f_{i'}(G)
\end{gathered}$$

Part (B): In this case only i and the applicants in the neighborhood of firm j are affected. Specifically,

$$
\begin{aligned}
m(\hat{G}) < m(G) &\Leftrightarrow \\
f_i(\hat{G}) + \sum_{k\in N_j} f_k(\hat{G}) < f_i(G) + \sum_{k\in N_j} f_k(G) &\Leftrightarrow \\
-\left(1-\frac{1}{1+d_j}\right)\left\{\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) + \sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right)\right\} < -\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) - \left(1-\frac{1}{d_j}\right)\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
\frac{1}{1+d_j}\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) < \left(\frac{1}{d_j}-\frac{1}{1+d_j}\right)\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
\frac{1}{1+d_j}\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) < \frac{1}{d_j(1+d_j)}\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) < \frac{1}{d_j}\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
1-f_i(G) < \underbrace{\frac{1}{d_j}\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right)}_{\equiv 1-\bar{f}_{N_j}(G)}
\end{aligned}
$$

To go from line 2 to line 3 we used the expression for an applicant's job-finding probability, canceled the 1's from all terms, and factored out $\left(1-\frac{1}{1+d_j}\right)$ on the left hand side for compactness. To go from line 3 to line 4 we collected terms on the two sides. □

C Section 5, Matching in random graphs

Theorem. *The matching function in our stochastic network model, defined as* $m = \mathbb{E}_G[m(G)]$, *is given by*

$$m = U\left(1-\sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U}\right) \tag{2}$$

where $\phi = \frac{1-z_0}{\bar{d}_V}$. $z_0 = (1-\lambda)^U$ *is the probability a firm receives no applications.*

Proof. We have that

$$Pr\{i \text{ receives at least one offer} \mid N_i, d_j \forall j\in N_i\} = 1-\prod_{j\in N_i}\left(1-\frac{1}{d_j}\right)$$

Therefore the (ex-ante) probability an applicant finds a job is given by

$$f = 1-\mathbb{E}_{N_i}\left\{\mathbb{E}_{(d_j)_{j\in N_i}}\left\{\prod_{j\in N_i}\left(1-\frac{1}{d_j}\right)\middle| N_i\right\}\right\}$$

$$
\begin{aligned}
&= 1 - \sum_{N_i} p_{N_i} \mathbb{E}_{(d_j)_{j\in N_i}} \left\{ \prod_{j\in N_i} \left(1 - \frac{1}{d_j}\right) \middle| N_i \right\} \\
&= 1 - \sum_{N_i} p_{N_i} \prod_{j\in N_i} \left(1 - \mathbb{E}_{d_j} \left\{ \frac{1}{d_j} \middle| N_i \right\}\right) \\
&= 1 - \sum_{N_i} p_{N_i} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{|N_i|} \\
&= 1 - \sum_{d_U} \sum_{N_i:|N_i|=d_U} p_{d_U} \frac{1}{\binom{V}{d_U}} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \frac{1}{\binom{V}{d_U}} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{d_U} \sum_{N_i:|N_i|=d_U} 1 \\
&= 1 - \sum_{d_U} p_{d_U} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \left(1 - \sum_{k=0}^{U-1} \frac{1}{1+k} \frac{(1+k)z_{1+k}}{\bar{d}_V}\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \left(1 - \frac{1}{\bar{d}_V} \sum_{k=1}^{U} z_k\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \left(1 - \frac{1 - z_0}{\bar{d}_V}\right)^{d_U}
\end{aligned}
$$

Since this is the probability of each applicant finding a job, the expected number of matches is given by $m = \sum_i f = Uf$. □

In the derivation of f, to go from the 2nd to the 3rd line we used the fact that d_j's are independently distributed within i's neighborhood (part (a) of lemma), and to go from the 3rd to the 4th line we used that d_j's are also identically distributed within any neighborhood (also part (a) of lemma). To go to the 5th line, we enumerate the neighborhoods by their size and use the fact that the probability to generate a particular neighborhood N_i of size d_U is to draw a degree of d_U and then choose the one among the $\binom{V}{d_U}$ neighborhoods of such size. To go to the 6th line we notice that nothing depends on the exact neighborhood N_i, only its size, thus we factor everything out of the second sum. To go to the 7th line we use the fact that the number of neighborhoods with d_U members is precisely $\binom{V}{d_U}$, hence this term cancels. To go to the 8th line we use part (b) of the lemma.

