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Do people share opportunities?

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Abstract

A set of agents is aware of the existence of an economic opportunity, and compete for the associated prize. We study incentives to communicate about the existence of this economic opportunity to uninformed agents when the winner of the prize shares it with others, through some exogenous sharing rule. Communicating about the opportunity has two conflicting effects: it increases competition, but it can also increase the likelihood of receiving a large share of the prize. We find that, for any sharing rule, there is a minimum equilibrium, which Pareto dominates all other equilibria. We also find that under bilaterally symmetric sharing, more sharing generates more communication. We then discuss these results along several extensions. (JEL: C72; D83; D85)

Keywords: Rival Opportunity; Sharing Network; Communication; Investment.

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1. Introduction

Economic opportunities are a cornerstone of economic growth. Two salient features of opportunities, that they are often rival by nature and that they are not easily identified, appear in many economic contexts. For instance, firms / researchers competing over an innovation may lack information necessary to evaluate the riskiness of an innovative project, or even to select scientific direction to take. Competing traders in banks may have trouble identifying relevant information about assets' returns. Attorneys can lack the information required to evaluate the true benefit they might gain from defending a potential client. Or a farmer may have difficulty gathering the information necessary to evaluate how well rival banks can meet his needs. The limited amount of relevant information to identify valuable opportunities is a strong limitation to innovation and to economic development.

For this reason, understanding how information about economic opportunities spreads in society is of primary importance. In particular, in a purely competitive framework where only few agents will actually seize the opportunity in the end, information sharing is likely to be very limited. However, in many contexts, the value generated by the opportunity is shared among economic agents. For instance, spreading industrial innovations is of benefit to many firms[1]; wages in organizations often partly redistribute the value of aggregate output to employees; or in village economies, social networks may serve as a channel for resource-sharing (through cultural norms, diversification motives, favors, altruism, etc). When the value generated by the opportunity is indeed shared among agents through some "sharing rule", it might provide incentives to communicate about the existence of the opportunity. This raises the following question: How do the characteristics of the sharing rule affect incentives to communicate about the existence of rival opportunities?

1. More generally, such sharing device might occur when the opportunity concerns a good containing a public good aspect

We model this question through a simple normal-form game. The existence of the economic opportunity (which we refer to as a *pie*), as well as the sharing rule, that we call the *sharing network*[2], are common knowledge among a set of *initially informed* agents. Agents then simultaneously choose to inform a set of uninformed agents. Finally all informed - whether initially or through communication - agents compete for the pie, and redistribute shares according to the sharing rule.

Our findings are the following. Our first result pertains to equilibrium characterization. We show that best-response strategies possess the following monotonicity property: an agent's best response can only be increased when others communicate more. This property implies that the communication game admits a minimum equilibrium (in terms of the set of informed agents), which Pareto-dominates all other equilibria (whereas information receivers might be better off in other equilibria).

Second, we address the impact of network structure on communication, by exploring whether more sharing is always beneficial to communication. The answer depends on the sharing rule. However, when shares are bilaterally symmetric, more sharing always fosters communication. In particular, increasing shares symmetrically (but not necessarily homogeneously across links) necessarily augments the set of informed agents at the minimum equilibrium.

We then discuss the robustness of our results to a series of variations or extensions. First, we test the monotonicity property in various contexts: we consider a setting in which only those who inform get a share of the pie; we allow for probability of winning to be increasing in the number of informed agents; we explore incentives to exert effort to win the contest. The monotonicity property only holds in the two former

2. A sharing rule is simply a stochastic matrices, whose i, j entry is simply equal to the fraction of the prize agent i transfers to agent j. We assume that there is no enforcement issue. Social norms may explain commitment to sharing. In the economic literature, contract enforcement issue can be overcome through reciprocity (see Fehr et al. (1997)), or through reputation building motive (like in relational contract theory, see Baker et al. (1994)).

situations. Second, we examine the robustness of the positive relationship between symmetric sharing and communication, by examining endogenous (costly) investment to be informed about the opportunity. We find that, under low investment cost, more sharing never reduces communication in the presence of endogenous investment. By contrast, it can be the case under large investment cost, by two mechanisms: first, investment amplifies the temptation of free riding, and, second, incentives to invest are worsened when investment boosts communication.

