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Expectations, self-fulfilling prophecies and the business cycle

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Abstract: *Macroeconomic models in which exogenous, self-fulfilling changes in expectations play a significant role in output fluctuations are often discarded on two claims: they require implausible calibrations of structural parameters and they are enable to account for several empirical features associated with demand shocks. We show that these claims are only valid to the extent that they are applied to one-sector models. In contrast, we prove that two-sector models allow the existence of self-fulfilling prophecies for a large set of empirically realistic values for all the structural parameters, and that a two-sector model submitted to sunspot shocks can account not only for all the standard stylized facts associated with demand shocks, but also for other dimensions of the business cycle that standard RBC-type models cannot explain.*

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1 Introduction

This paper explores the existence of local indeterminacy and sunspot fluctuations in general infinite-horizon models with external effects leading to increasing returns to scale (IRS). We aim to determine whether expectation-driven business-cycles exist under empirically realistic values for the main structural parameters regarding utility and production functions. We also assess whether such sunspot-driven models can explain several defining features of observed business-cycles.

Up to now, the sunspot literature has failed to provide a fully credible alternative to DSGE models as an explanation of the business cycle, for two main reasons. First, in many models, fluctuations based on self-fulfilling changes in expectations occur at parameter values (degree of increasing returns to scale, markups, preference parameters, etc.) that are considered inconsistent with existing empirical estimates. Second, as early pointed out by Schmitt-Grohé [61], even models that allow sunspot fluctuations to arise under credible parameter values have difficulties accounting for several characteristic features of the observed fluctuations.

Our aim in this paper is thus to address the following two questions: (i) is there any standard (one-sector or two-sector) infinite-horizon model in which sunspot fluctuations arise under realistic calibration for *all* the structural parameters? (ii) if such a model exist, can it also reproduce the salient empirical properties of the business cycle when self-fulfilling changes in expectations are the main source of transitory fluctuations in output? We argue that the answer to these two questions is positive.

To address these questions, we combine a theoretical contribution and a data confrontation analysis. From a theoretical perspective, we consider the standard framework of RBC-type models with increasing returns to scale (IRS) based on productive externalities put forward in the literature by the seminal contributions of Benhabib and Farmer [7, 8]. Yet, we do not restrict the specifications of individual preferences and of the production function to have particular forms, imposing instead minimal sets of assumptions on these functions. A key novelty of our analysis is that we express the local stability conditions of the steady state in terms of five critical and economically interpretable parameters: the elasticity of intertemporal substitution in consumption (EIS), the Frisch elasticities of the labor supply curve with respect both to real wage and to marginal utility of wealth, the elasticity of substitution between capital and labor, and the degree of increasing returns to scale. As explained below, we argue that the Frisch elasticity with respect to marginal utility of wealth provides a relevant measure of the degree of "wealth effect" on labor supply decisions, which has recently been shown to play a major role in the local stability properties of dynamic macroeconomic models (see Dufourt *et al.* [20, 21], Jaimovich [41]). We show that our general formulation encompasses as special cases all the standard formulations for individual preferences proposed in the literature, such as the Greenwood *et al.* (GHH) [30] formulation with no income effect, Hansen's [34] formulation with separable consumption and labor, and the King *et al.* (KPR) [46] formulation with constant positive income effect. For each version of model (one-sector and two-sector), we can then derive the range of parameter values consistent with indeterminacy and compare it with the range of available empirical estimates.

We derive two important conclusions. First, we prove that expectation-driven fluctuations in one-sector RBC models are *ruled out* for any empirically plausible calibrations for structural parameters. Second, in sharp contrast, we prove that the existence of expectation-driven fluctuations is a robust property of two-sector models, in the sense that they arise for a wide range of empirically credible parameter values. For example, we show that sunspot fluctuations are compatible with any value of the wage elasticity of labor supply provided the other critical elasticities are in an appropriate range. Likewise, sunspot fluctuations can occur for an arbitrarily small value for the elasticity of intertemporal substitution (EIS) in consumption provided the degree of income effect is not too small.

Building on these theoretical results, our second step is to confront the two-sector model submitted to technological shocks and self-fulfilling prophecies with the data. Following the approach of Schmitt-Grohé [61], we first estimate a bivariate VAR model involving output in first-difference and the consumption-output ratio on quarterly US data over the period 1948:1 - 2019:4 (all variables are expressed in log). Following Blanchard and Quah [13], we identify two kinds of shocks in the data: permanent shocks and transitory shocks. The permanent shock is the only one having a permanent effect on the level of output while leaving the long-run consumption-output ratio unaffected. The transitory shock is the only one leaving both output and consumption-output ratio unaffected in the long-run. In the model, a permanent shock is interpreted as a permanent technological shock on the TFP level, while a transitory shock is interpreted as a sunspot shock resulting from an exogenous (and self-fulfilling) shift in agents' expectations. Since we allow changes in expectations to be correlated with technological shocks, we define the sunspot shock as the component in agents' expectations which is uncorrelated with the fundamental TFP shock.

Following a one-standard-deviation transitory shock, we find that the response of output is hump-shaped: output jumps when the shock occurs, reaches a peak after three quarters, and then slowly returns to its initial level. Consumption reacts very little when the shock occurs and then gradually increases over time before returning to its initial level. As predicted by standard permanent income theory, when a transitory shock in income is observed, the response of consumption is much smoother than the response of output.

We then use a Simulated Method of Moments – Minimum Distance (SMM–MD) approach to estimate our model and assess its ability to account for the data. We find that the model comes extremely close to replicating the empirical IRF for both consumption and output in response to both permanent and transitory disturbances. In particular, the model perfectly replicates the hump-shaped response of output to a transitory shock, even though the assumption of rational expectations requires the sunspot shock to be white noise. This implies, among other things, that the model generates significant endogenous persistence, in sharp contrast with standard RBC models. As a result, the model also replicates the positive autocorrelation of output growth over short horizons found in the data, another feature that standard RBC models cannot match (Cogley and Nason [17]). Finally, we show that the model does a very good job of accounting for the standard "stylized facts" of the business cycle emphasized in the RBC literature, even though these statistics are not targeted in our SMM-MD approach. We conclude that standard two-sector models in which self-fulfilling changes in agents' expectations are a key driving force are credible candidates for the explanation of the business cycle.

The rest of this paper is organized as follows. In Section 2, we present a literature review. In Section 3, we analyze the aggregate model. We define the general technology and the general utility function considered throughout the paper. We present a new and innovative way of decomposing all the elasticities that characterize preferences, focusing in particular on income effect. We then study the existence and uniqueness of the steady state and we prove that sunspot fluctuations are not a realistic outcome of standard aggregate models for all income effects. In Section 4, we consider the two-sector model under the same general specification of preferences and technologies as in Section 3. As a result the existence and uniqueness of the steady state is derived under the same basic conditions as in the aggregate case. We show that the existence of sunspot fluctuations is a generic property of two-sector models and fully compatible with empirically relevant values for all the structural parameters. Section 5 details the data confrontation for our two-sector model. Section 6 concludes. All the technical proofs are contained in an online Appendix.

2 Literature review

The endogenous fluctuations and sunspot literature was initiated by the seminal contributions of Azariadis [3], Cass and Shell [15], Grandmont [28], and Woodford [65]. Yet Benhabib and Farmer (BF) [7] is the first paper to analyze these issues in the standard infinite-horizon one-sector model with endogenous labor supply, the workhorse model of the RBC literature. They show that in this model, indeterminacy occurs under the assumption of a large amount of externalities leading to an upward-sloping labor demand function. While this model subjected to sunspot shocks has been shown to account for the main "stylized facts" of the business cycles at least as well as standard RBC models (see Farmer and Guo [24]), the assumption of large aggregate IRS in production was found to be inconsistent with the data.[1] Since this seminal contribution, a major challenge in the literature has been to find extensions of this benchmark model capable of generating expectation-driven business-cycles under empirically realistic values for all structural parameters. Two strands of the literature address this challenge. The first seeks to determine how the local indeterminacy properties of the benchmark one-sector model evolve when some assumptions on preferences or the production function are relaxed. The second adds new ingredients to the benchmark model and reconsiders the issue of indeterminacy in the extended models. We briefly review some critical papers in these two strands of the literature.

The BF one-sector model is based on an additively separable utility function and a Cobb-Douglas technology. Many contributions have tried to generalize this formulation to assess whether the existence of sunspot fluctuations could be compatible with a downward-sloping labor demand function. Pintus [55] introduced a general technology into the Benhabib-Farmer framework, while Bennett and Farmer [12] and Hintermaier [39] considered non-separable preferences as defined by King *et al.* (KPR) [46], still assuming a Cobb-Douglas technology. Pintus [56] generalized their formulation to a general production function. Lloyd-Braga *et al.* [51] considered general homogenous preferences and a general technology. The overall message from this literature is that preference and

[1] See e.g. Basu and Fernald [4] for empirical estimates of aggregate IRS in the US economy and Aiyagari [1] for a critique of macroeconomic sunspot models relying on a upward sloping aggregate labor demand curve.

technology parameters, like the elasticity of intertemporal substitution in consumption, the degree of income effect on labor supply, the degree of IRS in production, and the elasticity of substitution between capital and labor, all interact to influence the local stability properties of the model. Yet in all these models, the existence of expectation-driven business-cycles requires at least one structural parameter value which appears to be outside the range of available empirical estimates. Moreover, in most of these papers there is no attempt to confront the model subjected to sunspot shocks with the data.

In response to this critique of the sunspot literature, some authors have modified the production structure of the model. Wen [64] proposed a simple extension consisting in introducing a variable capital utilization rate into the Benhabib-Farmer setup, in the spirit of Greenwood *et al.* [30], which has been proved to be sufficient to allow for the existence of sunspot fluctuations under empirically plausible values for the structural parameters. Benhabib and Wen [11] also showed that when this model is subjected to correlated fundamental and sunspot shocks, it can explain many dimensions of observed business-cycles. However, although this extended model is reported to successfully explain the Great Recession of the 30s (see Harrison and Weder [37]), the impulse response of output to a sunspot shock is not *hump-shaped*, unlike the empirical response to a typical "demand shock" revealed by Blanchard and Quah [13]. Clearly, for an explanation of actual business-cycles based on sunspot shocks/self-fulfilling prophecies to be fully convincing, these models should be able to replicate *all* the main stylized facts associated with a canonical demand shock identified in the empirical literature. Considering a more general utility function and a general production function, Dufourt *et al.* [22] showed that a variable capital utilization model could generate a hump-shaped response of output to an i.i.d sunspot shock, but that the hump is far too persistent for the model to be considered satisfactory from an empirical perspective.

Following Benhabib and Farmer [7], monopolistic competition and endogenous markups have also been considered to generate expectations-driven fluctuations in aggregate models. Increasing returns are based on imperfect competition and not on productive externalities. Gali [26] considers a monopolistic competition model in which the aggregate markup rate depends on the composition of aggregate demand. Sunspot equilibria are then possible for large markup rates.[2] Jaimovich [40] analyzes firms' entry decision in an imperfect competition model in which the number of firms in each sector is small and influences the price elasticity of sectoral demand. A free entry condition is then used to

[2]Lloyd-Braga *et al.* [50] show that models with markup variability are basically equivalent to models with positive output externalities or to models with constant markup and increasing returns.

close the model. He shows that variable markups associated with firm entry can generate sunspot fluctuations when the aggregate labor demand curve is upward-sloping. Combined with "material usage" and a variable capital utilization rate, the model can exhibit sunspot fluctuations at low markup rates and can account for several features of observed business-cycles. However, like Benhabib and Wen [11], the paper does not consider the extent to which the model with i.i.d. sunspot shocks is able to match the autocorrelation function of output growth and the empirical IRF to demand shocks - the two areas in which sunspot equilibrium models fail, according to Schmitt-Grohé [61].[3] Dos Santos Ferreira and Dufourt [18] pursue an alternative route and propose a model where free-entry conditions lead to an indeterminate number of active firms at equilibrium, influenced by firms' mutually consistent conjectures on competitors' behavior. Sunspot shocks in this case are not required to be serially uncorrelated, and persistent output fluctuations triggered by correlated sunspot shocks can then be obtained. Wang and Wen [63] exploit this idea to show that a variable markup model can generate hump-shaped output dynamics in response to sunspot shocks. However, these two contributions do not rely on the local indeterminacy of the steady state to generate sunspot shocks. As such, they do not belong to the class of papers covered by Schmitt-Grohé [61]'s critique, while this paper focuses on models that feature this critical property.

Two-sector models have also been considered, again following the seminal contribution of Benhabib and Farmer [8]. In this paper, Benhabib and Farmer extend their initial formulation to a two-sector economy producing differentiated consumption and investment goods but with the same Cobb-Douglas technology characterized by sector-specific output externalities leading to increasing returns. Building on the fact that capital and labor can be freely allocated between sectors, they prove that the existence of local indeterminacy becomes compatible with a downward-sloping labor demand function. Unlike their one-sector contribution, it clearly appears that when external effects in each sector depend on that sector's aggregate output, factor reallocations across sectors can have strong effects on marginal products and indeterminacy can occur with much smaller externalities. Harrison [35] builds on these results to show that indeterminacy occurs for a minimum value of the externality in the investment sector, even with no externality in the consumption sector. All these conclusions have been recently confirmed by Dufourt *et al.* [20] considering GHH preferences instead of additively separable ones.[4] The model prop-

[3] In this paper, only the autocorrelation function for the (HP-filtered) level of output is considered, which is quite different from the autocorrelation function of output growth emphasized by Cogley and Nason [17].

[4] See also Guo and Harrison [33].

erly calibrated solves several empirical puzzles traditionally associated with two-sector RBC models, but is still not able to replicate the hump-shaped output dynamics. Guo and Harrison [32] also introduce a variable capital utilization rate into the Benhabib and Farmer model and confirm that lower externalities are required, without providing any data confrontation analysis.

Worthy of mention too is all the literature departing from the contribution of Benhabib and Nishimura [9] and Benhabib *et al.* [10], in which sector-specific externalities are introduced in two-sector models with differentiated private technologies and constant social returns. In such a framework, the existence of local indeterminacy relies on a capital-intensity reversal between the private and social levels and, as shown in Nishimura and Venditti [53], requires extreme values for the elasticity of intertemporal substitution in consumption (EIS) which are not in line with empirical estimates.

All in all, although sunspot fluctuations have been shown to arise much more easily in extended model formulations, there is still no general analysis in the literature. Actually, as this review shows, no general paper has so far managed to contradict the statement by Schmitt-Grohé [61] that models of endogenous fluctuations based on sunspots are not able to properly replicate the main salient features of the dynamics of macroeconomic variables under empirically realistic calibrations of the structural parameters.

3 A general aggregate model

We consider a closed economy framework in the spirit of Benhabib and Farmer [7] (BF). The economy is composed of a large number of identical infinitely-lived agents and a large number of identical producers. Agents consume, supply labor and accumulate capital. Firms produce the unique final good which can be used either for consumption or investment. All markets are perfectly competitive, but there are externalities in production.

3.1 The representative firm: a general technological structure

The production sector is composed of a large number of identical firms which operate under perfect competition. Output Y_t is produced by combining labor L_t and capital K_t. The technology of each firm exhibits constant returns to scale with respect to its own inputs and we consider that each of the many firms benefits from positive externalities due to the contribution of the average levels of labor $\bar{L}$ and capital $\bar{K}$. These external effects are exogenous and not traded in markets. The production function is

$$Y_t = f(K_t, L_t)e(\bar{K}_t, \bar{L}_t) \tag{1}$$

with $e(\bar{K}_t, \bar{L}_t)$ the externality variable. We follow BF by assuming that externalities affect the technology in a multiplicative way but we depart from them by not requiring the production function to be Cobb-Douglas. Rather, our production function is general and satisfies:

Assumption 1. *$f(K, L)$ is $\mathbf{C}^2$ over $\mathbb{R}^2_{++}$, increasing in (K, L), concave over $\mathbb{R}^2_{++}$ and homogeneous of degree one. $e(\bar{K}, \bar{L})$ is $\mathbf{C}^1$ over $\mathbb{R}_{++}$ and increasing in $(\bar{K}, \bar{L})$. Moreover, for any given $L > 0$,*

$$\lim_{K\to 0} f_1(K, L)e(K, L) = +\infty \text{ and } \lim_{K\to +\infty} f_1(K, L)e(K, L) = 0$$

and, for any given $K > 0$,

$$\lim_{L\to 0} f_2(K, L)e(K, L) = +\infty \text{ and } \lim_{L\to +\infty} f_2(K, L)e(K, L) = 0$$

Firms rent capital units at the real rental rate r_t and hire labor at the unit real wage w_t. The profit maximization program of the representative firm,

$$\max_{\{Y_t, L_t, K_t\}} Y_t - w_t L_t - r_t K_t,$$

leads to the standard demand function for capital K_t and labor L_t:

$$r_t = f_1(K_t, L_t)e(\bar{K}_t, \bar{L}_t) \tag{2}$$

$$w_t = f_2(K_t, L_t)e(\bar{K}_t, \bar{L}_t) \tag{3}$$

As will become clear, the production function and the optimal decisions of firms influence the local dynamics of the model through four crucial elasticities: the elasticity of output with respect to capital stock $s(K, L)$ (which, at equilibrium, is also the share of capital in total income), the elasticity of capital-labor substitution $\sigma(K, L)$, and the elasticities of the externality variable with respect to labor, $\varepsilon_{eL}(\bar{K}, \bar{L})$, and capital, $\varepsilon_{eK}(\bar{K}, \bar{L})$:

$$s(K, L) = \frac{Kf_1(K,L)}{f(K,L)} \in (0, 1), \quad \sigma(K, L) = -\frac{(1-s(K,L))f_1(K,L)}{Kf_{11}(K,L)} > 0 \tag{4}$$

$$\varepsilon_{eK}(\bar{K}, \bar{L}) = \frac{e_1(\bar{K},\bar{L})\bar{K}}{e(\bar{K},\bar{L})}, \quad \varepsilon_{eL}(\bar{K}, \bar{L}) = \frac{e_2(\bar{K},\bar{L})\bar{L}}{e(\bar{K},\bar{L})} \tag{5}$$

Obviously, the choice of a Cobb-Douglas production function, as in BF, implies $\sigma(K, L) = 1$, whereas the use of a general production function entails $\sigma(K, L) \in (0, +\infty)$. To simplify notation, we will for now denote by s, σ, ε_{eK} and ε_{eL} the corresponding elasticities evaluated at the steady state. In order to allow for a direct comparison with BF, the externalities are also expressed as follows:

$$\varepsilon_{eK} = s\Theta_k \quad \varepsilon_{eL} = (1 - s)\Theta_l \tag{6}$$

where $\Theta_k, \Theta_l \geq 0$ are the degrees of increasing returns to scale in capital and labor. BF assume output externalities, implying $\Theta_k = \Theta_l = \Theta$. We allow for a more general formulation in which external effects can be factor-specific and independent of each-other.

Finally, we make a standard assumption requiring that the aggregate (i.e., taking external effects into account) labor demand and capital demand functions are decreasing in the real wage and in the rental rate of capital, respectively:

Assumption 2. $\Theta_k < (1-s)/s\sigma \equiv \bar{\Theta}_k$ *and* $\Theta_l < s/(1-s)\sigma \equiv \bar{\Theta}_l$

It is well known from BF analysis that when the slope of the aggregate labor demand curve is positive and greater than the slope of the aggregate labor supply curve, indeterminacy and sunspot fluctuations can occur in the one-sector infinite-horizon model. We rule out this possibility here because it entails extremely high degrees of increasing returns to scale that are at odds with the data. We will be more specific about realistic degrees of IRS later in the paper.

3.2 The representative household: a general utility function

The economy is composed of a continuum of mass 1 of identical households. In each period, the representative household is endowed with ℓ units of time. Given the real wage w_t and the rental rate of capital r_t, the household decides how much of its available time to allocate to leisure time $\mathcal{L}_t$ and hours worked l_t, and how much to consume c_t. It also rents its capital stock k_t to the representative firms, and accumulates capital according to the following capital accumulation constraint:

$$k_{t+1} = (1 - \delta + r_t)k_t + w_t l_t + d_t - c_t \tag{7}$$

where $\delta \in (0, 1)$ is the capital depreciation rate, and d_t are potential dividends redistributed ex-post by firms.