Corollary. *In the case of the Erdös-Rényi model the matching function is given by*

$$m = U\left(1 - \left[1 - \frac{1-(1-\mu)^U}{U}\right]^V\right)$$

Proof. The matching function is $m = U \cdot f$. We will work with the job-finding probability f. Theorem 2 specializes in this case as

$$\begin{aligned}
f &= 1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U} \\
&= 1 - \sum_{d_U=0}^{V} \binom{V}{d_U}\mu^{d_U}(1-\mu)^{V-d_U}(1-\phi)^{d_U} \\
&= 1 - \sum_{d_U=0}^{V} \binom{V}{d_U}[\mu(1-\phi)]^{d_U}(1-\mu)^{V-d_U} \\
&= 1 - [1-\mu+\mu(1-\phi)]^V \\
&= 1 - [1-\mu\phi]^V \\
&= 1 - \left[1 - \mu\frac{1-z_0}{\mu U}\right]^V \\
&= 1 - \left(1 - \frac{1-(1-\mu)^U}{U}\right)^V
\end{aligned}$$

□

D Section 6, The importance of network scaling

Proposition. *Take a function $\tilde{m}(U,V)$ such that $\tilde{m}(U,V) < U, V$. Then, in the Erdös-Rényi network, if*

$$\bar{d}_U = -\frac{V}{U}\ln\left(1 + \frac{U}{V}\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right)\right) \tag{**}$$

it will hold that $f = \frac{\tilde{m}(U,V)}{U}$, and $\bar{d}_U > 0$.

Proof.

$$
\begin{aligned}
\bar{d}_U &= -\frac{V}{U}\ln\left(1+\frac{U}{V}\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right)\right) \Leftrightarrow \\
e^{-U\bar{d}_U/V} &= 1+\frac{U}{V}\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
1-e^{-U\bar{d}_U/V} &= -\frac{U}{V}\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
\frac{V}{U}(1-e^{-U\bar{d}_U/V}) &= -\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
\frac{V}{U}(1-z_0) &= -\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
f &= \frac{\tilde{m}(U,V)}{U}
\end{aligned}
$$

Now, since $\bar{d}_U$ corresponds to a mean, it has to be that $\bar{d}_U > 0$; this is indeed the case. As long as $\tilde{m}(U,V) < U$ sufficiently, so that $\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \approx -\frac{\tilde{m}(U,V)}{U}$ is a good approximation, from the 3rd line above we get $1-e^{-U\bar{d}_U/V} = \frac{\tilde{m}(U,V)}{V}$, and since $\frac{\tilde{m}(U,V)}{V} < 1$ from our assumption on $\tilde{m}(U,V)$, this equation pins down a unique $\bar{d}_U > 0$. $\square$

The Matching Function:
A Unified Look into the Black Box

Online Appendix

Georgios Angelis Yann Bramoullé

June, 2023*

This Online Appendix provides supplementary proofs for results not central to the paper.

Section 3, Matching in an arbitrary graph

Proposition. *The matching function in the case of the complete graph is increasing and concave in its two arguments*

Proof. The matching function in this case is

$$m(U,V) = U\left(1 - \left(1 - \frac{1}{U}\right)^V\right)$$

Its derivatives are of the respective signs:

$$\frac{\partial m}{\partial V} = -U\big(1 - \frac{1}{U}\big)^V \ln\big(1 - \frac{1}{U}\big) > 0, \text{ and } \frac{\partial^2 m}{\partial V^2} = -U\big(1 - \frac{1}{U}\big)^V \big[\ln\big(1 - \frac{1}{U}\big)\big]^2 < 0$$
$$\frac{\partial m}{\partial U} = 1 - \big(1 - \frac{1}{U}\big)^{V-1}\big(\frac{U-(1+V)}{U}\big) > 0, \text{ and } \frac{\partial^2 m}{\partial U^2} = -\frac{V^2}{U^3}\big(1 - \frac{1}{U}\big)^{V-2} < 0$$

□

*Angelis, georgios.angelis@unibocconi.it: Bocconi University, IGIER; Bramoullé, yann.bramoulle@univ-amu.fr: Aix-Marseille University, CNRS, Aix-Marseille School of Economics.