Related literature. This paper relates to at least two strands of literature on networks: resource sharing on networks and strategic communication on networks.[3]

The first, and perhaps closest, examines resource-sharing on networks, focusing mainly on informal sharing. There is a huge empirical literature about informal transfers on networks (see for instance Fafchamp and co-authors, like Fafchamps and Gubert (2007) or De Weerdt and Fafchamps (2011)), and several theoretical works: Bramoullé and Kranton (2007a) and Bramoullé and Kranton (2007b) examine the formation of risk-sharing networks, where people share equally in communities. Bloch et al. (2008) study sharing under the threat of opportunism. Ambrus et al. (2014) examine risk-sharing under capacity constraints, and Ambrus et al. (2022) explore the role of local information as a limit to contracting. Bourlès et al. (2017) study the role of altruism as a redistributive mechanism in networks. Ambrus and Elliott (2021) model the formation of risk-sharing networks under division of surplus in social networks related to the Myerson value. In our model, the network is a channel for both transfers and information. With respect to that literature, one novelty of our work is the nature of the information transmitted on networks: the information here concerns a rival opportunity. Ultimately, this paper complements that literature by examining private incentives to invest in a technology allowing people to recognize

3. There is also a somewhat related literature on strategic experimentation and social learning (Keller et al. (2005)). Heidhues et al. (2015) introduce privacy of payoffs, and agents can communicate via cheap-talk messages. Marlats and Ménager (2021) introduce strategic costly observation of actions and outcomes. In contrast to that literature, we suppose that the value of the opportunity is known with certainty.

valuable opportunities.[4] In that sense, the revenues to be shared on the network in this model are endogenous to individual investments in a screening technology.

The second strand of literature addresses strategic communication, with a main focus on organizational economics[5] or political economy[6]. In that literature, the need for communication comes from seeking to influence others' actions under differentiated individual preferences and, in some contexts, coordination issues. Recent extensions to networks include Hagenbach and Koessler (2010), Galeotti et al. (2013), Calvó-Armengol et al. (2015). The two former focus on costless, non-verifiable information (cheap talk model as in Crawford and Sobel (1982)), whereas the latter model the endogenous acquisition of a communication technology (for both being able to communicate and to be able to receive messages) under costly and verifiable information. In our model, information is verifiable and communication is costless. Indeed, in our context, there is no need for costly verification because agents do not have incentives to lie.[7] Our model is orthogonal to that literature, our main contribution being to propose a new rationale for strategic communication. We reveal that strategic communication about the existence of a rival opportunity can emerge when the captured resource is shared with those who ignore its existence. We believe that this situation can arise in many applications, both inside and outside the field of the economics of organizations.

The paper is organized as follows. The communication game is exposed in Section 2, the characterization of the equilibria of he communication game, as well as welfare properties, are presented in Section 3. Section 4 discusses the robustness of the

4. Regarding endogenous revenues, Belhaj and Deroïan (2012) explore farmers' incentives to take (investment) risk on risk-sharing networks, where farmers share part of their revenues with their neighbors.

5. For decentralized decisions making within organizations, see Dessein and Santos (2006), Alonso et al. (2008), or Rantakari (2008).

6. See Dewan and Myatt (2008) for a study related to political parties.

7. In that sense, our model does not correspond to a model of conflict of interest, in which agents can be incited to lie.

results and Section 5 concludes. All proofs are relegated in Appendix A. Appendix B examines the sub-game perfect equilibria of the extensive form game that can be built from the one-shot communication game, Appendix C explores a linear Tullock contest, and Appendix D introduces endogenous investment to be informed.

2. The communication game

2.1. The model

Agents compete for a pie of value normalized to 1, called an *opportunity*. An agent is initially either aware of the existence of the opportunity, or not. Hence, the set of agents - $\mathcal{N} = \{1, 2, \cdots, n\}$ - is partitioned as follows: $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$ where $\mathcal{I}$, of cardinal I, is the set of agents informed before the communication stage (called *players*), and $\mathcal{J}$ is the set of agents who are not informed before communication (called *regular* agents). The sharing rule is represented by an n-square row-stochastic matrix $\mathbf{\Sigma} = (\sigma_{ij})_{i,j \in \mathcal{N}}$, where entry $\sigma_{ij} \geq 0$ is the share of the pie agent i transfers to agent j, if agent i wins the contest.