In each period, the utility that the household derives from consumption and leisure is described by a general instantaneous utility function $u(c, \mathcal{L})$. As in the case of the productive side of the economy, we want our analysis to be as general as possible. We thus make the following minimum standard assumptions on the utility function:

Assumption 3. *$u(c, \mathcal{L})$ is $\mathbf{C}^2$ over $\mathbb{R}^2_{++}$ increasing in each argument, strictly quasi-concave in $(c, \mathcal{L})$, and satisfies the Inada conditions*

$$\lim_{c\to 0} u_1(c, \mathcal{L}) = +\infty, \quad \lim_{c\to +\infty} u_1(c, \mathcal{L}) = 0$$
$$\lim_{\mathcal{L}\to 0} u_2(c, \mathcal{L}) = +\infty \text{ and } \lim_{\mathcal{L}\to +\infty} u_2(c, \mathcal{L}) = 0.$$

The Inada conditions are introduced to ensure an interior optimum. Furthermore, to avoid basing our analysis of the local stability conditions and of the occurrence of sunspot fluctuations on exotic features regarding individual preferences, we introduce the following standard assumption on consumption and leisure:

Assumption 4. *Consumption c and leisure $\mathcal{L}$ are normal goods.*

Assuming that the intertemporal utility function is additively separable over time, the representative consumer solves the following lifetime utility maximization program (where $\beta \in (0,1)$ is the subjective discount factor):

$$\begin{array}{ll} \max\limits_{\{c_t,\, l_t,\, k_{t+1}\}_{t=0...\infty}} & \sum\limits_{t=0}^{+\infty} \beta^t u(c_t, \ell - l_t) \\ s.t. & k_{t+1} = (1-\delta + r_t)k_t + w_t l_t + d_t - c_t, \quad t = 0...\infty \\ & k_0 \text{ given} \end{array} \tag{8}$$

Denoting by λ_t the Lagrange multiplier on constraint (7) and $R_t = 1 - \delta + r_t$ the net return factor on capital, the first-order conditions can be written as

$$u_1(c_t, \ell - l_t) = \lambda_t, \tag{9}$$

$$\frac{u_2(c_t, \ell - l_t)}{u_1(c_t, \ell - l_t)} = w_t \tag{10}$$

$$\lambda_t = \beta R_{t+1} \lambda_{t+1} \tag{11}$$

Equation (10) describes the optimal consumption-leisure trade-off, while equations (9) and (11) jointly describe the optimal arbitrage between consumption and saving (i.e., the Euler equation). An optimal path must also satisfy the transversality condition:

$$\lim_{t \to +\infty} \beta^t \lambda_t k_{t+1} = 0 \tag{12}$$

Following Rotemberg and Woodford [59], we can rewrite the optimality conditions (9-10) in terms of time-invariant Frisch consumption-demand and labor-supply curves involving the real wage w_t and the marginal utility of wealth λ_t :

$$c_t = c(w_t, \lambda_t), \quad l_t = l(w_t, \lambda_t) \tag{13}$$

As was the case for the productive size of the economy, we show later that the local dynamics of the model around the steady state is determined by a limited number of critical elasticities. Denote by

$$\epsilon_{cw} = \frac{c_1(w,\lambda)w}{c}, \ \epsilon_{c\lambda} = \frac{c_2(w,\lambda)\lambda}{c}, \ \epsilon_{lw} = \frac{l_1(w,\lambda)w}{l}, \ \epsilon_{l\lambda} = \frac{l_2(w,\lambda)\lambda}{l}, \tag{14}$$

the Frisch elasticities of the demand and supply functions (13), and by

$$\epsilon_{cc} = -\frac{u_1(c,\mathcal{L})}{u_{11}(c,\mathcal{L})c}, \tag{15}$$

the elasticity of intertemporal substitution in consumption. We can easily prove the following Lemma:

Lemma 1. *The three critical elasticities ϵ_{cc}, ϵ_{lw} and $\epsilon_{l\lambda}$ are related to the individual utility function by*

$$\epsilon_{cc} = -\frac{u_1}{u_{11}c}, \quad \epsilon_{lw} = \frac{1}{l}\left(\frac{-u_{11}u_2}{u_{11}u_{22}-u_{12}u_{21}}\right), \quad \epsilon_{l\lambda} = \frac{1}{l}\left(\frac{u_{21}u_1-u_{11}u_2}{u_{11}u_{22}-u_{12}u_{21}}\right) \tag{16}$$

Moreover, the remaining two elasticities ϵ_{cw} and $\epsilon_{c\lambda}$ are related to ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$ through the following equations

$$\epsilon_{cw} = \mathcal{C}\left(\epsilon_{lw} - \epsilon_{l\lambda}\right), \quad \epsilon_{c\lambda} = -\epsilon_{cc} + \mathcal{C}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\epsilon_{l\lambda} \tag{17}$$

where $\mathcal{C} = \theta(1-s)/(\theta - s\beta\delta) < 1$ is the steady state ratio of wage income over consumption (wl/c), which is independent of the specification of the individual utility function, and $\theta = 1 - \beta(1-\delta)$.

Proof. See Appendix 7.1.

An implication of this Lemma is that, as far as the representative consumer's decision is concerned, the dynamic properties of the model are completely determined by the three elasticities ϵ_{lw}, $\epsilon_{l\lambda}$, and ϵ_{cc}, in addition to the parameter $\mathcal{C}$, which is independent of the specification of individual preferences.

Using this Lemma, we can immediately derive the following Proposition:

Proposition 1. *Under Assumption 3, $\epsilon_{cc} > 0$ and $\epsilon_{lw} > 0$. Moreover, under Assumption 4, $\epsilon_{l\lambda} \geq 0$, $\epsilon_{c\lambda} \leq 0$, and thus*

$$\epsilon_{cc} \geq \frac{\mathcal{C}\epsilon_{l\lambda}(\epsilon_{lw}-\epsilon_{l\lambda})}{\epsilon_{lw}} \equiv \epsilon_{cc}^N. \tag{18}$$

Proof. See Appendix 7.2.

The importance of this Proposition is that it shows how assumptions on preferences (namely Assumptions 3 and 4) naturally translate into *restrictions* on the critical elasticities ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$. Note that these restrictions ($\epsilon_{cc} > 0$, $\epsilon_{lw} > 0$, $\epsilon_{l\lambda} \geq 0$, and $\epsilon_{cc} \geq \epsilon_{cc}^N$) are actually much simpler than working directly with the standard concavity and normality assumptions based on the utility function.

Another nice feature of considering these elasticities, instead of considering the first-order and second-order derivatives of $u(\cdot)$, is that the former have a clear economic interpretation. The EIS in consumption ϵ_{cc} and the Frisch elasticity of labor supply

ϵ_{lw} have a well-known interpretation that requires no further discussion. On the other hand, since λ_t is the marginal utility of wealth, the elasticity $\epsilon_{l\lambda}$ captures the extent to which a change in the household's expected wealth over its entire lifetime affects the current labor supply decision. Indeed, when, for any reason, lifetime income decreases, the intertemporal budget constraint obtained from aggregating (7) over time becomes more restrictive, and λ_t increases as the household's consumption choices are more constrained. As implied by the normality assumption, it follows that consumption and leisure decrease while hours worked increase. The elasticity $\epsilon_{l\lambda}$ captures the extent to which such a change in lifetime income affects the current labor supply decision.

In short, $\epsilon_{l\lambda}$ is a properly defined measure of the *wealth effect on labor supply.* This elasticity is particularly important because the recent literature has shown that the intensity of this wealth effect plays a significant role in the local stability properties of many dynamic macroeconomic models (see Dufourt *et al.* [20, 21], Jaimovich [41]). However, this literature only addressed particular utility functions in which the wealth effect is known to be either "positive" (but not precisely defined) or zero. Moreover, these specific utility functions also require the introduction of other cross-restrictions on the three critical elasticities defined above. With our analysis, however, the intensity of the wealth effect on labor supply can be chosen independently of the values given to the other two elasticities (ϵ_{lw} and ϵ_{cc}).

To better illustrate these points, the following Proposition clarifies the restrictions implied by some of the most widely used classes of utility functions:

Proposition 2. *Under KPR preferences,*

$$u(c,\mathcal{L}) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}v(\mathcal{L}), & \textit{with } \gamma > 0,\ \gamma \neq 1 \\ \log(c) + \log(v(\mathcal{L})), & \textit{with } \gamma = 1 \end{cases}$$

with $\mathcal{L} = \ell - l$ *and* $v(\mathcal{L})$ *increasing and concave (if* $\gamma \leq 1$*) or decreasing and convex (if* $\gamma > 1$*), the critical elasticities satisfy:*

$$\epsilon_{cc} = \frac{1}{\gamma},\ \epsilon_{lw} = \frac{\mathcal{L}}{l}\frac{1}{(1-\epsilon_{cc})v'(\mathcal{L})\mathcal{L}/v(\mathcal{L}) - v''(\mathcal{L})\mathcal{L}/v'(\mathcal{L})} > 0,\ \epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}.$$

Under generalized GHH preferences,

$$u(c,l) = \frac{1}{1-\gamma}\left(c - \frac{l^{1+\chi}}{1+\chi}\right)^{1-\gamma},$$

with $\gamma > 0$ *and* $\chi \geq 0$*, the critical elasticities satisfy:*

$$\epsilon_{cc} = \frac{1}{\gamma}\left(1 - \frac{\mathcal{C}}{1+\chi}\right),\ \epsilon_{lw} = \frac{1}{\chi},\ \epsilon_{l\lambda} = 0.$$

Under Generalized Hansen [34] preferences,

$$u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\chi}}{1+\chi},$$

with $\gamma > 0$ and $\chi \geq 0$, the critical elasticities satisfy:

$$\epsilon_{cc} = \frac{1}{\gamma},\ \epsilon_{lw} = \epsilon_{l\lambda} = \frac{1}{\chi}. \tag{19}$$

According to Proposition 3, in the case of KPR preferences, only two of the three critical elasticities ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$ are independent since they are related through the equation $\epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}$. In the case of generalized GHH preferences, the restriction $\epsilon_{l\lambda} = 0$ is well-known, having been introduced on purpose to eliminate the wealth effect on labor supply. However, there is often far less awareness that, with this class of preferences, changing the calibration of the preference parameter χ to change the value of the wage elasticity of the labor supply, $\epsilon_{lw} = 1/\chi$, meanwhile generates a change in the EIS in consumption, ϵ_{cc}. In fact, Proposition 2 implies that changing the calibration of χ to $\chi' \neq \chi$ also requires adjusting the calibration of γ to $\gamma' = \left(1 - \frac{\mathcal{C}}{1+\chi'}\right)\epsilon_{cc}$ if one wants to keep the initial value of the EIS unchanged. Finally, under generalized Hansen preferences, a strong restriction relating the two Frisch elasticities of the labor supply curve is introduced, since we have in this case: $\epsilon_{lw} = \epsilon_{l\lambda}$.

The particular case of the Jaimovich-Rebelo formulation

Jaimovich and Rebelo (JR) [42] were the first to discuss the importance of the income effect on the occurrence of indeterminacy and sunspot fluctuations. Their discussion is based on the following specification of the instantaneous utility function:

$$u(c_t, l_t, X_t) = \frac{\left[c_t - \frac{l_t^{1+\chi}}{1+\chi}X_t\right]^{1-\gamma} - 1}{1-\gamma} \tag{20}$$

with $X_t = c_t^{\phi}X_{t-1}^{1-\phi}$ and $\phi \in [0,1]$. As they state, this specification nests as polar cases the GHH utility function (when $\phi = 0$) and the KPR utility function (when $\phi = 1$) formulations. The magnitude of the income effect is therefore controlled by varying the value of γ between these two extremes.

Assuming $\gamma = 1$ and $\Theta_k = \Theta_l = \Theta$, Jaimovich [41] shows that, for some values of Θ compatible with a negatively-slopped labor demand function, there exist two bounds $0 < \underline{\phi} < \overline{\phi} < 1$ such that when $\phi \in (\underline{\phi}, \overline{\phi})$ local indeterminacy and sunspot fluctuations occur under realistic values for all the structural parameters. The conclusion is that the income effect has a non-linear effect on the range of values consistent with indeterminacy, which appears to arise under intermediate amounts of income effect, and to be ruled out with either low or high amounts.

A difference between the JR specification and our specification is that, except for the two polar cases $\phi = 0$ and $\phi = 1$, the JR utility function assumes that an additional state variable X_t enters the utility function:

$$X_t = \prod_{s=0}^{t-1} c_{t-s}^{\phi(1-\phi)^s} X_0^{(1-\phi)^s}$$

It follows that the utility function at time t depends on the whole history of past consumption decisions. The result is that the consumption demand and labor supply decisions no longer write $c(w_t, \lambda_t)$ and $l(w_t, \lambda_t)$ but $c(w_t, \lambda_t, X_t)$ and $l(w_t, \lambda_t, X_t)$. In short, compared to our specification, such a formulation introduces two additional elasticities ϵ_{cX} and ϵ_{lX} associated with consumption habits which generate a dynamic link between current and future consumption and labor supply decisions – what Jaimovich refers to as a form of "dynamic" income effects.[5]

To avoid the complexities of introducing additional state variables, Nourry *et al.* [54] and Dufourt *et al.* [21] consider a modified JR utility function which only involves current-period variables, namely

$$u(c_t, l_t) = \frac{\left[c_t - \frac{l_t^{1+\chi}}{1+\chi} c_t^{\phi}\right]^{1-\gamma} - 1}{1-\gamma} \tag{21}$$

We recover the two polar cases of a GHH and a KPR utility function associated with $\phi = 0$ and $\phi = 1$, respectively. Moreover, it is now possible to vary the values of the three critical elasticities ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$ by considering alternative calibrations for the three parameters γ, χ, and ϕ. However, the critical elasticities are very cumbersome combinations of these parameters, and when one parameter is adjusted so as to change the value of one elasticity, the other parameters also have to be adjusted to maintain the value of the other elasticities constant. It is therefore much better to work with a general utility function and calibrate the three critical elasticities directly, as we do in this paper.

3.3 Intertemporal equilibrium and steady state

At a symmetric general equilibrium of the economy, prices $\{w_t, r_t, \lambda_t\}$ adjust so that all markets clear at any date t, with the externality variables satisfying $(\bar{k}_t, \bar{l}_t) = (k_t, l_t)$ for any t. Imposing the latter equalities in the set of physical constraints and optimality conditions (1)-(3), (11), and (13), we obtain that a symmetric general equilibrium satisfies in any t,

[5] As we will show later on, absent these additional ingredients, sunspot fluctuations cannot occur under empirically relevant calibrations for any size of income effect.

$$\begin{aligned}
\lambda_t &= \beta(1-\delta+r_{t+1})\lambda_{t+1} \\
k_{t+1} &= (1-\delta)k_t + y_t - c_t \\
r_t &= f_1(k_t, l_t)e(k_t, l_t) \\
w_t &= f_2(k_t, l_t)e(k_t, l_t) \\
c_t &= c(w_t, \lambda_t) \\
l_t &= l(w_t, \lambda_t) \\
y_t &= f(k_t, l_t)e(k_t, l_t)
\end{aligned} \tag{22}$$

together with the initial condition k_0 given and the transversality condition (12).

From these dynamic equations, we immediately derive that if a steady state exists, the rental rate of capital at the steady state is

$$r^* = \frac{1-\beta(1-\delta)}{\beta} \equiv \frac{\theta}{\beta}$$

In order to study the existence and uniqueness of a steady state, we analyze the existence of a 6-uple $(k^*, y^*, l^*, c^*, w^*, \lambda^*)$ solution to the set of equations

$$f_1(k^*, l^*)e(k^*, l^*) = \frac{\theta}{\beta} \tag{23}$$

$$f_2(k^*, l^*)e(k^*, l^*) = \frac{u_2(c^*, \ell - l^*)}{u_1(c^*, \ell - l^*)} \tag{24}$$

$$c^* = f(k^*, l^*)e(k^*, l^*) - \delta k^* \tag{25}$$

$$w^* = f_2(k^*, l^*)e(k^*, l^*) \tag{26}$$

$$y^* = f(k^*, l^*)e(k^*, l^*) \tag{27}$$

$$c^* = c(w^*, \lambda^*) \tag{28}$$

Note that, for analytical convenience, instead of considering the Frisch labor supply equation $l^* = l(w^*, \lambda^*)$, we reintroduce the initial optimality condition involving the marginal rate of substitution between consumption and labor.

We first prove the following Lemma:

Lemma 2. *At the steady state, the ratios y^*/k^*, c^*/k^* and w^*l^*/c^* satisfy*

$$\frac{y^*}{k^*} = \frac{\theta}{s\beta}, \quad \frac{c^*}{k^*} = \frac{\theta - s\beta\delta}{s\beta}, \quad \textit{and} \quad \frac{w^*l^*}{c^*} = \frac{\theta(1-s)}{\theta - s\beta\delta} \equiv \mathcal{C}$$

Proof. See Appendix 7.3.

Using this Lemma, we derive the following Proposition:

Proposition 3. *Under Assumptions 1-4, a unique steady state generically exists. Moreover, for any given calibration of structural parameters, there always exists a value $\ell^* > 0$*

such that when $\ell = \ell^*$, the steady state is constant across calibrations with $l^* = \bar{l}^* < \ell^*$.

Proof. See Appendix 7.4.

An implication of Proposition 3 is that, when analyzing how alternative calibrations for the structural parameters affect the stability properties of the model, it is possible to maintain the steady state $(k^*, y^*, l^*, c^*, w^*, \lambda^*)$ unchanged by adjusting the value for ℓ^* accordingly. In other words, we can follow the usual practice of "calibrating" the level of hours worked at the steady state without difficulty.

3.4 Local stability analysis

We now carry out a thorough analysis of the local stability properties of the steady state when the dynamics is defined by (22). In order to do so, we log-linearize the set of equations in (22) around the unique steady state. Using Lemmata 1 and 2, we obtain (where hatted variables denote percentage deviations from the steady state):

$$\begin{aligned}
\widehat{\lambda}_t &= \widehat{\lambda}_{t+1} + \theta \widehat{r}_{t+1} \\
\widehat{k}_{t+1} &= (1-\delta)\widehat{k}_t + \tfrac{\theta}{s\beta}\widehat{y}_t - \left(\tfrac{\theta - s\beta\delta}{s\beta}\right)\widehat{c}_t \\
\widehat{r}_t &= \left(-\tfrac{1-s}{\sigma} + s\Theta_k\right)\widehat{k}_t + \left(\tfrac{1-s}{\sigma} + (1-s)\Theta_l\right)\widehat{l}_t \\
\widehat{w}_t &= \left(\tfrac{s}{\sigma} + s\Theta_k\right)\widehat{k}_t + \left(-\tfrac{s}{\sigma} + (1-s)\Theta_l\right)\widehat{l}_t \\
\widehat{c}_t &= \mathcal{C}\left(\epsilon_{lw} - \epsilon_{l\lambda}\right)\widehat{w}_t + \left(-\epsilon_{cc} + \mathcal{C}\left(1 - \tfrac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\epsilon_{l\lambda}\right)\widehat{\lambda}_t \\
\widehat{l}_t &= \epsilon_{lw}\widehat{w}_t + \epsilon_{l\lambda}\widehat{\lambda}_t \\
\widehat{y}_t &= s(1+\Theta_k)\widehat{k}_t + (1-s)\left(1+\Theta_l\right)\widehat{l}_t
\end{aligned}$$

This is a system of seven equations in seven variables, only two of these equations being dynamic. To analyze the local stability properties of the model, we first reduce the system by using the five static equations to eliminate five variables, $\widehat{y}_t$, $\widehat{c}_t$, $\widehat{l}_t$, $\widehat{w}_t$, and $\widehat{r}_t$, from the dynamic equations. The obtained system of *minimal dimension* – two dynamic equations in two variables, $\widehat{k}_t$ and $\widehat{\lambda}_t$ – can be expressed as:

$$\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix} \equiv J \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix}$$

where J is the Jacobian matrix of the underlying non-linear 2-dimensional system evaluated at the steady state, which is given in the following Proposition:

Proposition 4. *The elements of the Jacobian matrix J are:*

$$J_{11} = \frac{1}{\beta}\left\{1+\theta\Theta_k+\frac{\theta(1-s)\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\},\ J_{12} = \frac{1}{s\beta}\left\{\frac{\theta(1-s)\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}+(\theta-s\beta\delta)\epsilon_{cc}\right\}$$

$$J_{21} = \theta\frac{\frac{1-s}{\sigma}-s\Theta_k-\frac{\epsilon_{lw}}{\sigma}[\Theta_l(1-s)+s\Theta_k]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}J_{11},\ J_{22} = \frac{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\theta\left\{\frac{1-s}{\sigma}-s\Theta_k-\frac{\epsilon_{lw}}{\sigma}[\Theta_l(1-s)+s\Theta_k]\right\}J_{12}}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}$$

Proof. See Appendix 7.5.

The local dynamics of the model is thus determined by the nine structural parameters constituting the matrix J: four of them concern the productive side of the economy, namely s, σ, Θ_k, and Θ_l, four of them concern individual preferences, namely β, ϵ_{cc}, ϵ_{lw}, $\epsilon_{l\lambda}$, and finally there is the depreciation rate of capital δ.

Using the geometrical methodology of Grandmont *et al.* [29] as presented in the online Appendix 7.6, we prove that there exist two bifurcation loci in the parameter space such that, when ϵ_{cc} is increased from 0 to $+\infty$, a change in the stability properties of the steady state occurs when ϵ_{cc} crosses any of the two loci. These results are formally summarized in the following Lemma:

Lemma 3. *Under Assumptions 1-4, let $\Omega = (\beta, \delta, s, \sigma, \Theta_k, \Theta_l)$ be the set of structural parameters. For any $\omega \in \Omega$ such that $\sigma \leq \bar{\sigma} \equiv \theta/(1-\beta)$, there exist two* bifurcation curves *crossing the 3-dimensional plane $(\epsilon_{lw}, \epsilon_{l\lambda}, \epsilon_{cc})$ and generating a change in the local stability properties of the steady state:*

– *a* flip bifurcation curve $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ *associated with one real eigenvalue of J crossing -1,*

– *a (degenerate)* transcritical bifurcation curve $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ *associated with one real eigenvalue of J crossing 1.*

These bifurcation curves appear for any $\epsilon_{lw} > \underline{\epsilon}_{lw}$, with

$$\underline{\epsilon}_{lw} \equiv \frac{1-s-\sigma s\Theta_k}{(1-s)\Theta_l+s\Theta_k}.$$

There also exists one critical bound $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ *such that $\mathcal{D} = 1$ when $\epsilon_{l\lambda} = \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$. This critical bound exists for any (Θ_k, Θ_l) such that $\Theta_k \in [0, \underline{\Theta}_k)$ and $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$, with*

$$\underline{\Theta}_k \equiv \frac{s\beta}{(1-s)\sigma-(1-\beta)}, \qquad \underline{\Theta}_l \equiv \frac{1-\beta+\Theta_k}{\beta}.$$

The formal expressions of the bifurcation curves and the critical bound are given in Appendix 7.6.