Proposition. *The matching function m for a double regular graph exhibits constant returns to scale, and it is increasing and concave in each of its arguments.*

Proof. The matching function is

$$m(U, V; d_U) = U \left[1 - \left(1 - \frac{1}{d_U} \frac{V}{U} \right)^{d_U} \right]$$

Constant returns to scale follow from the definition, as $\forall \lambda > 0$

$$\begin{aligned} m(\lambda U, \lambda V; d_U) &= \lambda U \left[1 - \left(1 - \frac{1}{d_U} \frac{\lambda V}{\lambda U} \right)^{d_U} \right] \\ &= \lambda m(U, V; d_U) \end{aligned}$$

For the rest it helps to express m in terms of the job-finding probability $m(U, V) = U f(\theta)$, where $f(\theta) = \left[1 - \left(1 - \frac{1}{d_U} \theta \right)^{d_U} \right]$, and we dropped the parameter d_U as an argument of the functions for notational convenience.

So for monotonicity and concavity we check the derivatives:

$$\frac{\partial m}{\partial V} = f'(\theta) \tag{i}$$

$$\frac{\partial^2 m}{\partial V^2} = f''(\theta) \frac{1}{U} \tag{ii}$$

$$\frac{\partial m}{\partial U} = f(\theta) - f'(\theta)\theta \tag{iii}$$

$$\frac{\partial^2 m}{\partial U^2} = f''(\theta)\theta^2 U^{-1} \tag{iv}$$

We first show that f is increasing and concave, signing conditions (i), (ii), (iv):

$$\begin{aligned} f'(\theta) &= (1 - \theta)^{d_U - 1} \geq 0 \\ f''(\theta) &= -(d_U - 1)(1 - \theta)^{d_U - 2} \leq 0 \end{aligned}$$

We also note that $f(0) = 0$, and $f(1) = 1 - \left(1 - \frac{1}{d_U} \right)^{d_U} \leq 1$.

To get the sign of (iii) we show that $f(\theta) - f'(\theta)\theta \geq 0$: Define $Q(\theta) = f(\theta) - f'(\theta)\theta$. But $Q' = -\theta f'' \geq 0$. And since $Q(0) = 0$, $Q(\theta) \geq 0$. □

We note the elasticity of $m(\cdot)$ is not constant.[1] Specifically, denote $\eta(\theta) \equiv \frac{\partial m}{\partial V}\frac{V}{m}$, then

$$\eta(\theta) = \frac{f'(\theta)\theta}{f(\theta)}$$

Of course, from CRS we have that $\frac{\partial m}{\partial U}\frac{U}{m} = 1 - \eta(\theta)$. It follows from the concavity of m w.r.t U that $\eta(\theta) < 1$, as we showed above that $f'(\theta)\theta \leq f(\theta)$.

Proposition. *For tightness $\theta = \frac{V}{U}$ around 1, the matching function m for a double regular graph is equal to a Cobb-Douglas function up to 1st-order. Specifically, one can write*

$$m(U, V) \approx m_0 V^{\tilde{\eta}} U^{1-\tilde{\eta}},$$

where $m_0 = f(1) \leq 1$, and $\tilde{\eta} = \eta(1) < 1$.

Proof. We take logs of the matching function

$$\ln(m) = \ln(U) + \ln\left(1 - \left[1 - \frac{1}{d_U} e^{\ln\left(\frac{V}{U}\right)}\right]^{d_U}\right)$$

Define $L(x) \equiv \ln\left(1 - \left[1 - \frac{1}{d_U}e^x\right]^{d_U}\right)$, where $x \equiv \ln\left(\frac{V}{U}\right)$. We can take the Taylor expansion of $L(x)$ around any $x_0 \in (-\infty, \ln(d_U))$; we choose to do so around $x_0 = 0$:

$$L(x) = \sum_{n=0}^{\infty} \frac{L^{(n)}(0)}{n!} x^n$$

The 1st-order approximation yields

$$L(x) \approx L(0) + L'(0)x$$

And hence the matching function is (approximately) of the Cobb-Douglas form:

$$\ln(m) \approx L(0) + (1 - L'(0))\ln(U) + L'(0)\ln(V)$$

where $L(0) = \ln\left(1 - \left(1 - \frac{1}{d_U}\right)^{d_U}\right)$, $L'(0) = \frac{\left(1-\frac{1}{d_U}\right)^{d_U - 1}}{1-\left(1-\frac{1}{d_U}\right)^{d_U}}$. $\square$

[1] Constant elasticity is not considered one of the characteristic properties of the matching function. For example the Cobb-Douglas has constant elasticity, while the specification $m(V, U) = \left[V^{-\gamma} + U^{-\gamma}\right]^{-\frac{1}{\gamma}}, \gamma > 0$ does not.