Proof. See Appendix 7.6.

The following Theorem now provides a complete picture of the local stability properties of the aggregate model.

Theorem 1. *Under Assumptions 1-4, let $\sigma \leq \bar{\sigma} \equiv \theta/(1-\beta)$ and consider the bifurcation curves, critical bound, and thresholds defined in Lemma 3. Then the following results hold:*

Case 1 - Low wage elasticity of labor supply: $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$.

The steady state is a saddle-point.

Case 2 - High wage elasticity of labor supply: $\epsilon_{lw} > \underline{\epsilon}_{lw}$.

– *Under* low capital externalities $\Theta_k \in [0, \underline{\Theta}_k)$*, the steady state is*

i) for $\Theta_l \in [0, \underline{\Theta}_l)$,

- *a saddle-point if* $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$,
- *a source if* $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.

ii) for $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$,

- *a saddle-point if* $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$,
- *a source if* $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$ *and* $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$,
- *a sink if* $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$ *and* $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

– *Under* large capital externalities $\Theta_k \in (\underline{\Theta}_k, \bar{\Theta}_k)$*, the steady state is*

- *a saddle-point if* $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$,
- *a source if* $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.

Proof. See Appendix 7.7.

Theorem 1 – characterizing the local stability properties of the one-sector model for any specification of individual preferences, any specification for the production function, and any degrees of IRS in capital and labor consistent with downward-sloping labor and capital demand curves – considerably generalizes all existing results in the literature.[6] Two main conclusions can be drawn from this Theorem. First, from a theoretical standpoint, it is possible to identify an area in the parameter space such that the steady state is locally indeterminate and sunspot fluctuations exist, even though externalities are mild enough to ensure downward-sloping capital demand and labor demand curves (see case

[6] The literature shows that sunspot fluctuations require a sufficiently large (possibly larger than one) elasticity of capital-labor substitution and/or a sufficiently large elasticity of intertemporal substitution in consumption. Moreover, local indeterminacy is ruled out under GHH preferences with no-income effect and KPR preferences (see Bennett and Farmer [12], Hintermaier [39], Lloyd-Braga *et al.* [51], and Pintus [55, 56]).

2(ii) of the Theorem). Second, as we show below, this area of indeterminacy nonetheless does not occur for any empirically plausible calibrations for structural parameters. The robust conclusion is then that in the one-sector model, *indeterminacy and the existence of sunspot fluctuations are ruled out for any plausible parameter configuration.*

To illustrate this statement, it is now useful to introduce some empirical restrictions on the nine structural parameters s, σ, Θ_k, Θ_l, β, ϵ_{cc}, ϵ_{lw}, $\epsilon_{l\lambda}$ and δ, that influence the local stability properties of the steady state. We take advantage of the fact that a narrow range of empirical estimates exist for several of these parameters so as to concentrate our analysis on the remaining critical elasticities. In particular, it is widely accepted in the literature that, at a quarterly frequency, the subjective discount factor is close to $\beta = 0.99$, consistent with a long-run annual return on capital of around 4%. Likewise, empirical estimates of the annual depreciation rate of capital are typically around 10%, implying $\delta = 0.025$. In the US, the share of capital income in total income is typically estimated around 30%, implying a capital elasticity in the production function close to $s = 0.3$. Estimates for other critical elasticities are often more variables across empirical studies, but a range including most available empirical estimates can nonetheless be defined. For example, based on the recent empirical literature (see e.g. León-Ledesma *et al.* [49], Klump *et al.* [48], Duffy and Papageorgiou [19] and Karagiannis *et al.* [44]), we consider that a plausible range for the capital-labor elasticity of substitution is $\sigma \in (0, 2)$. Likewise, using the various empirical estimates provided by Campbell [14], Vissing-Jorgensen [62], and Gruber [31], we consider that a plausible range for the EIS in consumption is $\epsilon_{cc} \in (0, 2)$. Finally, estimates of increasing returns to scale by Basu and Fernald [4] for US manufacturing industry provide a value of around 10% with standard deviation 0.33, which enables us to define a range of empirically credible values for the aggregate degree of IRS in the model, $\Theta = (1-s)\Theta_l + s\Theta_k$, of $\Theta \in (0, 0.43)$. Regarding the Frisch wage-elasticity of the labor supply curve, it is well known from the literature that for various reasons this elasticity can be large at the aggregate level even though it is small at the individual level (see e.g. Rogerson and Wallenius [58], and Prescott and Wallenius [57] for a discussion). Our choice here is to not restrict this elasticity *a priori* in order to include Hansen's [34] specification of individual preferences – associated with an infinitely elastic aggregate labor supply curve – into the analysis, since these preferences are widely used in the DSGE literature. This leads us to introduce the following Assumption:

Assumption 5. ***Realistic structural parameters****:* $\beta = 0.99$*,* $\delta = 0.025$*,* $s = 0.3$*,* $\sigma \in (0, 2)$*,* $\epsilon_{cc} \in (0, 2)$ *and* $\Theta \in (0, 0.43)$ *with* $\Theta = (1-s)\Theta_l + s\Theta_k$*.*

We then obtain the following Proposition which follows directly from Theorem 1:

Proposition 5. *Under Assumptions 1-5, the steady state is a saddle-point.*

Proof. See Appendix 7.8.

Proposition 5 illustrates the practical importance of Theorem 1 since it shows that, even though from a theoretical standpoint some parameter configurations exist for which the steady-state is locally indeterminate (a sink) or fully unstable (a source), the steady-state is *always* a saddle point when we restrict parameters to the range of empirically credible values. This is true for *any specification of the production function or of the individual utility function.*

It is also worthwhile to emphasize that even though Proposition 5 is obtained for specific values for (β, δ, s), it actually remains valid in a significantly large neighborhood of these values covering all empirically interesting cases. In particular, corollary 5 holds for any plausible value close to $\beta = 0.99$ and $\delta = 0.025$, and for the whole interval $s \in (0.25, 0.4)$ of empirically available estimates for the capital income share across industrialized countries.[7] On the other hand, we can also completely get rid off Assumption 5 and consider instead other restrictions on structural parameters frequently imposed in the literature to show again that indeterminacy is not a realistic outcome of the one-sector model. The following Proposition summarizes these implications:

Proposition 6. *Under Assumptions 1-4, for any $\sigma > 0$, local indeterminacy is ruled out in the following cases: i) $\Theta_k = \Theta_l$, ii) $\epsilon_{lw} = 0$, iii) $\epsilon_{l\lambda} = 0$, iv) $\Theta_l = 0$.*

Proof. See Appendix 7.9.

Case i) of Proposition 6 corresponds to the standard specification of Benhabib and Farmer [7] with output externalities and was initially proved by Hintermaier [39] under the assumption of a Cobb-Douglas technology. Here, we extend this result to any production function. Cases ii), iii), and iv) correspond, respectively, to the case of an inelastic labor supply, to the case of GHH preferences with no-income effect on labor supply, and to the case in which externalities occur solely through the aggregate capital. In all these cases, indeterminacy is ruled out under all otherwise standard assumptions regarding the utility and production functions defined in Assumptions 1-4.

[7] Cross-country estimates of the capital income share are in the range 25%-40%. See for instance Karabarbounis and Neiman [43].

4 A general two-sector model

As emphasized by Jaimovich and Rebelo [42], aggregate and sectoral comovement are central features of business-cycles. We now assess whether indeterminacy and sunspot fluctuations are a more likely outcome of multisector infinite horizon models. There are of course many possible ways of constructing multisector economies. To facilitate comparison with the existing literature, we choose to focus our analysis on a two-sector model similar to the one analyzed by Benhabib and Farmer [8], except that we do not restrict the specifications of the utility and the production functions.[8]

Thus, we consider a two-sector economy in which firms produce differentiated consumption and investment goods using capital and labor. As in Benhabib and Farmer [8], we assume that capital and labor are perfectly mobile across sectors, and that both sectors produce their goods with the same technology at the private level. However, we assume, as in Dufourt *et al.* [20], that only the firms in the investment good sector are affected by productive externalities. This choice is based on the fact that empirical estimates for the degree of IRS in the consumption sector are close to zero, while they are positive and significant in the investment sector (see e.g. Harrison [36]).

Given these assumptions, firms in the consumption sector produce output Y_{ct} according to the production function:

$$Y_{ct} = f(K_{ct}, L_{ct}) \tag{29}$$

where K_{ct} and L_{ct} are capital and labor allocated to the consumption sector.

In the investment sector, output Y_{It} is also produced according to the same production function, but is affected by a productive externality

$$Y_{It} = f(K_{It}, L_{It})e(\bar{K}_{It}, \bar{L}_{It}) \tag{30}$$

where K_{It} and L_{It} are the numbers of capital and labor units used in the production of the investment good, and $e(\bar{K}_{It}, \bar{L}_{It})$ is the externality variable. The functions $f(.,.)$ and $e(.,.)$ of course satisfy Assumption 1. Following Benhabib and Farmer [8], we also restrict the specifications of externalities to consider *output externalities,* satisfying $\Theta_k = \Theta_l = \Theta \geq 0$.[9] Recall from Proposition 6 that under such a restriction, local indeterminacy is completely ruled out in the aggregate model.

Assuming that factor markets are perfectly competitive and that capital and labor inputs are perfectly mobile across the two-sectors, the first-order conditions for profit

[8] In Benhabib and Farmer [8], consumers have Hansen's type of individual preferences and the production functions are Cobb-Douglas.

[9] This assumption allows us to avoid having to consider a much larger number of cases.

maximization of the representative firm in each sector are:

$$r_t = f_1(K_{ct}, L_{ct}) = p_t f_1(K_{It}, L_{It}) e(\bar{K}_{It}, \bar{L}_{It}), \quad (31)$$

$$w_t = f_2(K_{ct}, L_{ct}) = p_t f_2(K_{It}, L_{It}) e(\bar{K}_{It}, \bar{L}_{It}) \quad (32)$$

where r_t, p_t, and w_t are respectively the rental rate of capital, the price of the investment good, and the real wage rate at time t, all in terms of the price of the consumption good, which is chosen here as the numeraire.

As in the previous section, we restrict the degree of IRS to ensure that the capital and labor demand functions are *negatively sloped.* Under output externalities, this is ensured by the following Assumption, replacing Assumption 2 above:

Assumption 6. $\Theta < s/(1-s)\sigma$

Denoting by i_t the investment, the budget constraint faced by the representative household is

$$c_t + p_t i_t = r_t k_t + w_t l_t + d_t, \quad (33)$$

where again dividends d_t are zero at equilibrium. The law of motion of the capital stock is:

$$k_{t+1} = (1-\delta)k_t + i_t \quad (34)$$

The household then maximizes its present discounted lifetime utility

$$\max_{\{k_{t+1}, c_t, l_t, i_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t, \ell - l_t) \quad (35)$$

subject to (33), (34), and k_0 given. The first-order conditions and the transversality condition are the same as (9)-(12), with the return factor now defined as:

$$R_{t+1} = \frac{(1-\delta)p_{t+1} + r_{t+1}}{p_t} \quad (36)$$

4.1 Intertemporal equilibrium and steady state

We consider symmetric rational expectation equilibria which consist of prices $\{r_t, p_t, w_t\}_{t\geq 0}$ and quantities $\{c_t, l_t, i_t, k_t, Y_{ct}, Y_{It}, K_{ct}, K_{It}, L_{ct}, L_{It}\}_{t\geq 0}$, with the externality variables satisfying $(\overline{K}_{It}, \overline{L}_{It}) = (K_{It}, L_{It})$ for any t, thereby satisfying the household's and the firms' first-order conditions as given by (9)-(11) and (31)-(32), the technological and budget constraints (29)-(30) and (33)-(34), the market equilibrium conditions for the consumption and investment goods

$$c_t = Y_{ct}, \quad i_t = Y_{It}, \quad (37)$$

with GDP defined as $y_t = c_t + p_t i_t$, the market equilibrium conditions for capital and labor

$$K_{ct} + K_{It} = k_t, \quad L_{ct} + L_{It} = l_t, \tag{38}$$

and the transversality condition (12).

Combining (29)-(30) and firms' first-order conditions (31)-(32), we derive $p_t e(K_{It}, L_{It}) = 1$ and that the equilibrium capital-labor ratios in the consumption and the investment sectors are identical and equal to $k_t/l_t = K_{ct}/L_{ct} = K_{It}/L_{It} = sw_t/((1-s)r_t)$. Combining this with Assumption 1, aggregate output y_t can be rewritten as $y_t = f(k_t, l_t)$, and the first-order conditions with respect to capital and labor in the consumption and investment sectors can be expressed as $r_t = f_1(k_t, l_t)$ and $w_t = f_2(k_t, l_t)$. It follows that a symmetric general equilibrium satisfies in any t,

$$\begin{aligned}
\lambda_t &= \beta\lambda_{t+1}\left[\frac{(1-\delta)p_{t+1} + r_{t+1}}{p_t}\right] & (39)\\
k_{t+1} &= (1-\delta)k_t + \frac{y_t - c_t}{p_t} & (40)\\
r_t &= f_1(k_t, l_t) & (41)\\
w_t &= f_2(k_t, l_t) & (42)\\
p_t &= \frac{1}{e(K_{It}, L_{It})} & (43)\\
c_t &= c(w_t, \lambda_t) & (44)\\
l_t &= l(w_t, \lambda_t) & (45)\\
y_t &= f(k_t, l_t) & (46)\\
K_{ct} &= \frac{sc_t}{r_t} & (47)\\
L_{ct} &= \frac{(1-s)r_t K_{ct}}{sw_t} & (48)\\
k_t &= K_{It} + K_{ct} & (49)\\
l_t &= L_{It} + L_{ct} & (50)
\end{aligned}$$

together with the initial condition k_0 given and the transversality condition (12).

It is easy to show that the same conclusion as in Proposition 3 applies here: under Assumptions 1, 3, 4, and 6, there exists a unique steady state. Moreover, this steady state can be maintained constant across calibrations by adjusting the value of ℓ accordingly.[10]

4.2 Local stability analysis

As in the former section, we log-linearize the system of equations (39)-(50) around the steady state. Once again, this system contains only two dynamic equations, so that it

[10] A proof of this statement can be provided upon request.

can be reduced to a system of *minimal dimension,* i.e. a system involving two dynamic equations in two variables $\widehat{k}_t$ and $\widehat{\lambda}_t$. This reduced system can be expressed as:

$$\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix} \equiv J \begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix}$$

where the Jacobian matrix J is given in the following Proposition:

Proposition 7. *The elements of the Jacobian matrix J are:*

$$J_{11} = \frac{A_{22}B_{11}+B_{21}}{A_{21}}, \quad J_{12} = \frac{A_{22}B_{12}-B_{22}}{A_{21}}, \quad J_{21} = B_{11}, \quad J_{22} = -B_{22}$$

with

$$A_{21} = 1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} - \frac{(1-\delta)\Theta}{s\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right], \; A_{22} = \frac{\frac{\theta(1-s)}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \frac{\theta(1-\delta)\Theta}{\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}$$

$$B_{11} = \frac{1+\Theta}{s\beta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right], \; B_{12} = \frac{1}{\beta}\left[1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \theta\Theta\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}\right]$$

$$B_{21} = 1 - \frac{\Theta}{s\beta\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right], \; B_{22} = \frac{\theta\Theta}{\beta\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}$$

Proof. See Appendix 7.10.

We can thus carry out the same kind of analysis as in Section 2 and provide a detailed local stability analysis of the steady state, considering a family of economies parameterized by the three elasticities (ϵ_{cc}, ϵ_{lw}, and $\epsilon_{l\lambda}$) that govern the EIS in consumption, the wage elasticity, and the income effect, and by the technological parameters σ and Θ governing the elasticity of capital-labor substitution and the degree of increasing returns to scale (IRS) in the investment sector.

Similar to the previous section, we prove in the online Appendix 7.11 that there exist three bifurcation loci in the parameter space such that, when ϵ_{cc} is increased from 0 to $+\infty$, a change in the stability properties of the steady state occurs when ϵ_{cc} crosses any of the three loci. We establish the following Lemma:

Lemma 4. *Under Assumptions 1, 3, 4 and 6, let* $\Omega = (\beta, \delta, s, \sigma, \Theta)$ *be the set of structural parameters. For any* $\omega \in \Omega$, *there exist three* bifurcation curves *crossing the 3-dimensional plane* $(\epsilon_{lw}, \epsilon_{l\lambda}, \epsilon_{cc})$ *and generating a change in the local stability properties of the steady state:*

- *a* flip bifurcation curve $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ *combined with one real eigenvalue of J crossing -1,*

- *a* Hopf bifurcation curve $\epsilon_{cc}^{H}(\epsilon_{lw},\epsilon_{l\lambda})$ *combined with two complex conjugate eigenvalues of* J *crossing the unit circle,*

- *a (degenerate)* transcritical bifurcation curve $\epsilon_{cc}^{T}(\epsilon_{lw},\epsilon_{l\lambda})$ *combined with one real eigenvalue of* J *crossing 1.*

There also exist four critical bounds $\bar{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$ *and* $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ *such that:*

- $\epsilon_{cc}^{H}(\epsilon_{lw},\epsilon_{l\lambda}) = 0$ *when* $\epsilon_{l\lambda} = \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$,

- $\epsilon_{cc}^{F}(\epsilon_{lw},\epsilon_{l\lambda}) = 0$ *when* $\epsilon_{l\lambda} = \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$,

- $\epsilon_{cc}^{H}(\epsilon_{lw},\epsilon_{l\lambda}) = \epsilon_{cc}^{T}(\epsilon_{lw},\epsilon_{l\lambda})$ *when* $\epsilon_{l\lambda} = \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$,

- $\epsilon_{cc}^{F}(\epsilon_{lw},\epsilon_{l\lambda}) = \epsilon_{cc}^{T}(\epsilon_{lw},\epsilon_{l\lambda})$ *when* $\epsilon_{l\lambda} = \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

The formal expressions of these bifurcation curves and critical bounds are given in Appendix 7.11.

Proof. See Appendix 7.11.

The critical bounds help us to define areas in the 3-dimensional plane where the bifurcation curves exist (or not) when ϵ_{cc} is gradually increased from 0 to $+\infty$. For example, the flip bifurcation exists whenever $\epsilon_{l\lambda} \in (0, \tilde{\epsilon}_{l\lambda})$.[11] The transcritical bifurcation has a vertical asymptote at $\underline{\epsilon}_{lw} = (\tilde{\sigma} - \sigma)/s$, so that the transcritical bifurcation exists whenever $\epsilon_{lw} > \underline{\epsilon}_{lw}$. Finally, if $\epsilon_{lw} \leq \underline{\epsilon}_{lw}$, the Hopf bifurcation exists whenever $\epsilon_{l\lambda} \in (0, \bar{\epsilon}_{l\lambda})$, whereas if $\epsilon_{lw} > \underline{\epsilon}_{lw}$, the Hopf bifurcation exists whenever $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$.

It is also easy to show that whenever both curves exist, the flip and Hopf bifurcations satisfy $0 < \epsilon_{cc}^{F} < \epsilon_{cc}^{H} < \infty$. Likewise, whenever both curves exist, the flip and transcritical bifurcations satisfy $0 < \epsilon_{cc}^{F} < \epsilon_{cc}^{T} < \infty$ if $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}$, and $0 < \epsilon_{cc}^{T} < \epsilon_{cc}^{F} < \infty$ if $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}$.

We can now establish the following Theorem, providing a complete picture of the stability properties of the 2-sector model.

Theorem 2. *Under Assumptions 1, 3, 4, 5, and 6, let* $\sigma < \bar{\sigma} \equiv \min\{2, \tilde{\sigma}\}$ *with* $\tilde{\sigma} = (1-s)(1+\Theta)/\Theta$. *Consider the bifurcation curves and critical curves defined by Lemma 4, and define by* $\underline{\epsilon}_{lw} = (\tilde{\sigma} - \sigma)/s$ *and by* $\bar{\epsilon}_{lw}$ *the unique solution of* $\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}) = \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$. *We have:*

Case 1 - Low wage elasticity of labor supply: $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$.

[11] To simplify notations, from now on we no longer explicitly mention the dependence of the critical bounds on ϵ_{lw} and the dependence of the bifurcation curves on ϵ_{lw} and $\epsilon_{l\lambda}$.

i) when $\epsilon_{l\lambda} \in [0, \tilde{\epsilon}_{l\lambda})$*, the steady state is*

- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$*,*

- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$*,*

- a source for any $\epsilon_{cc} > \epsilon_{cc}^H$*.*

ii) when $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$*, the steady state is*

- a sink for any $\epsilon_{cc} \in [0, \epsilon_{cc}^H)$*,*

- a source for any $\epsilon_{cc} > \epsilon_{cc}^H$*.*

iii) when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}$*, the steady state is a source for any* $\epsilon_{cc} \geq 0$*.*

Case 2 - Intermediate wage elasticity of labor supply: $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$.

i) when $\epsilon_{l\lambda} \in [0, \underline{\epsilon}_{l\lambda})$*, the steady state is*

- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$*,*

- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$*,*

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$*.*

ii) when $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}, \hat{\epsilon}_{l\lambda})$*, the steady state is*

- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$*,*

- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$*,*

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$*.*

iii) when $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}, \tilde{\epsilon}_{l\lambda})$*, the steady state is*

- a saddle-point for any $\epsilon_{cc} \in [0, \epsilon_{cc}^F)$*,*

- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^H)$*,*

- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^H, \epsilon_{cc}^T)$*,*

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$*.*

iv) when $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$*, the steady state is*

- a sink for any $\epsilon_{cc} \in (0, \epsilon_{cc}^H)$*,*

- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^H, \epsilon_{cc}^T)$*,*

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$*.*

v) when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}$*, the steady state is*

- a source for any $\epsilon_{cc} \in (0, \epsilon_{cc}^T)$*,*

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$*.*

Case 3 - High wage elasticity of labor supply: $\epsilon_{lw} > \bar{\epsilon}_{lw}$.

i) when $\epsilon_{l\lambda} \in [0, \underline{\epsilon}_{l\lambda})$*, the steady state is*

- a saddle for any $\epsilon_{cc} \in (0, \epsilon_{cc}^F)$*,*

- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$,

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

ii) when $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}, \tilde{\epsilon}_{l\lambda})$*, the steady state is*

- a saddle for any $\epsilon_{cc} \in (0, \epsilon_{cc}^F)$,

- a sink for any $\epsilon_{cc} \in (\epsilon_{cc}^F, \epsilon_{cc}^T)$,

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

iii) when $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}, \hat{\epsilon}_{l\lambda})$*, the steady state is*

- a sink for any $\epsilon_{cc} \in (0, \epsilon_{cc}^T)$,

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

iv) when $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}, \bar{\epsilon}_{l\lambda})$*, the steady state is*

- a sink for any $\epsilon_{cc} \in (0, \epsilon_{cc}^H)$,

- a source for any $\epsilon_{cc} \in (\epsilon_{cc}^H, \epsilon_{cc}^T)$,

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

v) when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}$*, the steady state is*

- a source for any $\epsilon_{cc} \in (0, \epsilon_{cc}^T)$,

- a saddle-point for any $\epsilon_{cc} > \epsilon_{cc}^T$.

Proof. *See Appendix 7.12.*

4.3 Comments and interpretation

Theorem 2 is the second central result of our paper. To illustrate its practical importance, Figure 1 displays the local stability properties of the two-sector model in the 3-dimensional plane defined by $(\epsilon_{l\lambda}, \epsilon_{lw}, \epsilon_{cc})$, when a benchmark calibration for the remaining parameters is used : namely $\beta = 0.99$, $\delta = 0.025$, $s = 0.3$, $\sigma = 1$ (Cobb-Douglas production function) and $\Theta = 0.3$, a value which is close to the point estimate obtained by Harrison [36] for the degree of increasing returns to scale in the investment sector in the US economy (4-digit data).[12] We see that, in sharp contrast with the one-sector model, there now exists a wide range of values for $(\epsilon_{lw}, \epsilon_{l\lambda}, \epsilon_{cc})$ such that the steady state is locally indeterminate and sunspot equilibria emerge. In fact, indeterminacy is only robustly excluded in the following three cases: (i) when a large Frisch labor supply elasticity ϵ_{lw} is combined with a small or a large wealth effect on labor supply $\epsilon_{l\lambda}$ (cases 2(i) and 3(i) and cases 2(v)

[12] As a robustness check, we also assess how these local stability properties vary when other calibrations for σ and Θ are considered, and we show that our main conclusions hold. These last results are reported in the separate appendix.

and 3(v) in the Theorem, respectively), (ii) when a small labor supply elasticity ϵ_{lw} is combined with a large wealth effect $\epsilon_{l\lambda}$ (case 1 in the theorem), and (iii) in all other cases, when the EIS in consumption ϵ_{cc} is too large.

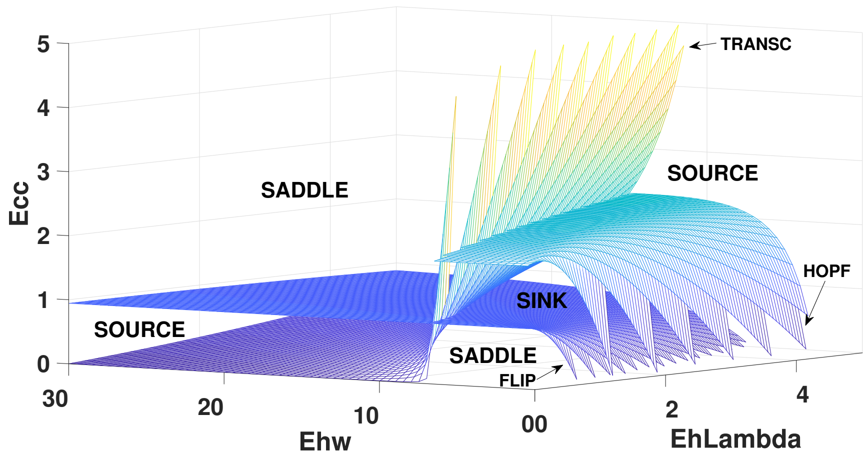

Figure 1: *Local stability properties of the two-sector model. Benchmark calibration with* $\sigma = 1$ *and* $\Theta = 0.3$.

Clearly, the range of values associated with indeterminacy is entirely consistent with empirical estimates: Indeterminacy robustly occurs for a large set of values of the EIS in consumption in the range $\epsilon_{cc} \in (0, 2)$ and for various configurations for the wage and wealth elasticities of the labor supply curve $(\epsilon_{lw}, \epsilon_{l\lambda})$. This includes very large values for the latter two elasticities (as in Hansen's type of preferences with infinitely elastic labor supply, $\epsilon_{lw} = \epsilon_{l\lambda} = +\infty$) or very small values for the wage-elasticity ϵ_{lw} consistent with micro-level estimates.[13] To further illustrate this point, Figure 2 displays the stability property areas when the range of values considered for ϵ_{lw} and $\epsilon_{l\lambda}$ is restricted to (0,5).[14] As can be seen, indeterminacy continues to arise in this configuration for a very wide range of values for ϵ_{cc} in the realistic interval (0,2), including values that are arbitrarily

[13] As is well kown, a lengthy discussion exists in the literature about how to calibrate the Frisch wage-elasticity of the aggregate labor supply curve. Both theoretical considerations and empirical evidence point toward small values at the individual level but greater values at the aggregate level (see for example Rogerson and Wallenius [58] for a discussion). Meanwhile, it is well known that standard RBC-DSGE models do not perform well when the wage elasticity of the aggregate labor supply is too low, which explains the popularity of the class of preferences suggested by Hansen [34].

On the other hand, there is very little empirical evidence on the wealth-elasticity $\epsilon_{l\lambda}$. The main difficulty is that this elasticity captures the effect of a *marginal* increase in intertemporal wealth on labor supply, and that exogenous variations enabling this elasticity to be identified are very difficult to find in the data (see the lengthy discussion and relevant references to the literature in Kimball and Shapiro, [45]). Analyzing data from a thought-experiment survey conducted by the Health and Retirement Study (HRS), Kimball and Shapiro [45] tend toward the conclusion that the elasticities ϵ_{lw} and $\epsilon_{l\lambda}$ are rather small. Yet, their estimations assume an equality between both elasticities (as imposed by Hansen's type of preferences), while from a theoretical standpoint there is no reason to assume that they are equal.

[14] Rogerson and Wallenius [58], and Prescott and Wallenius [57] suggest benchmark values around 3 to calibrate the aggregate wage elasticity of labor supply.

close to 0. This substantiates our conclusion that *indeterminacy and the existence of sunspot equilibria are very robust properties of the competitive two-sector model.*

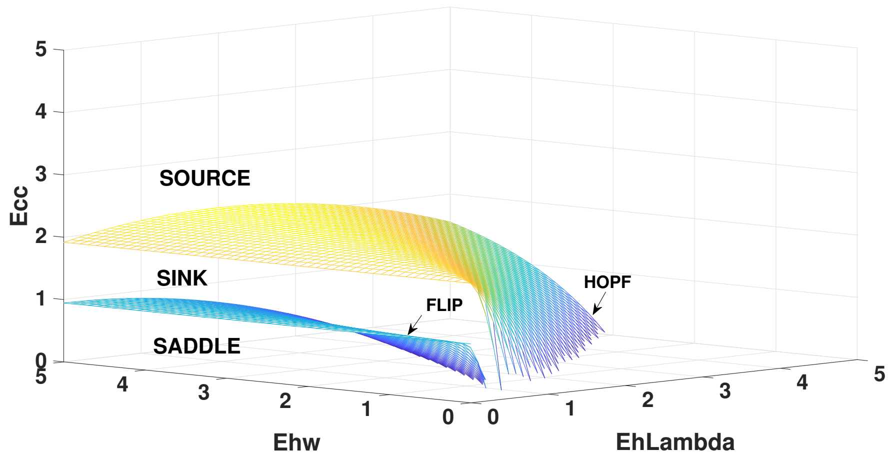

Figure 2: *Local stability properties for low labor supply elasticities (benchmark calibration with $\sigma = 1$ and $\Theta = 0.3$).*

Another contribution of Theorem 2 lies in determining whether indeterminacy can arise (or not) when particular specification of individual preference frequently are used. To illustrate this, Figure 3 displays the area in the $(\epsilon_{lw}, \epsilon_{l\lambda})$ plane for which local indeterminacy emerges for some values of ϵ_{cc} in the appropriate range $(0, 2)$. The figure also indicates the restrictions imposed by specific classes of utility functions: GHH utility functions associated with $\epsilon_{l\lambda} = 0$, KPR utility functions associated with $\epsilon_{l\lambda} = \epsilon_{cc}\epsilon_{lw}$, and generalized Hansen preferences associated with $\epsilon_{l\lambda} = \epsilon_{lw}$.[15]

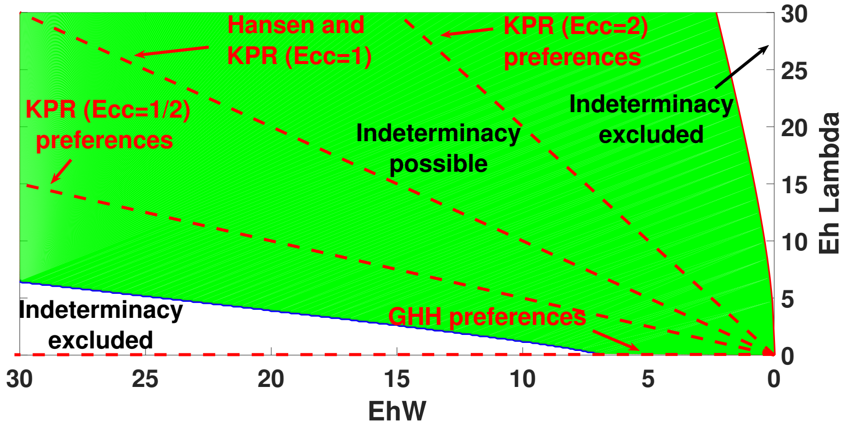

Figure 3: *Potential indeterminacy area and standard specification of individual preferences.*

As can be seen, local indeterminacy and the existence of sunspot fluctuations can be

[15]Note that in the particular case of KPR preferences with $\epsilon_{cc} = 1$, we obtain the same same restriction $\epsilon_{l\lambda} = \epsilon_{lw}$ as with Hansen preferences, even though their implications for ϵ_{cc} differ (see Proposition 2).

obtained for all classes of preferences. In the case of GHH preferences, indeterminacy requires that the wage elasticity ϵ_{lw} remains sufficiently small (less than 7 in our benchmark calibration). In the case of KPR and generalized Hansen preferences, indeterminacy can occur for a very wide set of values for ϵ_{lw} and $\epsilon_{l\lambda}$, including very large ones. In our benchmark calibration, indeterminacy can even be obtained with an infinitely elastic aggregate labor supply ($\epsilon_{lw} = \epsilon_{l\lambda} = +\infty$)), which corresponds to the initial specification of Hansen preferences.

5 Confronting the 2-sector model to the data

Is a standard stochastic growth model with random changes in agents' expectations (sunspot shocks) able to account for the bulk of observed business-cycles? So far, the response to this question has been mostly negative. In a well-known contribution, using a two-sector model very similar to ours but assuming KPR preferences and Cobb-Douglas production functions, Schmitt-Grohé [61] concludes that such models are inconsistent with several defining features of actual fluctuations. These include the positive autocorrelation of output growth, the hump-shaped response of output to transitory shocks, and the pattern of correlations between the forecastable components in output, consumption, investment, and hours worked. On the other hand, Benhabib and Wen [11] show that a standard one-sector model with a variable capital utilization rate is easily prone to indeterminacy for very low degrees of IRS, and that their model subjected to exogenous productivity and government spending shocks can explain many features of the business cycle. However, in the case of sunspot shocks alone, the model cannot account for the hump-shaped response of output to transitory shocks that is found in the data. Dufourt *et al.* [22] show that this last deficiency can be overcome by considering a larger class of additively separable utility functions and a general class of production functions. Yet the paper also shows that the model's impulse response functions fall very short of replicating those obtained from the data in response to a transitory shock.

In this section, we reconsider this issue in the light of our results. As in Dufourt *et al.* [22] , we concentrate our analysis on what are considered the most challenging tests among those in Schmitt-Grohé [61], namely the ability to replicate the empirical impulse response functions (IRF) to shocks on both the permanent and the transitory components of output obtained from an identified structural VAR model.

5.1 An estimated bivariate VAR model

We follow the approach of Schmitt-Grohé [61] and estimate a bivariate VAR model involving output and a second macroeconomic variable. One minor difference from Schmitt-Grohé [61] is that we include the consumption-output ratio, instead of hours worked, as the second variable in the VAR. This avoids the debate in the literature on whether hours worked should be considered a stationary or a first-order integrated variable, the results of the identified VAR being somewhat sensitive to this assumption (Gali [27]). By contrast, as amply demonstrated in the literature, consumption and output are typically best described as first-order cointegrated variables with a cointegrating vector equal to $(1, -1)$, implying that the consumption-output ratio is stationary. King *et al.* [47] show that this feature of the data is consistent with the predictions of standard stochastic growth models (like those considered in this paper) when the TFP level is subject to permanent productivity shocks.

We thus estimate a bivariate model involving output in first-difference and the consumption-output ratio, using quarterly US data over the period 1948:1 - 2019:4 (all variables are expressed in log).[16] As shown in Figure 4, the data suggests that the consumption-output ratio remained stationary until the beginning of the 90s but that since then it has been increasing at a roughly constant pace. In our benchmark estimation, we thus include a linear trend in the consumption-output ratio starting in 1990:Q1 to take this fact into account.[17] However, our results are only marginally affected when this deterministic trend is not introduced. Our choice of two lags is based on the BIC criterion. Following Blanchard and Quah [13], we then identify two kinds of shocks in the data: permanent shocks and transitory shocks. Permanent shocks are the only ones having a permanent effect on the level of output, while leaving the long-run consumption-output ratio unaffected. By contrast, transitory shocks are the only ones leaving output and the consumption-output ratio unaffected in the long-run. In our model, a permanent shock is interpreted as a permanent technological shock affecting the TFP level. A transitory shock is interpreted as a sunspot shock resulting from an exogenous (and self-fulfilling) shift in agents' expectations. As explained below, we allow changes in expectations to be

[16]We use the Federal Reserve Economic Data (FRED) database. Consumption is defined as the sum of personal consumption expenditures on nondurable goods and services. Investment is the sum of gross private domestic investment and personal consumption expenditures on durable goods, all variables divided by the GDP deflator. To obtain per capita variables, we divided the obtained series by the population aged 16 and over. Output is defined as the sum of per capita consumption and investment.

[17]As shown in Forrester [25], a simple way to explain a deterministic trend in the consumption-output ratio is to assume that the long-run growth rate of TFP in the market sector exceeds the long-run growth rate of TFP in the home sector, a quite realistic assumption.

correlated with technological shocks, but we define the sunspot shock as the component in agents' expectations which is uncorrelated with the fundamental TFP shock.

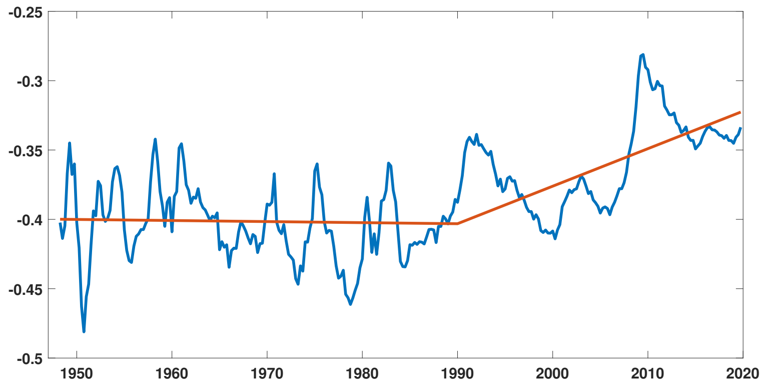

Figure 4: *US postwar consumption-output ratio (in log), with* $y = c + i$.

Figure 5 displays the impulse response functions of output and consumption to one-standard-deviation permanent and transitory shocks, as identified from our bivariate VAR model. The figure also displays 90% bootstrapped confidence intervals. The results are similar to those obtained in the literature. Following a positive permanent shock, output jumps and gradually increases over time to reach a peak after 5 quarters, then gradually converges to its new long-run level. According to our estimation, output "overshoots" its new long-run level during the transition. Consumption also jumps when the shock occurs and then quickly converges to its new long-run level. Figure 5 also shows that following a one-standard-deviation transitory shock, the response of output is hump-shaped: output jumps when the shock occurs, reaches a peak after three quarters, and then slowly returns to its initial level. Meanwhile, consumption reacts very little when the shock occurs and then gradually increases over time before returning to its initial level. The response of consumption is much smoother than the response of output, in accordance with the predictions of the permanent income theory following transitory income shocks.

This pattern of output fluctuations in response to permanent and transitory disturbances was found in numerous papers and proves to be robust to various changes in the VAR specification, including changing the number of lags, changing the second variable (for example, taking investment, hours worked or the unemployment rate as the second variable), or considering more than two variables in the VAR. For the interested reader, some alternative specifications are presented in a separate appendix. For this reason, the ability to reproduce the "hump-shaped" response of output to a transitory shock is considered one of the criteria defining appropriate business cycle models. In line with this approach, we therefore investigate the ability of our two-sector model to account for these

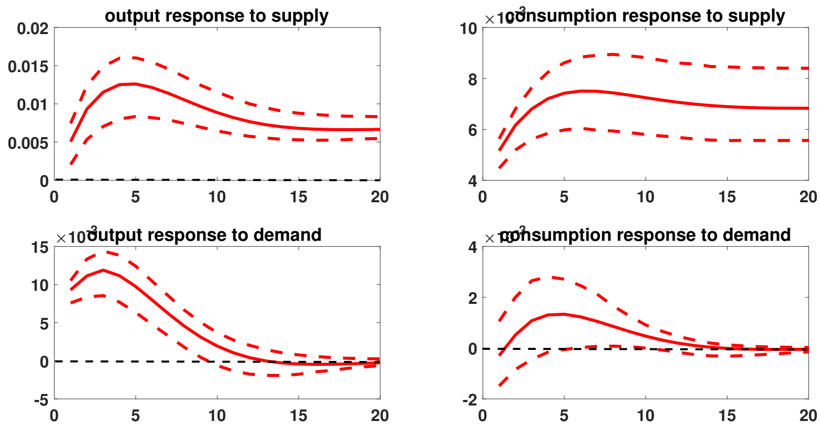

Figure 5: *Empirical impulse response functions to 1-std-deviation supply (permanent) and demand (transitory) shocks.*

empirically estimated IRFs when the aggregate TFP level is subject to permanent productivity shocks and agents' expectations shift randomly in response to non-fundamental sunspot shocks.