We also notice that $L'(0) = \frac{f'(1)\cdot 1}{f(1)} = \eta(1)$, and $L(0) = \ln(f(1))$. Thus we have shown that at a 1st-order, for cases when $V \approx U$, and all applicants and firms are symmetric we can write

$$m(U,V) \approx m_0 V^{\tilde{\eta}} U^{1-\tilde{\eta}},$$

where $m_0 = f(1) \leq 1$, and $\tilde{\eta} = \eta(1) < 1$.

Section 4, Structure and match efficacy

In this section we provide some background material on mean-preserving spreads over arbitrary vectors which are not necessarily a probability distribution relating to proposition 4 in the main text.

Definition *A vector x' is **a mean preserving spread** (MPS) of vector x if they have the same mean $\sum_i x_i = \sum_i x'_i$ and if x can be obtained from x' by a series of **Pigou-Dalton transfers**, ignoring the identities of the agents.*

Definition *A transfer $t > 0$ from one agent to another when the two agents are endowed with x_1, x_2 respectively of some quantity, is **a Pigou-Dalton transfer** if $x_1 > x_2$ AND $x_1 - t \geq x_2 + t$.*

That is a Pigou-Dalton transfer between two agents is one such that an amount is transferred from the richer to the poorer agent preserving their relative positions. The quantity under consideration can be anything, e.g. wealth, number of friends etc.

We make the following observations following straight from the definitions.

Remark 1: Any sequence of Pigou-Dalton transfers is mean-preserving.

Remark 2: It can be helpful to think of a mean-preserving spread (MPS) x' of a vector x, as created from x doing "inverse" Pigou-Dalton transfers. "Inverse" Pigou-Dalton transfers are transfers where the rich become richer and the poor poorer.

Remark 3: A mean-preserving spread (MPS) increases inequality in the outcomes, while a Pigou-Dalton transfer decreases it.

For a MPS we can show the following basic result:

Proposition. *For a strictly convex function $h(\cdot)$, if x' is a MPS of x, then it holds that*

$$\sum_i h(x_i') > \sum_i h(x_i)$$

Proof. It suffices to show the inequality holds for a single Pigou-Dalton transfer (up to relabeling). Then by applying it repetitively, we can show it holds for any sequence of such transfers. Let us assume a transfer occurs between agents 1 and 2.

Assume $x_2' = x_2 + t, x_1' = x_1 - t$, and $x_2 \geq x_1$, where $t > 0$. Then

$$\begin{aligned} \sum_i h(x_i') > \sum_i h(x_i) &\Leftrightarrow \\ h(x_1 - t) + h(x_2 + t) > h(x_1) + h(x_2) &\Leftrightarrow \\ h(x_2 + t) - h(x_2) > h(x_1) - h(x_1 - t) &\Leftrightarrow \\ \frac{h(x_2 + t) - h(x_2)}{t} > \frac{h(x_1) - h(x_1 - t)}{t} & \end{aligned}$$

But we know from the mean value theorem there exist $\tilde{c} \in (x_2, x_2 + t)$ and $\tilde{\tilde{c}} \in (x_1 - t, x_1)$ s.t.

$$\begin{aligned} h'(\tilde{c}) &= \frac{h(x_2 + t) - h(x_2)}{t} \\ h'(\tilde{\tilde{c}}) &= \frac{h(x_1) - h(x_1 - t)}{t} \end{aligned}$$

We also know that since $h(\cdot)$ is convex, $h'(\cdot)$ is increasing thus $h'(\tilde{c}) > h'(\tilde{\tilde{c}})$, completing the proof. $\square$

Note: Even though typically the outcome vectors x, x' are taken to be positive in applications (e.g. income redistribution), this is not a requirement. The result holds equally well for positive and negative outcome vectors.