5.2 Estimation method

We use a Simulated Method of Moments – Minimum Distance (SMM–MD) approach to estimate our model and assess its ability to account for the data. More precisely, we define by $\Omega = (\Omega_1, \Omega_2)$ the vector of structural parameters and let Ω_1 be the vector of calibrated parameters and Ω_2 the vector of estimated parameters. For each candidate value of Ω, we repeat $Nsim = 300$ times the following procedure : we draw a set of $T = 287$ (the length of our dataset) vectors of shocks $\epsilon_t = (\epsilon_t^z, \epsilon_t^s)$, $t = 1, ..., T$, where ϵ_t^z is the technological shock and ϵ_t^s is the sunspot shock at period t, and we compute the equilibrium trajectory $(y_t, c_t, i_t, ...)$, $t = 1, ..., T$, obtained from the model.[18] We then apply to these simulated time series the same bivariate estimation that we applied to the data, collecting the obtained impulse response functions to permanent and transitory disturbances. We obtain a set of $Nsim$ simulated IRFs, and compute their median by collecting the median value at each lag. We compare this median IRF $\Psi^m(\Omega_2)$ to the one estimated from the data, Ψ. The vector of estimated parameters $\widehat{\Omega}_2$ is obtained as that minimizing the distance between the simulated and the empirical IRFs, according to the following criterion:

$$\widehat{\Omega}_2 = \arg\max_{\Omega_2} \left(\Psi^m(\Omega_2) - \Psi\right)' W \left(\Psi^m(\Omega_2) - \Psi\right) \tag{51}$$

[18]We actually draw $T' = 307$ shocks and remove the first 20 observations to eliminate the influence of initial conditions.

where W is a weighting matrix. Following common practice in the literature (see e.g. Rotemberg and Woodford, [60], Amato and Laubach, [2]), we set W as the identity matrix, implying that the weights associated with each period in the IRFs are the same.

5.3 Stochastic version of the model and estimated parameters

To obtain some randomness in our model, shocks are introduced. In accordance with the RBC-DSGE literature, we assume that the production functions in the consumption and the investment sectors are:

$$Y_{ct} = f(K_{ct}, z_t L_{ct}) \tag{52}$$

$$Y_{It} = f(K_{It}, z_t L_{It}) e(\bar{K}_{It}/z_t, \bar{L}_{It}) \tag{53}$$

where z_t is a labor-augmenting technical progress, assumed to follow a logarithmic random walk:

$$\ln z_t = \ln z_{t-1} + \sigma_z \epsilon_t^z$$

with $\epsilon_t^z \sim N(0,1)$ and $\sigma_z > 0$ is a variance parameter. With such a specification for TFP, a general specification for individual preferences consistent with balanced growth is

$$u\left(\frac{c_t}{z_t}, \ell - l_t\right)$$

i.e., preferences are based on the "productivity-adjusted" consumption level. In our view, using this specification for individual preferences, rather than the standard specification $u\left(c_t, \ell - l_t\right)$ usually considered in the literature, has two main advantages. First, from a theoretical point of view, this specification allows us to preserve the balanced-growth property while considering a much larger set of individual preferences than those defined in King *et al.* [46] – in particular, all the utility functions satisfying Assumptions 3 and 4 above. Notably, the balanced growth property is preserved even under the GHH and Hansen utility functions frequently considered in the literature. Second, from en empirical point of view, this specification is consistent with the well-known "Easterlin paradox" (Easterlin [23]) that happiness or individual lifetime satisfaction levels are not increasing over time in spite of long-run growth.

Under this specification, it is easy to show that the set of dynamic equations (22) in the one-sector model and the set of dynamic equations (39)-(50) in the two-sector model are the same, with all variables except hours worked expressed in their productivity-adjusted form, i.e. for any variable x_t, we define $\widetilde{x}_t \equiv \frac{x_t}{z_t}$. Therefore, the steady state and the local stability properties of the steady state are also the same (with all the variables expressed in their productivity-adjusted form), and all the propositions and Theorems previously derived remain valid.

Denote by $\widehat{\widetilde{x}}_t$ the productivity-adjusted variable $\widetilde{x}_t$ expressed in percentage deviation points from the steady state. Under local indeterminacy, the reduced (minimum dimension) log-linearized stochastic model can be expressed as

$$\begin{pmatrix} \widehat{\widetilde{k}}_{t+1} \\ \widehat{\widetilde{\lambda}}_{t+1} \end{pmatrix} = J \begin{pmatrix} \widehat{\widetilde{k}}_t \\ \widehat{\widetilde{\lambda}}_t \end{pmatrix} + R \begin{pmatrix} \epsilon_t^z \\ \epsilon_t^s \end{pmatrix}$$

with

$$R = \begin{bmatrix} \sigma_z & 0 \\ \rho & \sigma_s \end{bmatrix}$$

the matrix formed with the variances σ_z and σ_s of the productivity and sunspot shocks, respectively, with the shocks $(\epsilon_t^z, \epsilon_t^s)$ drawn is the standard normal distribution. In matrix R, the parameter ρ captures how the marginal utility of consumption (and thus consumption itself) reacts to a technological innovation. Equivalently, ρ captures the extent to which agents' expectations adjust to a positive or negative productivity shock. In the literature, this parameter is often arbitrarily set to 0 – implying that the matrix R is diagonal –, but there is actually no reason why this should be the case. We choose instead to estimate this parameter. Thus, the sunspot shock ϵ_t^s truly captures the notion of a sunspot shock, i.e. a change in agents' expectations which is unrelated to economic fundamentals but which, under local indeterminacy, turns out to be self-fulfilling and thus consistent with rational expectations.

Of course, once a dynamic trajectory for $\left(\widetilde{k}_t, \widetilde{\lambda}_t\right)$, $t = 1, ..., T$, is computed, we can easily recover the corresponding dynamics for all the other endogenous variables $(\widetilde{y}_t, \widetilde{c}_t, \widetilde{i}_t, ...)$ using the static equations of the model. This allows us to apply to these simulated data the same structural VAR estimation that we used with actual US data. In the stochastic version of the two-sector model, the vector of structural parameters is $\Omega = (\beta, \delta, s, \sigma, \Theta, \epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \rho, \sigma_z, \sigma_s)$. As in the previous section, we set $\Omega_1 = (\beta, \delta, s)$ the vector of calibrated parameters and we use the same benchmark calibration as before (see Assumption 5). We estimate the remaining eight structural parameters, which include the "critical parameters" crucial for the emergence of sunspot fluctuations and the three parameters governing the stochastic processes. This gives $\Omega_2 = (\sigma, \Theta, \epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \rho, \sigma_z, \sigma_s)$. The range of admissible values for Ω_2 – over which estimation is performed – is the range of empirically credible values as defined in Assumption 5, namely : $\sigma \in (0, 2)$, $\Theta \in (0, 0.43)$ and $\epsilon_{cc} \in (0, 2)$, where Θ now represents the degree of IRS in the investment sector only.

5.4 Estimation results

In Figure 6 we report the median IRF obtained from the model at the solution of the minimization problem above. The model comes incredibly close to replicating the empirical IRF for both consumption and output in response to both permanent and transitory disturbances. Notably, the model perfectly replicates the hump-shaped response of output to a transitory shock even though in the model, this shock (the sunspot shock) is white noise. This implies among other things that the model generates significant endogenous persistence, in sharp contrast with standard RBC models. The model also very closely replicates the IRF of output to the permanent shock, as well as the response of consumption to both shocks. The only noticeable difference between the empirical and the model-implied IRFs concerns the instantaneous response of consumption to the permanent income shock, which is slightly too small in the model when compared to the data. Overall, the estimation results show that a standard two-sector model with productivity and sunspot shocks does an excellent job of replicating the empirical IRFs to both demand and supply shocks, in sharp contrast to previous models in the literature (Schmitt-Grohé [61], Benhabib and Wen [11], Dufourt *et al.* [22]).

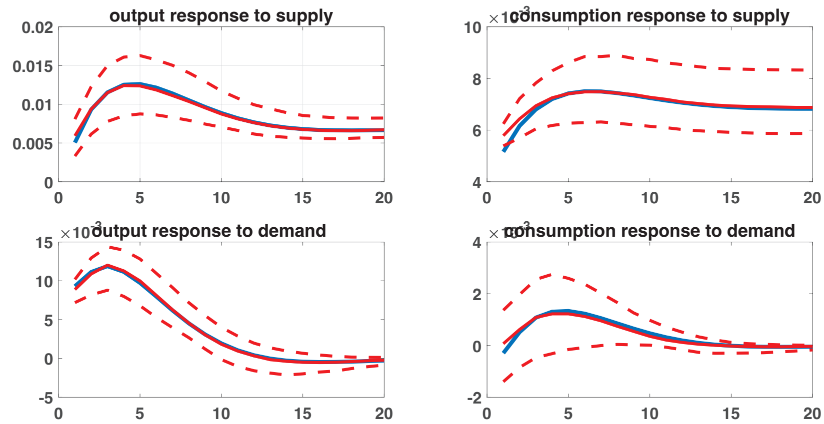

Figure 6: *Model versus empirical impulse response functions. Red curve: data, blue curve: model.*

Table 1 reports the parameter estimates obtained from our SMM-MD procedure. The estimated value for the EIS in consumption ϵ_{cc} is smaller than 1, consistent with the majority of estimates in the empirical literature.[19] The degree of IRS in the investment sector is roughly 11%, showing that the model can closely mimic the data even for very moderate levels of increasing returns to scale. The estimated value for the capital-labor

[19] See discussion above. In a meta-analysis of 2375 estimates of the EIS in consumption, Havranek [38] concludes that the mean value of micro estimates is around 0.4 and that most estimates are equal or less than 0.8.

elasticity of substitution is equal to 1.83. Finally, estimation results confirm that, like most RBC-DSGE models, our two-sector model performs better when the Frisch labor supply elasticities ϵ_{lw} and $\epsilon_{l\lambda}$ are "both large", in the spirit of Hansen-type preferences, but with $\epsilon_{lw} > \epsilon_{l\lambda}$ in our case. The variance of the TFP shock is 1%, and the variance of the sunspot shock is 0.6%, which both seem reasonable.

ϵ_{cc}	Θ	σ	ϵ_{lw}	$\epsilon_{l\lambda}$	ρ	σ_z	σ_s
0.83	0.11	1.83	1948	1145	0.64	0.01	0.006

Table 1: Estimated parameters

Of course, it would be problematic if the model succeeded in replicating the IRFs to permanent and transitory disturbances but failed in other important areas, e.g. if it failed to replicate the business cycle properties of other variables. To complete our model evaluation, we thus consider other statistics emphasized in the RBC literature. We start with standard second-order moments regarding the cyclical components of macroeconomic variables, updating the data set constructed in Dufourt *et al.* [20] to build not only empirical series for output, consumption, investment, aggregate hours worked, labor productivity, etc., but also series for labor in the consumption and the investment sectors. All series are then detrended using the HP filter. The model-implied statistics are computed following an approach similar to our SMM detailed above, i.e., we use the $Nsim$ model-generated series, we apply the HP filter to these series, and we compute standard business cycle statistics. The figures presented in Table 2 are the average of the second-order moments thus obtained.

Overall, the model does an excellent job of accounting for these standard "stylized facts" of the business cycle, even though these statistics were not the target of our SMM-MD approach. The relative standard deviations are of the right magnitude, except for a slight underestimation of the volatility of labor productivity and some overestimation of the variance of labor in the investment sector. The first-order autocorrelation coefficients are also all correctly accounted for, as are the contemporaneous correlation coefficients with output. It is noteworthy that the model is also able to explain the low correlation coefficient between output and labor productivity (0.16 in the data versus -0.17 in the model), whereas standard RBC models are recognized to fail dramatically in this respect (tending to generate a very large positive correlation coefficient). Thus, the sunspot-driven model appears able to closely reproduce not only the empirical impulse response functions to both permanent and transitory disturbances, but also the business cycle dynamics regarding the other common macroeconomic variables.

I. Volatility (relative standard deviations: σ_x/σ_y)							
(x)	y	c	pi	l	L_C	L_I	y/l
Data	1	0.39	2.53	1.05	0.69	1.61	0.52
Model	1	0.45	3.13	1.07	0.49	3.22	0.26
II. Persistence (first-order autocorrelation: ρ_x)							
(x)	y	c	pi	l	L_C	L_I	y/l
Data	0.89	0.85	0.87	0.93	0.92	0.93	0.84
Model	0.83	0.78	0.85	0.85	0.85	0.85	0.84
III. Covariations (correlations with output: $\text{corr}(x,y)$)							
(x)	y	c	pi	l	L_C	L_I	y/l
Data	1	0.80	0.98	0.87	0.87	0.86	0.16
Model	1	0.95	0.99	0.97	0.97	0.97	-0.17

Table 2: Standard additional statistics

As a final informal test, we consider the autocorrelation function of output growth. As first documented by Nelson and Plosser [52] and Cochrane [16], output dynamics is positively autocorrelated over short horizons and has weak negative autocorrelation over longer horizons. Cogley and Nason [17] show that, in addition to being unable to reproduce the hump-shaped dynamics of output to transitory disturbances, standard RBC models subjected to random-walk TFP shocks are unable to reproduce this autocorrelation function of output growth, which indicates that these models crucially fail to create endogenous persistence. It is thus interesting to assess how our sunspot-driven two-sector model performs in this respect Table 3 shows that the model performs very well in this area too, very accurately reproducing both the large positive autocorrelation coefficients over the first two quarters and the smaller negative autocorrelation coefficient over a longer horizon. This proves that, in contrast to the standard RBC model, our model has sufficiently strong propagation mechanisms to generate significant endogenous persistence.

Autocorrelation of output growth					
	$\rho_y(1)$	$\rho_y(2)$	$\rho_y(3)$	$\rho_y(4)$	$\rho_y(5)$
Data	0.44	0.23	-0.01	-0.16	-0.24
Model	0.44	0.25	0.10	-0.01	-0.09

Table 3: Autocorrelation function of output growth

6 Concluding comments

We have proved that two-sector infinite-horizon models with productive externalities and IRS provide a strong theoretical basis to explain business-cycle fluctuations driven by self-fulfilling prophecies when appropriate restrictions of the EIS in consumption, the wage elasticity of labor supply, and the elasticity of labor with respect to the marginal utility of wealth are considered. We have shown in particular that the existence a Hopf bifurcation allows to calibrate the economy with empirically relevant values for all the structural parameters and to provide enough persistence to replicate all the most salient features of the business cycles.

Our conclusions are in line with a recent trend of the literature, mainly driven by Beaudry *et al.* [5, 6], focusing on the fact that fluctuations can be the result of internal forces of the economy generating alternate periods of boom and bust. Indeed, Beaudry *et al.* [5] introduce simple nonlinearities in estimation of macroecomic aggregates and show that the local properties of the system switch from being locally stable when the nonlinear terms are excluded to being locally unstable when they are included. Beaudry *et al.* [6] then show the existence of recurrent peaks in several spectral densities in many US trendless macroeconomic data suggesting the presence of fluctuations of various periodicities. Moreover, they prove that the existence of a limit cycle through a Hopf bifurcation provides an appropriate theoretical support for these empirical conclusions.

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Expectations, self-fulfilling prophecies and the business cycle: Online appendix*

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7 Appendix

7.1 Proof of Lemma 1

From the definition of ϵ_{lw} and $\epsilon_{l\lambda}$ as given by (15)-(16), a total differentiation of the optimality conditions (9)-(10) gives

$$\begin{aligned} u_{11}dc_t - u_{12}dl_t &= d\lambda_t \\ u_{21}dc_t - u_{22}dl_t &= d\lambda_t w_t + \lambda_t dw_t \end{aligned}$$

Solving this system with respect to dl_t yields to the expressions (19) and (20). We also derive

$$\begin{aligned} \epsilon_{cw} &= -\frac{u_{12}u_1}{u_{11}u_{22}-u_{12}u_{21}}\frac{w}{c} = \frac{wl}{c}(\epsilon_{lw} - \epsilon_{l\lambda}) \\ \epsilon_{c\lambda} &= \frac{u_1}{u_{11}c} - \frac{u_{12}}{u_{11}c}\frac{u_{11}u_2-u_{21}u_1}{u_{11}u_{22}-u_{12}u_{21}} = -\epsilon_{cc} + \frac{wl}{c}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\epsilon_{l\lambda} \end{aligned}$$

We show in Appendix 7.3 below that at the steady state

$$\frac{wl}{c} = \frac{\theta(1-s)}{\theta - s\beta\delta} \equiv \mathcal{C} < 1 \tag{57}$$

The result follows.

□

7.2 Proof of Proposition 1

By Assumption 3, u is an increasing function over $\mathbb{R}^2_{++}$, implying $(c, l, u_1, u_2) > 0$. Moreover, the strict quasi-concavity of u implies $u_{11} < 0$ and $u_{11}u_{22} - u_{12}u_{21} > 0$. Using Lemma 1, we straightforwardly obtain $\epsilon_{cc} > 0$ and $\epsilon_{lw} > 0$. By Assumption 4, c and $\mathcal{L}$ are normal goods. The normality of $\mathcal{L}$ requires $u_{21}u_1 - u_{11}u_2 \geq 0$. Combined with the strict quasi-concavity of u, we straightforwardly obtain $\epsilon_{l\lambda} \geq 0$. Using a similar reasoning, we obtain that the normality of c requires $\epsilon_{c\lambda} \leq 0$ and therefore, using Lemma 1, $\epsilon_{cc} \geq \mathcal{C}\epsilon_{l\lambda}(\epsilon_{lw} - \epsilon_{l\lambda})/\epsilon_{lw} \equiv \epsilon_{cc}^N$.

□

7.3 Proof of Lemma 2

We know from constant-returns-scale of the technology at the private level that

$$\frac{rk}{y} = s \text{ and } \frac{wl}{y} = 1 - s$$

Considering that at the steady state we have $R^* = 1/\beta$ with $R^* = r^* + 1 - \delta = sy^*/k^* + 1 - \delta$ we get

$$\frac{y^*}{k^*} = \frac{\theta}{s\beta}$$

with $\theta = 1 - \beta(1-\delta)$. It follows from the capital accumulation equation evaluated at the steady state that $c = y - k$ and thus

$$\frac{c^*}{k^*} = \frac{\theta - s\beta\delta}{s\beta}$$

We conclude from this

$$\frac{w^* l^*}{c^*} = \frac{\theta(1-s)}{\theta - s\beta\delta} \equiv \mathcal{C} < 1 \tag{58}$$

□

7.4 Proof of Proposition 3

Considering again that $R^* = r^* + 1 - \delta = 1/\beta$, we get

$$f_1(k^*, l^*)e(k^*, l^*) \equiv g(k^*, l^*) = \frac{\theta}{\beta} \tag{59}$$

It is then easy to compute under Assumption 2

$$\frac{g_1(k^*,l^*)k^*}{g(k^*,l^*)} = s\Theta_k - \frac{1-s}{\sigma} < 0$$

Therefore, applying the implicit function theorem, we conclude that there exists a unique function $k(.)$ such that $k^* = k(l^*)$. Considering that

$$\frac{g_2(k^*,l^*)l^*}{g(k^*,l^*)} = (1-s)\Theta_l + \frac{1-s}{\sigma}$$

we conclude that

$$\frac{k'(l^*)l^*}{k(l^*)} = -\frac{(1-s)\Theta_l + \frac{1-s}{\sigma}}{s\Theta_k - \frac{1-s}{\sigma}} > 0$$

Recalling now that

$$\frac{y^*}{k^*} = \frac{\theta}{s\beta} \text{ and } c^* = \frac{\theta - s\beta\delta}{s\beta}k^* \tag{60}$$

we derive

$$c^* = c(l^*) = \frac{\theta - s(k(l^*),l^*)\beta\delta}{s(k(l^*),l^*)\beta}k(l^*) \equiv h(l^*)k(l^*)$$

Straightforward computations give

$$\frac{h'(l^*)l^*}{h(l^*)} = \frac{\theta(1-s)}{\theta - s\beta\delta}\left(1 - \frac{1}{\sigma}\right)\frac{(1-s)\Theta_l + s\Theta_k}{s\Theta_k - \frac{1-s}{\sigma}}$$

and we easily conclude under Assumption 2

$$\frac{c'(l^*)l^*}{c(l^*)} = -\frac{(1-s)\left\{\Theta_l\left[s(1-\beta) + \frac{\theta(1-s)}{\sigma}\right] + \frac{\theta s\Theta_k}{\sigma} + \frac{\theta - s\beta\delta}{\sigma} - \theta s\Theta_k\right\}}{(\theta - s\beta\delta)\left(s\Theta_k - \frac{1-s}{\sigma}\right)} > 0$$

Moreover we also get from (25)

$$w^* = w(l^*) = f_2(k(l^*), l^*)e(k(l^*), l^*)$$

and thus

$$\frac{w'(l^*)l^*}{w(l^*)} = -\frac{\frac{1-s}{\sigma}\Theta_l + \frac{s}{\sigma}\Theta_k}{s\Theta_k - \frac{1-s}{\sigma}} > 0$$

Consider then the third equation of (26) which becomes

$$\frac{u_2(c(l^*), \ell - l^*)}{u_1(c(l^*), \ell - l^*)} \equiv \psi(l^*) = w(l^*) \tag{61}$$

Under Assumptions 2, 3 and 4, we get

$$\frac{\psi'^*)l^*}{\psi(l^*)} = \frac{c'(l^*)l^*}{c(l^*)}\frac{\epsilon_{l\lambda}}{\epsilon_{cc}\epsilon_{lw}} + \frac{1}{\epsilon_{cc}\epsilon_{lw}}\left[\epsilon_{cc} - \mathcal{C}\epsilon_{l\lambda}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\right] \geq 0 \tag{62}$$

It follows that the existence of a unique steady state value l^* is obtained if $g'(l^*) \neq w'(l^*)$. Straightforward computations show that this condition is satisfied if

$$\frac{c'(l^*)l^*}{c(l^*)}\frac{\epsilon_{l\lambda}}{\epsilon_{cc}\epsilon_{lw}} + \frac{1}{\epsilon_{cc}\epsilon_{lw}}\left[\epsilon_{cc} - \mathcal{C}\epsilon_{l\lambda}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\right] - \frac{w'(l^*)l^*}{w(l^*)} \neq 0 \tag{63}$$

Such a condition is generically satisfied so that the existence and uniqueness of a steady state is generically ensured.

Now let us normalize the steady state considering the value $\bar{l}^*$ corresponding to the average amount of working hours relative to the total amount of time ℓ. Substituting $l^* = \bar{l}^*$ into equation (61), we get

$$\frac{u_2(c(\bar{l}^*), \ell - \bar{l}^*)}{u_1(c(\bar{l}^*), \ell - \bar{l}^*)} \equiv \phi(\ell) = w(\bar{l}^*) \tag{64}$$

Straightforward computations give

$$\frac{\phi'(\ell)\ell}{\phi(\ell)} = -\frac{\ell}{\bar{l}^*}\frac{1}{\epsilon_{cc}\epsilon_{lw}}\left[\epsilon_{cc} - \mathcal{C}\epsilon_{l\lambda}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\right]$$

It follows under Assumptions 3 and 4 that $\phi'(\ell)\ell/\phi(\ell) \leq 0$ and there exists a unique value $\ell^* > \bar{l}^*$ solution of equation (64). We conclude finally that if $\ell = \ell^*$, then the unique steady state (k^*, l^*, c^*) is such that $l^* = \bar{l}^*$.

□

7.5 Proof of Proposition 4

From the optimality conditions (9)-(10) and Lemma 1, we derive

$$\widehat{l}_t = \epsilon_{lw}\widehat{w}_t + \epsilon_{l\lambda}\widehat{\lambda}_t \tag{65}$$

$$\widehat{c}_t = \mathcal{C}\left(1 - \frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\widehat{l}_t - \epsilon_{cc}\widehat{\lambda}_t \tag{66}$$

and (25) implies

$$\widehat{w}_t = \left(s\Theta_k + \frac{s}{\sigma}\right)\widehat{k}_t + \left[(1-s)\Theta_l - \frac{s}{\sigma}\right]\widehat{l}_t$$

Using this expression in (65) yields

$$\widehat{l}_t = \frac{\epsilon_{lw}s\left(\frac{1}{\sigma}+\Theta_k\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\widehat{k}_t + \frac{\epsilon_{l\lambda}}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\widehat{\lambda}_t \tag{67}$$

Using (67) into (66) gives

$$\widehat{c}_t = \frac{\mathcal{C}s\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}-\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\widehat{k}_t - \left(\epsilon_{cc} - \frac{\epsilon_{l\lambda}\mathcal{C}\left(1-\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right)\widehat{\lambda}_t \tag{68}$$

Using (25), the system of difference equations describing the intertemporal equilibrium can be stated as follows

$$\begin{aligned} f(k_t, l(k_t,\lambda_t))e(k_t, l(k_t,\lambda_t)) + (1-\delta_t)k_t - c(k_t,\lambda_t) - k_{t+1} &= 0 \\ \beta\left[1-\delta+f_1(k_{t+1}, l(k_{t+1},\lambda_{t+1}))e(k_{t+1}, l(k_{t+1},\lambda_{t+1}))\right]\lambda_{t+1} - \lambda_t &= 0 \end{aligned} \tag{69}$$

Linearizing the first equation around the steady state using (60), (67) and (68) gives after simplifications

$$\begin{aligned} \widehat{k}_{t+1} = \widehat{k}_t\frac{1}{\beta}&\left\{1+\theta\Theta_k + \frac{\theta(1-s)\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\} \\ &+ \widehat{\lambda}_t\frac{1}{s\beta}\left\{\frac{\epsilon_{l\lambda}\theta(1-s)\left(\Theta_l+\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} + (\theta - s\beta\delta)\epsilon_{cc}\right\} \end{aligned} \tag{70}$$

Linearizing the second equation of (69) around the steady state gives

$$\widehat{\lambda}_{t+1} = \widehat{\lambda}_t + \widehat{k}_{t+1}\left[s\Theta_k - \frac{1-s}{\sigma}\right]\theta - \widehat{l}_{t+1}\left[\Theta_l + \frac{1}{\sigma}\right]\theta(1-s) \tag{71}$$

Using (67) finally gives

$$\widehat{\lambda}_{t+1}\left\{1 + \frac{\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\} - \widehat{k}_{t+1}\theta\frac{\frac{1-s}{\sigma} - s\Theta_k - \frac{\epsilon_{lw}}{\sigma}\left[s(\Theta_k-\Theta_l)+\Theta_l\right]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} = \widehat{\lambda}_t \tag{72}$$

Equations (70) and (72) can be expressed as follows

$$\begin{pmatrix} 1 & 0 \\ -A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix}$$

with

$$
\begin{aligned}
A_{21} &= \theta \frac{\frac{1-s}{\sigma}-s\Theta_k-\frac{\epsilon_{lw}}{\sigma}[s(\Theta_k-\Theta_l)+\Theta_l]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} \\
A_{22} &= 1+\frac{\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]} \\
B_{11} &= \frac{1}{\beta}\left\{1+\theta\Theta_k+\frac{\theta(1-s)\left(\frac{1}{\sigma}+\Theta_k\right)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\} \\
B_{12} &= \frac{1}{\beta s}\left\{\frac{\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\theta(1-s)(\epsilon_{lw}\Theta_l+\epsilon_{l\lambda})}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}+(\theta-s\beta\delta)\epsilon_{cc}\right\}
\end{aligned}
$$

The Jacobian matrix J follows after straightforward computations and simplifications. □

7.6 Proof of Lemma 3

We easily derive from Proposition 4 the following characteristic polynomial

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T}(\epsilon_{cc}) + \mathcal{D} \tag{73}$$

with

$$
\mathcal{D} = \frac{1}{\beta}\left\{1+\theta\frac{\Theta_k\left[1+s\frac{\epsilon_{lw}}{\sigma}+(1-s)\epsilon_{l\lambda}\right]+(1-s)\Theta_l\left(\frac{\epsilon_{lw}}{\sigma}-\epsilon_{l\lambda}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}\right\}
$$

$$
\mathcal{T}(\epsilon_{cc}) = 1+\mathcal{D}+\frac{\theta(\theta-s\beta\delta)(1-s)}{\beta s}\frac{\epsilon_{cc}\left[\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}-\frac{\epsilon_{lw}}{\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)\right]+\epsilon_{l\lambda}\left[\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}s\Theta_k}{1-s}\right)+\Theta_l\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}}{\sigma}\right)+\mathcal{C}\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)\right]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}
$$

The analysis of the local stability properties of the model is based on the geometrical methodology of Grandmont *et al.* [30]. In Figure 1, we draw a graph in the trace-determinant $(\mathcal{T}, \mathcal{D})$ space where three relevant lines are considered: line AC ($\mathcal{D} = \mathcal{T} - 1$) along which one eigenvalue of $\mathcal{D}$ is equal to 1, line AB ($\mathcal{D} = -\mathcal{T} - 1$) along which one eigenvalue of $\mathcal{D}$ is equal to -1 and segment BC ($\mathcal{D} = 1$, $|\mathcal{T}| < 2$) along which the two eigenvalues of $\mathcal{D}$ are complex conjugates with modulus equal to 1. These three lines divide the space $(\mathcal{T}, \mathcal{D})$ into three different types of regions according to the number of eigenvalues with modulus smaller than, equal to, and greater than 1. This determines whether the steady state is a sink (locally indeterminate), a source (locally unstable) or a saddle-point (see the corresponding areas in Figure 1).

Then, for any particular calibration of structural parameters, we can compute the trace and determinant using the expression for the Jacobian matrix obtained in Proposition 4 and assess in which area the model is located. We can also assess how these local stability properties change when the calibration of any particular parameter is varied over its admissible range.

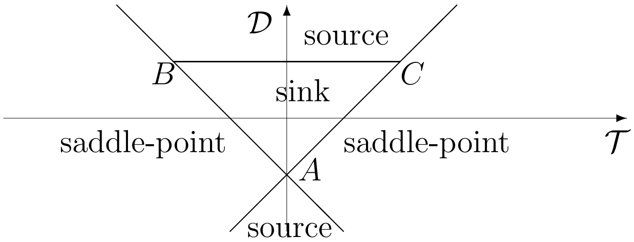

Figure 1: Area of local indeterminacy.

In the one-sector stochastic growth model considered so far, the analysis is greatly simplified by observing that $\mathcal{D}$ does not depend on ϵ_{cc}, implying that the pair $(\mathcal{T}(\epsilon_{cc}), \mathcal{D})$ describes an horizontal line in the $(\mathcal{T}, \mathcal{D})$ space when ϵ_{cc} increases from 0 to $+\infty$. As a result, any Hopf bifurcation related to a Determinant equal to 1 is generically ruled out.

To prove the possible existence of local indeterminacy we need to show that there exist some parameters' configurations such that $\mathcal{D} < 1$ and $1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} > 0$. It is easy to show from the expression of $\mathcal{T}(\epsilon_{cc})$ that a necessary condition to get $1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} > 0$ is

$$\epsilon_{lw} > \frac{1 - \frac{\sigma s \Theta_k}{1-s}}{\Theta_l + \frac{s\Theta_k}{1-s}} \equiv \underline{\epsilon}_{lw}$$

Let us now write the determinant as

$$\mathcal{D} = \frac{1}{\beta} \frac{1 + \epsilon_{lw}\left[\frac{s}{\sigma} - \Theta_l(1-s)\left(1 - \frac{\theta}{\sigma}\right)\right] + \epsilon_{l\lambda}\frac{\theta(1-s)}{\sigma} + \theta\Theta_k\left[1 + s\frac{\epsilon_{lw}}{\sigma} + (1-s)\epsilon_{l\lambda}\right]}{1 + \epsilon_{lw}\left[\frac{s}{\sigma} - \Theta_l(1-s)\right] + \epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma} + \Theta_l\right)}$$

Since under Assumption 2 the expression $\frac{s}{\sigma} - \Theta_l(1-s)\left(1 - \frac{\theta}{\sigma}\right)$ is necessarily positive for any $\sigma > 0$, we get $\mathcal{D} > 0$ for any $\sigma > 0$. Moreover, $\mathcal{D} \leq 1$ if and only if $\Theta_l > \underline{\Theta}_l$ and $\epsilon_{l\lambda} \geq \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ with

$$\underline{\Theta}_l \equiv \frac{1-\beta}{\beta} + \frac{\Theta_k}{\beta}, \ \underline{\epsilon}_{l\lambda}(\epsilon_{lw}) \equiv \frac{1 - \beta + \theta\Theta_k + \epsilon_{lw}\left[\frac{s(1-\beta+\theta\Theta_k)}{\sigma} - \Theta_l(1-s)\left(1 - \beta - \frac{\theta}{\sigma}\right)\right]}{\theta(1-s)(1-\beta+\Theta_k)(\Theta_l - \underline{\Theta}_l)} \tag{74}$$

Under $\sigma \leq \bar{\sigma} \equiv \theta/(1-\beta)$ we get $1 - \beta - \frac{\theta}{\sigma} \leq 0$ so that $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) > 0$ for any $\Theta_l > \underline{\Theta}_l$ and $\epsilon_{lw} \geq 0$. We need therefore to show that $\underline{\Theta}_l < \bar{\Theta}_l$ which is obtained if and only if

$$\Theta_k < \underline{\Theta}_k \equiv \frac{s\beta}{(1-s)\sigma} - (1-\beta)$$

with $\underline{\Theta}_k \in (0, \bar{\Theta}_k)$ under Assumption 5 and $\sigma \leq \bar{\sigma}$.

Obviously, we conclude that $\mathcal{D} > 1$ when $\Theta_l < \underline{\Theta}_l$ for any $\epsilon_{l\lambda} \geq 0$, or when $\Theta_k \in [0, \underline{\Theta}_k)$, $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ and $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

Let us compute the critical values $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ and $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ respectively associated with flip and transcritical bifurcations. The first one is obtained as the solution of $1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} = 0$, namely

$$\epsilon_{cc}^T \equiv \epsilon_{l\lambda} \frac{\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}s\Theta_k}{1-s}\right)+\Theta_l\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}}{\sigma}\right)+\mathcal{C}\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)}{\frac{\epsilon_{lw}}{\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)-\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)} \tag{75}$$

while the second one is obtained as the solution of $1+\mathcal{T}(\epsilon_{cc})+\mathcal{D}=0$, namely

$$\begin{aligned}\epsilon_{cc}^F \equiv\ & \frac{2\left\{1+\beta+\theta\Theta_k+\epsilon_{lw}\left[(1+\beta)\left(\frac{s}{\sigma}-\Theta_l(1-s)\right)+\frac{\theta\Theta_l(1-s)}{\sigma}\right]+\epsilon_{l\lambda}\theta(1-s)\left[\frac{1+\beta}{\sigma}+\beta\Theta_l+\Theta_k\right]\right\}}{\frac{\theta(\theta-s\beta\delta)(1-s)}{s\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)(\epsilon_{lw}-\underline{\epsilon}_{lw})} \\ & + \frac{\epsilon_{l\lambda}\frac{\theta(1-s)(\theta-s\beta\delta)}{s}\left[\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}s\Theta_k}{1-s}\right)+\Theta_l\left(\frac{s(1-\beta)}{\theta-s\beta\delta}+\frac{\mathcal{C}}{\sigma}\right)+\mathcal{C}\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\left(\frac{1}{\sigma}-\frac{s\Theta_k}{1-s}\right)\right]}{\frac{\theta(\theta-s\beta\delta)(1-s)}{s\sigma}\left(\Theta_l+\frac{s\Theta_k}{1-s}\right)(\epsilon_{lw}-\underline{\epsilon}_{lw})}\end{aligned} \tag{76}$$

□

7.7 Proof of Theorem 1

We immediately derive for any Θ_l:

$$1-\mathcal{T}(0)+\mathcal{D}<0 \text{ and } \lim_{\epsilon_{cc}\to+\infty}\mathcal{T}(\epsilon_{cc})=\pm\infty \text{ when } \epsilon_{lw} \lesseqgtr \underline{\epsilon}_{lw}$$

Case 1 - Let us consider first the case with a low wage elasticity for the labor supply, i.e. $\epsilon_{lw}<\underline{\epsilon}_{lw}$. We get the following two graphical configurations depending on the values of Θ_l, Θ_k and $\epsilon_{l\lambda}$:

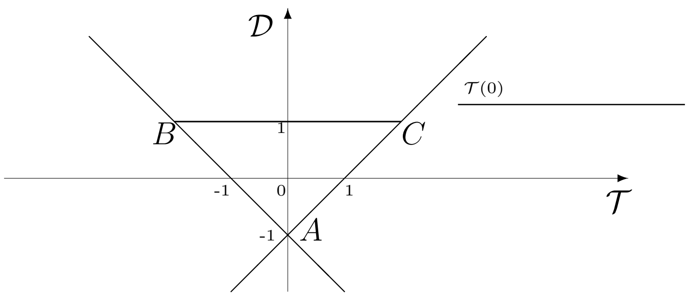

Figure 2: $\epsilon_{lw}<\underline{\epsilon}_{lw}$

As $\mathcal{D}$ does not depend on ϵ_{cc} and, when $\epsilon_{lw}<\underline{\epsilon}_{lw}$, the determinant $\mathcal{D}$ satisfies $\mathcal{D}>1$ or $\mathcal{D}\in(0,1)$ depending on the values of Θ_l and $\epsilon_{l\lambda}$, and we get an horizontal line characterizing the variation of $\mathcal{T}(\epsilon_{cc})$ when ϵ_{cc} is varied over $[0,+\infty)$. Obviously, this line cannot cross the line BC. Moreover, as $1-\mathcal{T}(0)+\mathcal{D}<0$, the starting point when $\epsilon_{cc}=0$ is located below the line AC and we have $\lim_{\epsilon_{cc}\to+\infty}\mathcal{T}(\epsilon_{cc})=+\infty$. The steady state is then a saddle-point for any $\epsilon_{cc}\geq 0$.

Case 2 - Let us consider now the case with a high wage elasticity for the labor supply, i.e. $\epsilon_{lw}>\underline{\epsilon}_{lw}$, and low capital externalities, i.e. $\Theta_k\in[0,\underline{\Theta}_k)$. We get the following graphical configuration:

When $\Theta_l<\underline{\Theta}_l$ or $\Theta_l\in(\underline{\Theta}_l,\bar{\Theta}_l)$ and $\epsilon_{l\lambda}<\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, we have $\mathcal{D}>1$ but now $\lim_{\epsilon_{cc}\to+\infty}\mathcal{T}(\epsilon_{cc})=-\infty$. Local indeterminacy cannot arise but the steady state is not

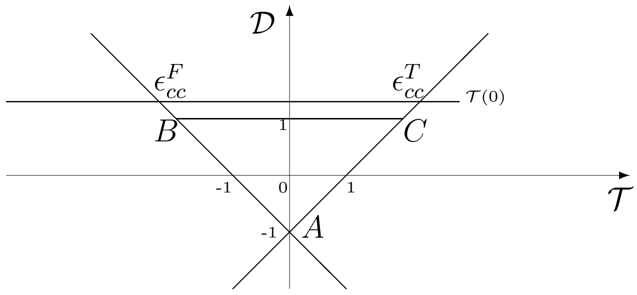

Figure 3: $\epsilon_{lw} > \underline{\epsilon}_{lw}$ and $\Theta_k \in [0, \underline{\Theta}_k)$, with $\Theta_l < \underline{\Theta}_l$ or $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ and $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$

always a saddle-point and can be a source. Indeed, the steady state is saddle-point stable for any $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$ and locally unstable when $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.

When $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ and $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, we get $\mathcal{D} \in (0,1)$ with $\lim_{\epsilon_{cc}\to+\infty} \mathcal{T}(\epsilon_{cc}) = -\infty$. It follows that the line now crosses the triangle ABC and we get indeterminacy:

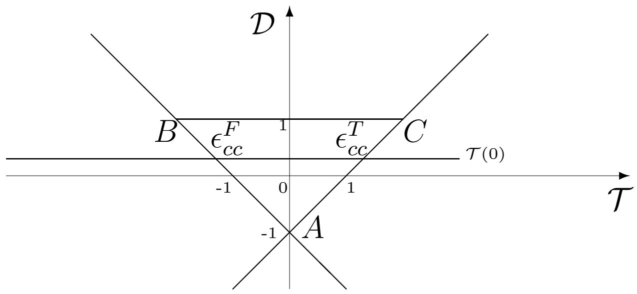

Figure 4: $\epsilon_{lw} > \underline{\epsilon}_{lw}$ and $\Theta_k \in [0, \underline{\Theta}_k)$, with $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$ and $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

As $1 - \mathcal{T}(0) + \mathcal{D} < 0$, the starting point when $\epsilon_{cc} = 0$ is located below the line AC and the steady state is saddle-point stable. As ϵ_{cc} increases, $\mathcal{T}(\epsilon_{cc})$ will cross the line AC when $\epsilon_{cc} = \epsilon_{cc}^T$, implying the existence of a degenerate transcritical bifurcation since the steady state is unique. When $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$, the steady state is locally indeterminate. When $\epsilon_{cc} = \epsilon_{cc}^F$, a flip bifurcation generically occurs leading to the existence of period-two cycles in a right or left neighborhood of ϵ_{cc}^F. Finally, when $\epsilon_{cc} > \epsilon_{cc}^F$, the steady state is again saddle-point stable.

Case 3 - Let us finally consider the case with a high wage elasticity for the labor supply, i.e. $\epsilon_{lw} > \underline{\epsilon}_{lw}$, and large capital externalities, i.e. $\Theta_k \in (\underline{\Theta}_k, \bar{\Theta}_k)$. Since in this case $\underline{\Theta}_l > \bar{\Theta}_l$, we have necessarily $\Theta_l \in (0, \underline{\Theta}_l)$ and thus $\mathcal{D} > 1$. We then get the same configuration as Figure 3. The steady state is saddle-point stable for any $\epsilon_{cc} \in [0, \epsilon_{cc}^T) \cup (\epsilon_{cc}^F, +\infty)$ and locally unstable when $\epsilon_{cc} \in (\epsilon_{cc}^T, \epsilon_{cc}^F)$.

Implausibility of indeterminacy:

Consider the necessary conditions required for indeterminacy underlined in Theorem 1: $\epsilon_{lw} > \underline{\epsilon}_{lw}$, $\Theta_l \in (\underline{\Theta}_l, \bar{\Theta}_l)$, $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ and $\epsilon_{cc} > \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$. Indeterminacy requires at the same time a sufficiently large wage elasticity of the labor supply curve, a sufficiently large degree of IRS in labor, a sufficiently large degree of income effect on labor supply, and a sufficiently large EIS in consumption. These four conditions can only be simultaneously satisfied for extremely high values of the last two elasticities. To see this, consider that the wage elasticity of the labor supply curve is close to the lower bound $\underline{\epsilon}_{lw}$ required for indeterminacy. From Appendix 7.6, we know that ϵ_{cc}^T tends to $+\infty$ when ϵ_{lw} tends to $\underline{\epsilon}_{lw}$, making the condition $\epsilon_{cc} > \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ impossible to satisfy for plausible EIS values. Conversely, consider that the aggregate labor supply curve is very elastic due, for example, to labor indivisibility at the individual level combined with perfect unemployment insurance, as in Hansen [37] and Rogerson [88]. We know that $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ is an increasing function of ϵ_{lw} so the condition $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ required for indeterminacy now imposes very large degrees of income effect on the labor supply. Moreover, $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ is increasing in $\epsilon_{l\lambda}$, so the large degree of income effect has a retroactive large effect on the value for the EIS in consumption required for indeterminacy ($\epsilon_{cc} > \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$). No empirically realistic calibration ensures this outcome.

To fix ideas, consider a simple calibration with $\sigma = 1$, $\Theta_k = 0.25$, Θ_l set to its upper bound $\overline{\Theta}_l$ (the most favorable case for indeterminacy), and $\epsilon_{lw} = 3$ (the value advocated by Rogerson and Wallenius [70] and Prescott and Wallenius [68] to calibrate the wage elasticity of the aggregate labor supply curve in standard RBC/DSGE models). Indeterminacy requires in this case $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda} \simeq 55$ and $\epsilon_{cc} > \epsilon_{cc}^T \simeq 1169$. If, at the other extreme, the wage elasticity of the labor supply curve is increased to 1000 for the same other parameter values (approximating Hansen's [37] type of preferences with an infinitely elastic labor supply curve), indeterminacy now requires $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda} \simeq 1259$ and $\epsilon_{cc} > \epsilon_{cc}^T \simeq 254$. Clearly, no configuration is empirically realistic.[1]

□

7.8 Proof of Proposition 5

The critical value ϵ_{cc}^T given in Lemma 3 provides a lower bound on ϵ_{cc} to get local indeterminacy. Since $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, we can derive the following lower bound for ϵ_{cc}^T:

$$\epsilon_{cc}^T > \underline{\epsilon}_{l\lambda}(\epsilon_{lw}) \frac{\frac{1}{\sigma}\left(\frac{s(1-\beta)}{\theta - s\beta\delta} + \frac{\mathcal{C}s\Theta_k}{1-s}\right) + \Theta_l\left(\frac{s(1-\beta)}{\theta - s\beta\delta} + \frac{\mathcal{C}}{\sigma}\right) + \mathcal{C}\frac{\underline{\epsilon}_{l\lambda}(\epsilon_{lw})}{\epsilon_{lw}}\left(\frac{1}{\sigma} - \frac{s\Theta_k}{1-s}\right)}{\frac{\epsilon_{lw}}{\sigma}\left(\Theta_l + \frac{s\Theta_k}{1-s}\right) - \left(\frac{1}{\sigma} - \frac{s\Theta_k}{1-s}\right)} \equiv \underline{\epsilon}_{cc}^T(\epsilon_{lw})$$

[1] Similar unrealistic values appear for the whole range of potential calibrations regarding σ, Θ_k and Θ_l.

$\underline{\epsilon}_{cc}^{T}(\epsilon_{lw})$ is a decreasing function of ϵ_{lw} over $(\underline{\epsilon}_{lw}, +\infty)$ with $\lim_{\epsilon_{lw} \to \underline{\epsilon}_{lw}} = +\infty$. Straightforward computations then show that under Assumption 5, $\underline{\epsilon}_{cc}^{T} > 2$ when $\epsilon_{l\lambda} > \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$. As a result local indeterminacy is ruled out and the steady state is always a saddle-point. □

7.9 Proof of Proposition 6

We easily conclude that under Assumptions 1-4, local indeterminacy is ruled out for any $\sigma > 0$ in the following cases:

i) when $\Theta_k = \Theta_l = \Theta$, we get

$$\mathcal{D} = \frac{1}{\beta}\left\{1 + \theta \frac{\Theta\left(1+\frac{\epsilon_{lw}}{\sigma}\right)}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]+\epsilon_{l\lambda}\theta(1-s)\left(\frac{1}{\sigma}+\Theta_l\right)}\right\} > \frac{1}{\beta}$$

ii) when $\epsilon_{lw} = 0$ we get

$$\lim_{\epsilon_{lw} \to 0} 1 - \mathcal{T}(\epsilon_{cc}) + \mathcal{D} = -\infty$$

iii) when $\epsilon_{l\lambda} = 0$ we get

$$\mathcal{D} = \frac{1}{\beta}\left\{1 + \theta \frac{\Theta_k + \frac{\epsilon_{lw}}{\sigma}[s\Theta_k + (1-s)\Theta_l]}{1+\epsilon_{lw}\left[\frac{s}{\sigma}-\Theta_l(1-s)\right]}\right\} > \frac{1}{\beta}$$

iv) when $\Theta_l = 0$ we get

$$\mathcal{D} = \frac{1}{\beta}\left\{1 + \theta \frac{\Theta_k\left[1+\frac{s\epsilon_{lw}}{\sigma}+(1-s)\epsilon_{l\lambda}\right]}{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{\epsilon_{l\lambda}\theta(1-s)}{\sigma}}\right\} > \frac{1}{\beta}$$

□

7.10 Proof of Proposition 7

From (45) we derive

$$\widehat{w}_t = \frac{s}{\sigma}\left(\widehat{k}_t - \widehat{l}_t\right) \tag{77}$$

Using this into (65) and (66) then gives

$$\begin{aligned} \widehat{l}_t &= \frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{\lambda}_t + \frac{\frac{s\epsilon_{lw}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t \\ \widehat{c}_t &= \left[\mathcal{C}\left(1-\frac{\epsilon_{l\lambda}}{\epsilon_{lw}}\right)\frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} - \epsilon_{cc}\right]\widehat{\lambda}_t + \frac{\mathcal{C}\frac{s}{\sigma}(\epsilon_{lw}-\epsilon_{l\lambda})}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t \end{aligned} \tag{78}$$

Equation (77) then becomes

$$\widehat{w}_t = \frac{s}{\sigma}\left[\frac{1}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t - \frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{\lambda}_t\right] \tag{79}$$

From the prices r_t and p_t as given by (44) and (46) we finally derive:

$$\begin{aligned}
\widehat{r}_t &= \frac{(1-s)}{\sigma}\left[\frac{\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{\lambda}_t - \frac{1}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t\right] \\
\widehat{p}_t &= -\frac{\Theta}{s\beta\delta}\left\{\frac{s\theta\left[1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}\right]}{1+\frac{s\epsilon_{lw}}{\sigma}}\widehat{k}_t + \left[\frac{\theta(1-s)\epsilon_{l\lambda}^2}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right]\widehat{\lambda}_t\right\}
\end{aligned}$$

Tedious computations based on these results allow to get from the system of difference equations (42)-(43):

$$\begin{pmatrix} 0 & 1 \\ A_{21} & -A_{22} \end{pmatrix}\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{\lambda}_{t+1} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & -B_{22} \end{pmatrix}\begin{pmatrix} \widehat{k}_t \\ \widehat{\lambda}_t \end{pmatrix}$$

with

$$\begin{aligned}
A_{21} &= 1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} - \frac{(1-\delta)\Theta}{s\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right] \\
A_{22} &= \frac{\frac{\theta(1-s)}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \frac{\theta(1-\delta)\Theta}{\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}} \\
B_{11} &= \frac{1+\Theta}{s\beta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right] \\
B_{12} &= \frac{1}{\beta}\left[1 + \frac{\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s\epsilon_{lw}}{\sigma}} + \theta\Theta\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}\right] \\
B_{21} &= 1 - \frac{\Theta}{s\beta\delta}\left[\frac{\theta(1-s)\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}{1+\frac{s\epsilon_{lw}}{\sigma}} + (\theta - s\beta\delta)\epsilon_{cc}\right] \\
B_{22} &= \frac{\theta\Theta}{\beta\delta}\frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}}
\end{aligned}$$

The Proposition follows.

□

7.11 Proof of Lemma 4

We easily derive from Proposition 5 the Determinant and Trace:

$$\begin{aligned}
\mathcal{D} &= \frac{B_{11}B_{22}+B_{12}B_{21}}{A_{21}} \\
\mathcal{T} &= 1 + \mathcal{D} + \frac{(B_{21}-A_{21})(1-B_{12})+B_{11}(A_{22}-B_{22})}{A_{21}}
\end{aligned}$$

The characteristic polynomial is then

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) \tag{80}$$

with

$$\begin{aligned}
\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= \tfrac{1}{\beta}\left\{1 + \Theta\theta \frac{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{(1-s)\epsilon_{l\lambda}}{\sigma}}{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{\theta(1-s)\epsilon_{l\lambda}}{\sigma}-\frac{\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left[\frac{\mathcal{C}\epsilon_{l\lambda}^2}{\epsilon_{lw}}+\epsilon_{cc}(1+\frac{s\epsilon_{lw}}{\sigma})\right]}\right\} \\
\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) \\
&+ \frac{\frac{\theta(\theta-s\beta\delta)}{s\beta}\left\{\epsilon_{l\lambda}\left[\frac{(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}}{\sigma}\right]+\frac{\mathcal{C}\epsilon_{l\lambda}^2}{\epsilon_{lw}}\left[\frac{1-s}{\sigma}-\Theta\left(1-\frac{1-s}{\sigma}\right)\right]+\epsilon_{cc}\left[\frac{1-s}{\sigma}-\Theta\left(1-\frac{1-s}{\sigma}\right)-\Theta\frac{s}{\sigma}\epsilon_{lw}\right]\right\}}{1+\frac{s\epsilon_{lw}}{\sigma}+\frac{\theta(1-s)\epsilon_{l\lambda}}{\sigma}-\frac{\Theta(1-\delta)(\theta-s\beta\delta)}{s\delta}\left[\frac{\mathcal{C}\epsilon_{l\lambda}^2}{\epsilon_{lw}}+\epsilon_{cc}(1+\frac{s\epsilon_{lw}}{\sigma})\right]} \\
&\equiv 1 + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{X}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)
\end{aligned}$$

It is easy to derive that as the parameter ϵ_{cc} is varied over the interval $(0, +\infty)$, $\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)$ and $\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)$ are linked through a linear relationship $\Delta(\mathcal{T})$ such that

$$\mathcal{D} = \Delta(\mathcal{T}) = \mathcal{S}(\epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)\mathcal{T} + \mathcal{M}$$

with

$$\mathcal{S}(\epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = \frac{\partial\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \Theta)/\partial\epsilon_{cc}}{\partial\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \Theta)/\partial\epsilon_{cc}}$$

which does not depend on ϵ_{cc}.

Straightforward computations show that $\partial\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)/\partial\epsilon_{cc} > 0$ and, under Assumptions 1, 3, 4, 5 and 6, with $\sigma < \bar{\sigma}$, with $\bar{\sigma} = \min\{3.3, \tilde{\sigma}\}$ and $\tilde{\sigma} = (1-s)(1+\Theta)/\Theta$ and $\Theta \in (0, 0.44)$, $\partial\mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)/\partial\epsilon_{cc} > 0$. It follows that $\mathcal{S}(\epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ and thus $\Delta(\mathcal{T})$ is a line in the space $(\mathcal{T}, \mathcal{D})$ with a positive slope. In order to locate this line, we need to compute the starting and end points $(\mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta), \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta))$ and $(\mathcal{T}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta), \mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta))$. We easily get

$$\begin{aligned}
\mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= \tfrac{1}{\beta}\left\{1 + \Theta\theta \frac{1+\frac{s}{\sigma}\epsilon_{lw}+\frac{(1-s)}{\sigma}\epsilon_{l\lambda}}{1+\frac{s}{\sigma}\epsilon_{lw}+\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}-\frac{\Theta(1-\delta)\theta(1-s)}{s\delta}\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}\right\} \equiv \mathcal{D}_0 \\
\mathcal{X}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= \frac{\frac{\theta(\theta-s\beta\delta)\epsilon_{l\lambda}[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}{s\beta\sigma\epsilon_{lw}}}{1+\frac{s}{\sigma}\epsilon_{lw}+\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}-\frac{\Theta(1-\delta)\theta(1-s)}{s\delta}\frac{\epsilon_{l\lambda}^2}{\epsilon_{lw}}}
\end{aligned}$$

with

$$\hat{\epsilon}_{lw} \equiv \frac{\Theta\mathcal{C}\epsilon_{l\lambda}(\sigma-\tilde{\sigma})}{(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}} \text{ and } \tilde{\sigma} \equiv \frac{(1-s)(1+\Theta)}{\Theta}$$

In the rest of the proof we assume that $\sigma < \tilde{\sigma}$ so that the bound $\hat{\epsilon}_{lw} < 0$ is no longer relevant. It follows that $\mathcal{D}_0$ satisfies:

- $\mathcal{D}_0 > 1/\beta$ if and only if $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})(\epsilon_{lw})$ with

$$\epsilon_{l\lambda}^0(\epsilon_{lw}) \equiv \frac{\frac{\theta(1-s)s\delta}{\sigma}\epsilon_{lw}+\sqrt{\left[\frac{\theta(1-s)s\delta}{\sigma}\epsilon_{lw}\right]^2+4\Theta(1-\delta)\theta(1-s)\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}}}{2\Theta(1-\delta)\theta(1-s)}$$

- $\mathcal{D}_0 \in (-\infty, 1)$ if and only if $\epsilon_{l\lambda} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$ with

$$\bar{\epsilon}_{l\lambda}(\epsilon_{lw}) \equiv \frac{\frac{\theta(1-s)s\delta\epsilon_{lw}(1-\beta+\Theta)}{\sigma}+\sqrt{\left[\frac{\theta(1-s)s\delta\epsilon_{lw}(1-\beta+\Theta)}{\sigma}\right]^2+4\Theta(1-\beta)(1-\delta)\theta(1-s)\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}(1-\beta+\Theta\theta)}}{2\Theta(1-\beta)(1-\delta)\theta(1-s)} > \epsilon_{l\lambda}^0(\epsilon_{lw})$$

- $\mathcal{D}_0 \in (1, 1/\beta)$ if and only if $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

Under $\sigma < \tilde{\sigma}$ we also immediately conclude that

$$1 - \mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = -\mathcal{X}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) < 0$$

if and only if $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})$. Moreover, we easily derive that

$$1 + \mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$$

if and only if $\epsilon_{l\lambda} \in (0, \epsilon_{l\lambda}^0(\epsilon_{lw})) \cup (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), +\infty)$ with

$$\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}) \equiv \frac{\frac{\theta\delta\epsilon_{lw}}{\sigma}\left\{2(1-s)s(1+\beta+\Theta)+(\theta-s\beta\delta)[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}]\right\}+\sqrt{\Delta}}{2\Theta\theta(1-s)\left[2(1-\delta)(1+\beta)+\frac{\theta\delta(\sigma-\tilde{\sigma})}{\sigma}\right]} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$$

and

$$\begin{aligned}\Delta = &\left(\frac{\theta\delta\epsilon_{lw}\left\{2(1-s)s(1+\beta+\Theta)+(\theta-s\beta\delta)[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}]\right\}}{\sigma}\right)^2 \\ &+ 8\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}(1+\beta+\Theta\theta)\Theta\theta(1-s)\left[2(1-\delta)(1+\beta)+\frac{\theta\delta(\sigma-\tilde{\sigma})}{\sigma}\right]\end{aligned} \tag{81}$$

while

$$1 + \mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) < 0$$

if and only if $\epsilon_{l\lambda} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

Finally, we have

$$\begin{aligned}\mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= \frac{1}{\beta} \equiv \mathcal{D}_\infty \in (1, \mathcal{D}_0) \\ \mathcal{X}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) &= \frac{\theta\delta}{\sigma\beta}\frac{s(\epsilon_{lw}-\underline{\epsilon}_{lw})}{(1-\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)}\end{aligned}$$

with

$$\underline{\epsilon}_{lw} \equiv \frac{\tilde{\sigma}-\sigma}{s}$$

We conclude here that under $\sigma < \tilde{\sigma}$, $\mathcal{X}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ and thus

$$1 - \mathcal{T}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = -\mathcal{X}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \Theta) < 0$$

if and only if $\epsilon_{lw} > \underline{\epsilon}_{lw}$.

Under $\sigma < \tilde{\sigma}$, we then get the following conclusions:

- if $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})$ then $1 - \mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) < 0$ for any $\epsilon_{lw} \geq 0$ while $1 - \mathcal{T}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ if and only if $\epsilon_{lw} < \underline{\epsilon}_{lw}$;

- if $\epsilon_{l\lambda} > \epsilon_{l\lambda}^0(\epsilon_{lw})$ then $1 - \mathcal{T}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(0, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ for any $\epsilon_{lw} \geq 0$ while $1 - \mathcal{T}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(+\infty, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > 0$ if and only if $\epsilon_{lw} < \underline{\epsilon}_{lw}$.

Let us compute the critical values $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ and $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ respectively associated with Hopf, transcritical and flip bifurcations. The first one $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$ is obtained solving the equality $\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = 1$ with respect to ϵ_{cc}. Straightforward computations yield the value

$$\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}) = \frac{\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)(1-\beta+\Theta\theta)+\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}(1-\beta+\Theta)-\Theta\theta(1-\beta)(1-\delta)(1-s)\frac{\epsilon_{l\lambda}^2}{s\delta\epsilon_{lw}}}{\Theta(1-\beta)\frac{(1-\delta)}{s\delta}(\theta-s\beta\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)}$$

It follows obviously that for any given ϵ_{lw}, $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}) \geq 0$ if and only if $\epsilon_{l\lambda} \leq \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

The critical value $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ is obtained as the solution of the equality $\mathcal{X}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = 0$, namely

$$\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}) \equiv \frac{\epsilon_{l\lambda}[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}{\Theta s\epsilon_{lw}(\epsilon_{lw}-\underline{\epsilon}_{lw})}$$

Under $\sigma < \tilde{\sigma}$, we have $\hat{\epsilon}_{lw} < 0$ and thus $\epsilon_{lw} - \hat{\epsilon}_{lw} > 0$. So, obviously, $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}) > 0$ if and only if $\epsilon_{lw} > \underline{\epsilon}_{lw}$. By convention, we will consider that $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}) = +\infty$ when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$. Note that since the steady state is generically unique, $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ corresponds to a degenerate transcritical bifurcation.

The critical value $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ is finally obtained as the solution of the equality $1 + \mathcal{T}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) + \mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = 0$, namely

$$\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda}) = \frac{\theta(1-s)\left[2(1-\delta)(1+\beta)+\frac{\theta\delta(\sigma-\tilde{\sigma})}{\sigma}\right]\left(\epsilon_{l\lambda}^-(\epsilon_{lw})-\epsilon_{l\lambda}\right)(\epsilon_{l\lambda}-\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))}{(\theta-s\beta\delta)\epsilon_{lw}\left[2(1+\beta)(1-\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)+\frac{\theta s\delta}{\sigma}(\epsilon_{lw}-\underline{\epsilon}_{lw})\right]}$$

with

$$\epsilon_{l\lambda}^-(\epsilon_{lw}) = \frac{\frac{\theta\delta\epsilon_{lw}}{\sigma}\left\{2(1-s)s(1+\beta+\Theta)+(\theta-s\beta\delta)[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}]\right\}-\sqrt{\Delta}}{2\Theta\theta(1-s)\left[2(1-\delta)(1+\beta)+\frac{\theta\delta(\sigma-\tilde{\sigma})}{\sigma}\right]}$$

and Δ as given by (81). Obviously, $\epsilon_{l\lambda}^-(\epsilon_{lw}) < 0$ and we conclude that $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda}) \geq 0$ if and only if $\epsilon_{l\lambda} \leq \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. This critical value corresponds to a flip bifurcation giving rise to the existence of period-two cycles.

As this will become obvious later on, we need now to check that $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}) \leq \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$. This inequality is satisfied if and only if $\epsilon_{l\lambda} \geq \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$ with

$$\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = \frac{-\sigma(1-s)\left\{(1-\beta+\Theta\theta)(1-\beta)(1-\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)-\frac{\theta}{\sigma}(1-\beta+\Theta)s\delta(\epsilon_{lw}-\underline{\epsilon}_{lw})\right\}+\sigma\sqrt{\hat{\Delta}}}{2\Theta\theta(1-\beta)(1-\delta)(1-s)\tilde{\sigma}}$$

and

$$\begin{aligned}\hat{\Delta} &= (1-s)^2\Big\{(1-\beta+\Theta\theta)(1-\beta)(1-\delta)\left(1+\tfrac{s}{\sigma}\epsilon_{lw}\right) - \tfrac{\theta}{\sigma}(1-\beta+\Theta)s\delta(\epsilon_{lw}-\underline{\epsilon}_{lw})\Big\}^2 \\ &+ \tfrac{4\Theta\theta(1-\beta)(1-\delta)(1-s)\tilde{\sigma}}{\sigma}\left(1+\tfrac{s}{\sigma}\epsilon_{lw}\right)(1-\beta+\Theta\theta)s\delta(\epsilon_{lw}-\underline{\epsilon}_{lw})\end{aligned}$$

Note that $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) > 0$ if and only if $\epsilon_{lw} > \underline{\epsilon}_{lw}$. By convention, we will consider that $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0$ when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$.

When $\epsilon_{l\lambda} < \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$, the Hopf bifurcation is always ruled out. In order to locate the $\Delta(\mathcal{T})$ line we need to check whether $\mathcal{D} = -1$ can occur, and if yes, we need to know the sign of $\mathcal{T}$ when $\mathcal{D} = -1$. If the sign is positive then the $\Delta(\mathcal{T})$ line is located below the triangle ABC and local indeterminacy is ruled out. On the contrary, if the sign is negative then the $\Delta(\mathcal{T})$ line may cross the triangle ABC leading to the possible existence of local indeterminacy. Solving $\mathcal{D}(\epsilon_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = -1$ with respect to ϵ_{cc} gives

$$\bar{\epsilon}_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) = \frac{\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)(1+\beta+\Theta\theta)+\frac{\theta(1-s)}{\sigma}\epsilon_{l\lambda}(1+\beta+\Theta)-\Theta\theta(1+\beta)(1-\delta)(1-s)\frac{\epsilon_{l\lambda}^2}{s\delta\epsilon_{lw}}}{\Theta(1+\beta)\frac{(1-\delta)}{s\delta}(\theta-s\beta\delta)\left(1+\frac{s}{\sigma}\epsilon_{lw}\right)}$$

Straightforward computations show that $\bar{\epsilon}_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}) > 0$ if and only if $\epsilon_{l\lambda} > \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ with

$$\begin{aligned}\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}) &\equiv \frac{\frac{\theta(1-s)s\delta\epsilon_{lw}(1+\beta+\Theta)}{\sigma}+\sqrt{\left[\frac{\theta(1-s)s\delta\epsilon_{lw}(1+\beta+\Theta)}{\sigma}\right]^2+4\Theta(1+\beta)(1-\delta)\theta(1-s)\left(1+\frac{s\epsilon_{lw}}{\sigma}\right)s\delta\epsilon_{lw}(1+\beta+\Theta\theta)}}{2\Theta(1+\beta)(1-\delta)\theta(1-s)} \\ &\in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))\end{aligned}$$

If $\epsilon_{l\lambda} > \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ we derive that

$$\mathcal{T}(\bar{\epsilon}_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = \chi(\bar{\epsilon}_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta)$$

and straightforward computations yield $\mathcal{T}(\bar{\epsilon}_{cc}, \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) \geq 0$ if and only if $\epsilon_{l\lambda} \geq \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ with

$$\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) \equiv \frac{-(1-s)\left\{\frac{(1-\beta+\Theta\theta)(1+\beta)(1-\delta)}{\delta}+\frac{\theta(1+\beta+\Theta)(\tilde{\sigma}-\sigma)}{\sigma}+\frac{s\epsilon_{lw}}{\delta\sigma}\left[(1+\beta)[\theta(1-\delta)-\delta]+\Theta\theta[(1+\beta)(1-\delta)-\delta]\right]\right\}+\sqrt{\underline{\Delta}}}{\frac{2\Theta(1+\beta)(1-\delta)\theta(1-s)\tilde{\sigma}}{\delta\sigma}} < \epsilon_{l\lambda}^0(\epsilon_{lw})$$

and

$$\begin{aligned}\underline{\Delta} &= (1-s)^2\Big\{\tfrac{(1-\beta+\Theta\theta)(1+\beta)(1-\delta)}{\delta} + \tfrac{\theta(1+\beta+\Theta)(\tilde{\sigma}-\sigma)}{\sigma} + \tfrac{s\epsilon_{lw}}{\delta\sigma}\big[(1+\beta)[\theta(1-\delta)-\delta] \\ &+ \Theta\theta[(1+\beta)(1-\delta)-\delta]\big]\Big\}^2 + 4\left(1+\tfrac{s}{\sigma}\epsilon_{lw}\right)(1+\beta+\Theta\theta)s(\epsilon_{lw}-\underline{\epsilon}_{lw})\tfrac{\Theta(1+\beta)(1-\delta)\theta(1-s)\tilde{\sigma}}{\delta\sigma}\end{aligned}$$

Note that $\underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ is obviously such that $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda}) = \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ when $\epsilon_{l\lambda} = \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

□

7.12 Proof of Theorem 2

Under Assumptions 1, 3, 4, 5 and 6, let $\Theta \in (0, \bar{\Theta})$ and $\sigma < \bar{\sigma}$, with $\bar{\Theta} = \min\{s/(1-s)\sigma, 0.44\}$, $\bar{\sigma} = \min\{2, \tilde{\sigma}\}$ and $\tilde{\sigma} = (1-s)(1+\Theta)/\Theta$. Straightforward computations show that if $\epsilon_{lw} > \underline{\epsilon}_{lw}$, $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$. We need now to check whether $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) \gtrless \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. Since $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0 < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$ when $\epsilon_{lw} = \underline{\epsilon}_{lw}$, obvious computations then show that there exists a unique value $\bar{\epsilon}_{lw} > \underline{\epsilon}_{lw}$ such that $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$ if and only if $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$ if and only if $\epsilon_{lw} = \bar{\epsilon}_{lw}$.

Let us consider now the value of the Trace when $\epsilon_{cc} = \epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$. We get

$$\mathcal{T}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \Theta) = 2 + \mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \Theta)$$

with

$$\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) = -\frac{\frac{\theta(1-\beta)(\theta-s\beta\delta)\epsilon_{l\lambda}}{\sigma s\beta} \frac{[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}{\epsilon_{lw}} \left\{1+\frac{\Theta s\epsilon_{cc}^H(\epsilon_{lw},\epsilon_{l\lambda})\epsilon_{lw}(\underline{\epsilon}_{lw}-\epsilon_{lw})}{\epsilon_{l\lambda}[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}\right\}}{\Theta\theta[1+s\epsilon_{lw}+(1-s)\epsilon_{l\lambda}]}$$

We easily derive that under $\sigma < \tilde{\sigma}$, $\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) < 0$ and thus $\mathcal{T}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \Theta) < 2$ if and only if $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

We need now to provide a condition to get $\mathcal{T}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > -2$ or equivalently $\mathcal{X}(\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{lw}, \epsilon_{l\lambda}, \sigma, \Theta) > -4$. We get

$$\frac{(1-\beta)(\theta-s\beta\delta)\epsilon_{l\lambda}}{\epsilon_{lw}} \frac{[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}{\sigma} \left\{1+\frac{\Theta s\epsilon_{cc}^H(\epsilon_{lw},\epsilon_{l\lambda})\epsilon_{lw}(\underline{\epsilon}_{lw}-\epsilon_{lw})}{\sigma\epsilon_{l\lambda}[(1-s)(1-\mathcal{C})+\Theta s\mathcal{C}](\epsilon_{lw}-\hat{\epsilon}_{lw})}\right\}$$
$$< 4\beta\Theta\left[1+\frac{s}{\sigma}\epsilon_{lw}+\frac{(1-s)}{\sigma}\epsilon_{l\lambda}\right]$$

Tedious but straightforward computations show that if $\epsilon_{l\lambda} < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$, the previous inequality is satisfied.

Case **1** - Let us consider in a first step the case $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ where $\hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0$. We get the following geometrical characterizations of the $\Delta(\mathcal{T})$ line depending on the value of $\epsilon_{l\lambda}$. When $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})$ or $\epsilon_{l\lambda} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$, we have:

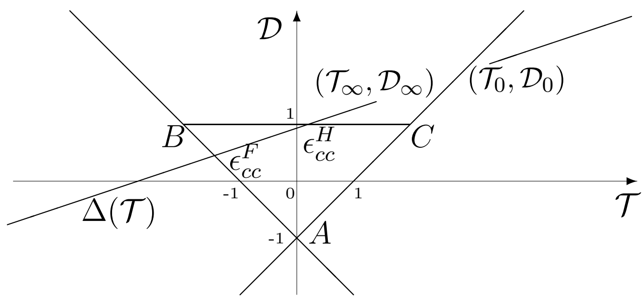

Figure 5: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} < \epsilon_{l\lambda}^0(\epsilon_{lw})$.

As shown by these Figures, increasing ϵ_{cc} from 0, the steady state is first saddle-point

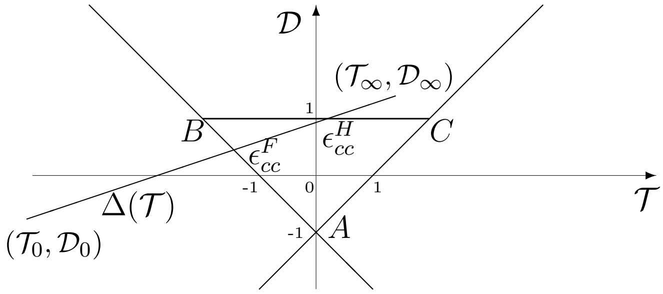

Figure 6: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\epsilon_{l\lambda}^0(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

stable. Still increasing ϵ_{cc} leads to the existence of a flip bifurcation giving rise to the existence of period-two cycles when ϵ_{cc} crosses $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$. Above $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ the steady state then becomes locally indeterminate and when ϵ_{cc} crosses the bound $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, a Hopf bifurcation occurs giving rise to the existence of periodic cycles. Above $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, the steady state is totally unstable for any $\epsilon_{cc} \in (\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), +\infty)$.

When $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$, we obviously get the following case where the flip bifurcation no longer exists, i.e. $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$:

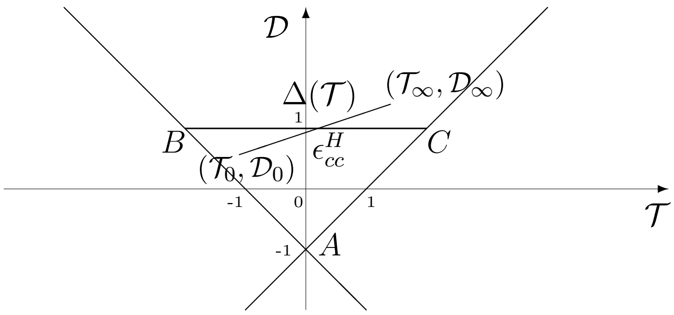

Figure 7: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

Finally, when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$, we get the following case where the Hopf bifurcation no longer exists, i.e. $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$:

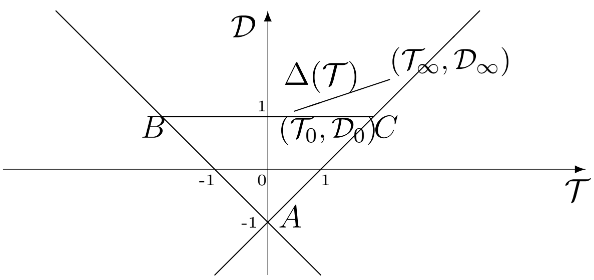

Figure 8: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (0, \underline{\epsilon}_{lw})$ and $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

Case **2** - Let us consider now the case $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ where $0 < \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. We get $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) < \epsilon^0_{l\lambda}(\epsilon_{lw}) < \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}) < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$ and $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) < \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \tilde{\epsilon}_{l\lambda}(\epsilon_{lw})$. However, the bounds $\epsilon^0_{l\lambda}(\epsilon_{lw})$ and $\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ may be lower or larger than $\hat{\epsilon}_{l\lambda}(\epsilon_{lw})$ but this does not really impact the local stability results. It is worth noticing that when $\epsilon_{lw} = \underline{\epsilon}_{lw}$, $\underline{\epsilon}_{l\lambda}(\epsilon_{lw}) = \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) = 0$. We then obtain the following geometrical characterizations of the $\Delta(\mathcal{T})$ line depending on the value of $\epsilon_{l\lambda}$ and ϵ_{lw}. Recall that $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and consider for now that $\epsilon^0_{l\lambda}(\epsilon_{lw}) < \bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}) < \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$. When $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$ we have

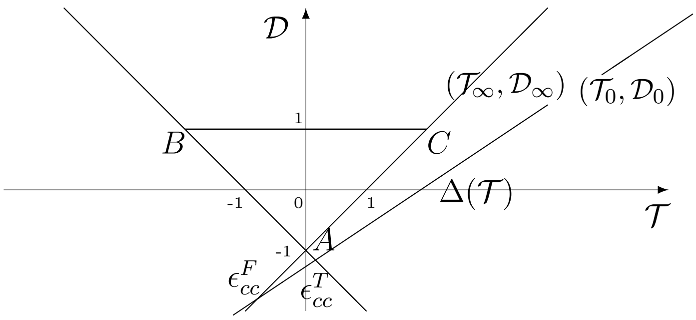

Figure 9: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$.

When $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$ we get the following three cases:

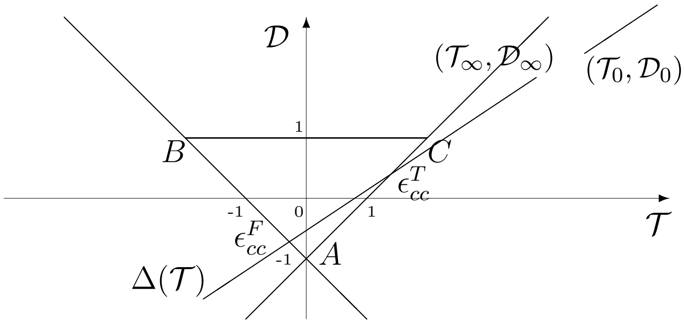

Figure 10: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}(\epsilon_{lw}), \epsilon^0_{l\lambda}(\epsilon_{lw}))$.

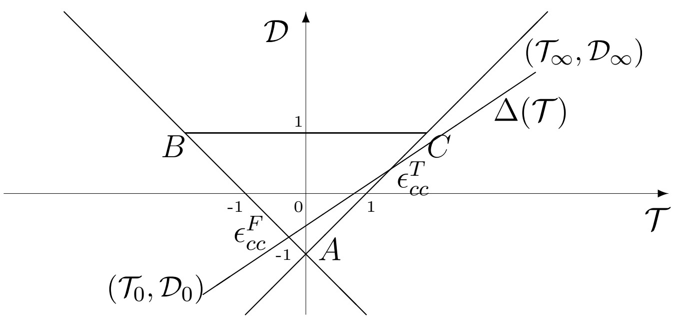

Figure 11: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\epsilon^0_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

In these three cases, local indeterminacy arises if $\epsilon_{cc} \in (\epsilon^F_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon^T_{cc}(\epsilon_{lw}, \epsilon_{l\lambda}))$ and saddle-point stability holds outside this interval. The same conclusion would be obtained is $\epsilon^0_{l\lambda}(\epsilon_{lw})$ and/or $\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw})$ were larger than $\hat{\epsilon}_{l\lambda}(\epsilon_{lw})$.

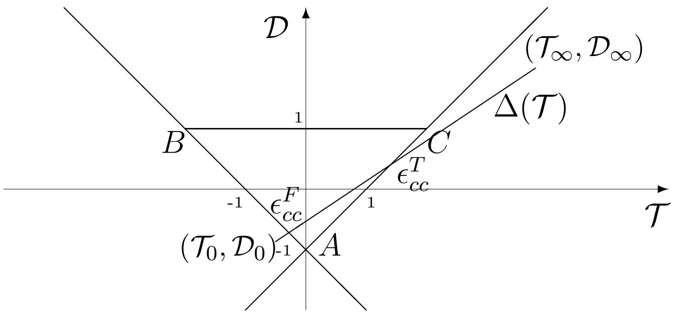

Figure 12: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\bar{\bar{\epsilon}}_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

When $\epsilon_{l\lambda} > \hat{\epsilon}_{l\lambda}(\epsilon_{lw})$, the Hopf bifurcation may again occur as long as $\epsilon_{l\lambda} < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$. We get indeed:

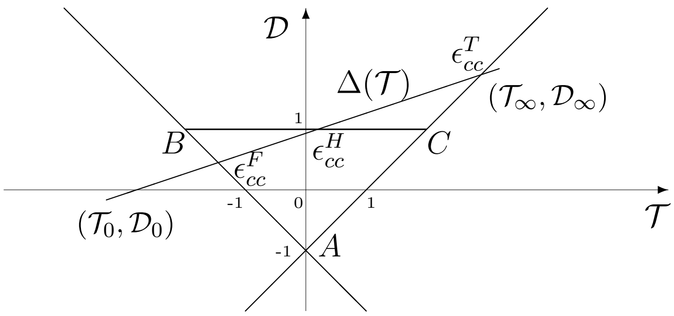

Figure 13: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

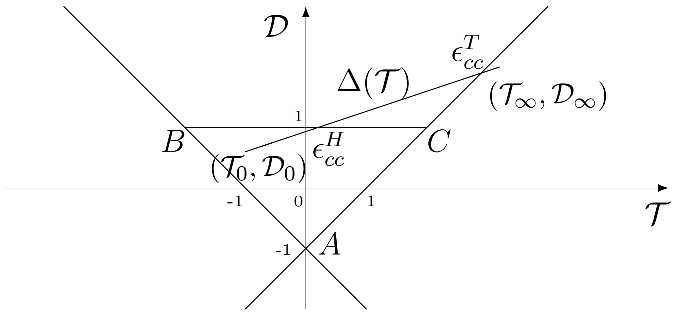

Figure 14: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

As shown by these Figures, when $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$, increasing ϵ_{cc} from 0, the steady state is first saddle-point stable. Still increasing ϵ_{cc} leads to the existence of a flip bifurcation giving rise to the existence of period-two cycles when ϵ_{cc} crosses $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$. Above $\epsilon_{cc}^F(\epsilon_{lw}, \epsilon_{l\lambda})$ the steady state then becomes locally indeterminate and when ϵ_{cc} crosses the bound $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, a Hopf bifurcation occurs giving rise to the existence of periodic cycles. Above $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda})$, the steady state is locally unstable when $\epsilon_{cc} \in (\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}), \epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}))$ and saddle-point stable when $\epsilon_{cc} \in (\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}), +\infty)$. Since the steady state is generically unique, $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda})$ corresponds to a degenerate transcritical bifurcation. When $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$, we get $\epsilon_{cc}^T(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$ and when $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$ we get $\epsilon_{cc}^H(\epsilon_{lw}, \epsilon_{l\lambda}) = 0$.

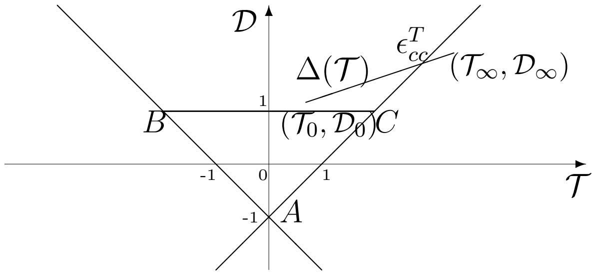

Figure 15: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} \in (\underline{\epsilon}_{lw}, \bar{\epsilon}_{lw})$ and $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$.

Case **3** - Let us consider finally the case $\epsilon_{lw} > \bar{\epsilon}_{lw}$ where $\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}) < \hat{\epsilon}_{l\lambda}(\epsilon_{lw}) < \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$. Therefore, using all our previous results, we get the following geometrical characterizations of the $\Delta(\mathcal{T})$ line depending on the value of $\epsilon_{l\lambda}$. When $\epsilon_{l\lambda} < \underline{\epsilon}_{l\lambda}(\epsilon_{lw})$, local indeterminacy is ruled out as the $\Delta(\mathcal{T})$ does not cross the triangle ABC as in Figure 9.

On the contrary, when $\epsilon_{l\lambda} \in (\underline{\epsilon}_{l\lambda}(\epsilon_{lw}), \tilde{\epsilon}_{l\lambda}(\epsilon_{lw}))$, local indeterminacy can arise as we get the following three cases where the $\Delta(\mathcal{T})$ crosses the triangle ABC as in Figures 10, 11 and 12.

When $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$ the flip bifurcation no longer exists.

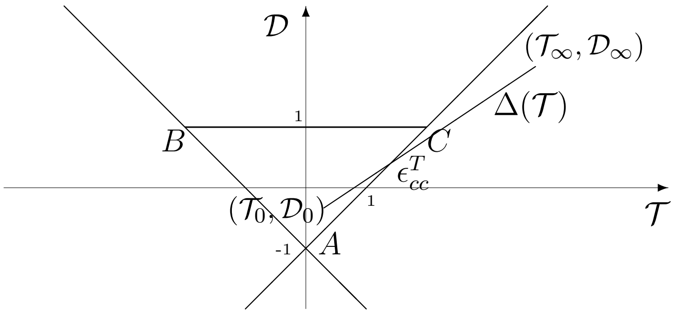

Figure 16: $\Delta(\mathcal{T})$ line when $\epsilon_{lw} > \bar{\epsilon}_{lw}$ and $\epsilon_{l\lambda} \in (\tilde{\epsilon}_{l\lambda}(\epsilon_{lw}), \hat{\epsilon}_{l\lambda}(\epsilon_{lw}))$.

When $\epsilon_{l\lambda} \in (\hat{\epsilon}_{l\lambda}(\epsilon_{lw}), \bar{\epsilon}_{l\lambda}(\epsilon_{lw}))$, the Hopf bifurcation exists as the same configuration as Figure 14 occurs. When $\epsilon_{l\lambda} > \bar{\epsilon}_{l\lambda}(\epsilon_{lw})$, the Hopf bifurcation no longer exists and local indeterminacy is again ruled out as the same configuration as Figure 15 occurs.

□