Given a sharing network $\mathbf{\Sigma}$ as well as a subset of initially informed agents $\mathcal{I}$, we define a normal-form game $(\mathcal{I}; (\mathcal{S}_i)_{i \in \mathcal{I}}; (\pi_i)_{i \in \mathcal{I}})$ as follows: agent i chooses a set $\mathbf{s}_i \in \mathcal{S}_i := \mathcal{P}(\mathcal{J})$. Let $\mathbf{S} := (\mathbf{s}_i)_{i \in \mathcal{I}}$ be an action profile. For simplicity, we also denote by $\mathbf{S}$ the set $\underset{i \in \mathcal{I}}{\cup} \mathbf{s}_i$, i.e. the set of agents who have been informed of the opportunity through communication. We let $\mathcal{M}(\mathbf{S}) := \mathcal{I} \cup \mathbf{S}$ and $m(\mathbf{S}) := |\mathcal{M}(\mathbf{S})|$.

Let $\mathbf{S}_{-i} := (\mathbf{s}_j)_{j \neq i}$ be the profile of actions of all players, except for i.[8] Then

$$\pi_i(\mathbf{s}_i, \mathbf{S}_{-i}) = \frac{1}{m(\mathbf{S})} \left(1 - \sum_{j \in \mathcal{N}} \sigma_{ij} + \sum_{j \in \mathcal{M}(\mathbf{S}) \setminus i} \sigma_{ji} \right) = \frac{1}{m(\mathbf{S})} \sum_{j \in \mathcal{M}(\mathbf{S})} \sigma_{ji}$$

This is the average expected share among all informed agents after the communication phase (including herself).

2.2. Refining best responses

A best response $\mathbf{s}_i$ to a profile of actions of other players $\mathbf{S}_{-i}$ is such that $\pi_i(\mathbf{s}_i, \mathbf{S}_{-i}) \geq \pi_i(\mathbf{s}'_i, \mathbf{S}_{-i})$ for all $\mathbf{s}'_i$. Given that the expected payoff is the average incoming share among informed agents, it is profitable for agent i to inform a regular agent j whenever σ_{ji} exceeds the average incoming share among already informed agents:

RESULT 1. *Given $i \in \mathcal{I}$ and $\mathbf{S}_{-i} \in \mathcal{S}_{-i}$, $\mathbf{s}_i$ is a best response against $\mathbf{S}_{-i}$ iif*[9]

- $\sigma_{ji} \geq Mean\{\sigma_{ki} : k \in \mathcal{M}(\mathbf{S})\}\ \forall j \in \mathbf{s}_i \setminus \mathbf{S}_{-i}$;
- $\sigma_{ji} \leq Mean\{\sigma_{ki} : k \in \mathcal{M}(\mathbf{S})\}\ \forall j \in \mathcal{J} \setminus \mathbf{S}$.

As a consequence, the set $Br_i(\mathbf{S}_{-i})$ is stable by intersection.

If agent j has not been informed by any player, including player i, the incoming share from agent j to agent i cannot be strictly larger than agent i's payoff. Moreover if j is not informed by other players, but is informed by player i, then it must be the case that the incoming share from j to i is larger than his payoff. By Result 1 the current payoff can be viewed as a threshold above which incoming shares

8. From the point of view of player i, all that matters in this game is the set of agents to which other players transmitted their knowledge. We characterize equilibria in terms of their set of informed agents. However, there can be many equilibrium strategies generating a given set of informed agents (through appropriate permutations on the label of the informer of a given informed agent). We disregard those permutations in the paper.

9. For convenience $\mathbf{S}_{-i}$ also denotes the set $\bigcup_{j \in \mathcal{I}, j \neq i} \mathbf{s}_j$.

entail profitable communication, but this threshold is endogenous to the agent's communication strategy.

The set of best responses is never empty. However it is typically not a singleton, because if one player informs an agent then any other player is indifferent between informing this agent or not. To eliminate this source of best-response multiplicity, a natural refinement is considered: [10]

DEFINITION 1 (*Tight best response*). Let $\mathbf{s}_i \in Br_i(\mathbf{S}_{-i})$. Then we say that $\mathbf{s}_i$ is a tight best response against $\mathbf{S}_{-i}$ if, for any $\mathbf{t}_i \subseteq \mathbf{s}_i$ such that $\mathbf{t}_i \neq \mathbf{s}_i$, we have

$$\pi_i(\mathbf{t}_i, \mathbf{S}_{-i}) < \pi_i(\mathbf{s}_i, \mathbf{S}_{-i})$$

In short, a best response is tight if none of the current communications of an agent to a set of neighbors can be cut without penalizing the agent's payoff. Note that, if the empty set is a best response, it is tight by construction. Moreover, since the best-response set is stable by intersection, the tight best response is the intersection of all best responses and is therefore unique. By a slight abuse of notation, we denote this set $\text{TBR}_i(\mathbf{S}_{-i})$. We have:

RESULT 2. *Let $\mathcal{J} \setminus \mathbf{S}_{-i} = \{j_1, j_2, ...j_L\}$ be such that $\sigma_{j_1,i} \geq ... \geq \sigma_{j_L,i}$. Then $TBR_i(\mathbf{S}_{-i}) = \{j_1, ...j_l\}$ iif*

$$\sigma_{j_l i} > Mean\,\{\sigma_{ji} : j \in \{j_1, ..., j_l\} \cup \mathbf{S}_{-i} \cup \mathcal{I}\} \geq \sigma_{j_{l+1},i}$$

By Result 2, agent i's tight best response is easily identified: agent i ranks the incoming shares of all uninformed agents in the society. Then, she examines the profitability of informing the agent with the highest incoming share in that pool, say agent 1. If informing this agent is not strictly profitable, the empty set is the tight

10. Since communication is costless, communication strategies can generate information redundancy, which is an irrelevant source of equilibrium multiplicity. We thus introduce the notion of Tight Best Response, which always proves to be unique.

best response. Otherwise, agent i should inform agent 1. Then, agent i examines the possibility of informing the agent with the second largest share in the pool, say agent 2. If this is not profitable, the tight best response consists in informing agent 1. Otherwise, agent i should inform agent 2. Etc. The full process involves no more than $n-1$ steps. To sum up, at every stage of this process, agent i's expected utility, which is the threshold on incoming shares above which informing is preferred to not informing, is strictly increasing; when the process stops, all incoming shares, and only these shares, exceed agent i's expected utility at the tight best response.

By Result 2, the tight best-response map

$$\text{TBR} : \mathcal{J}^n \to \mathcal{J}^n,\ \text{TBR}(\mathbf{S}) = (\text{TBR}_1(\mathbf{S}_{-1}), ..., \text{TBR}_n(\mathbf{S}_{-n}))\,.$$

is well defined and one-to-one. A tight Nash equilibrium (or TNE) is a fixed point of the tight best-response map.

3. Results

In this section, we characterize the equilibria of the communication game, and then we turn to welfare considerations.

3.1. Equilibria of the communication game

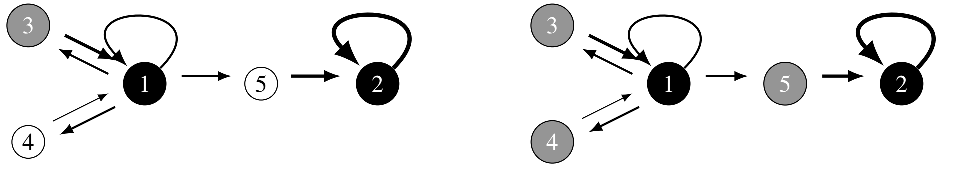
If $\mathcal{I}$ is a singleton, the initially informed agent never wants to communicate, because not communicating is sure to yield the whole pie, which is the highest possible return (thus the current payoff necessarily exceeds the incoming share of any neighbor). However, as soon as $\mathcal{I} = 2$, communication can emerge. We illustrate this with the following example.

EXAMPLE 1 (*Non-uniqueness*). Consider the 5-agent society $\mathcal{N} = \{1, 2, 3, 4, 5\}$, where players are $\mathcal{I} = \{1, 2\}$. Suppose that the sharing network is

$$\mathbf{\Sigma} = \begin{bmatrix} 3/10 & 0 & 7/30 & 7/30 & 7/30 \\ 0 & 1 & 0 & 0 & 0 \\ 3/10 & 0 & 7/10 & 0 & 0 \\ 1/6 & 0 & 0 & 5/6 & 0 \\ 0 & 3/10 & 0 & 0 & 7/10 \end{bmatrix}.$$

There are two TNEs: one where only regular agent 3 is informed by player 1: $\mathbf{s}_1^* = \{3\}, \mathbf{s}_2^* = \emptyset$, with payoffs $(1/5, 1/3)$. If agent 1 informs regular agent 4, his payoff becomes $23/120 < 1/5$. If he ceases informing agent 3, his payoff becomes $3/20 < 7/30$. On the other hand, player 2 gets $13/40 < 1/3$ if he chooses to inform agent 5. Hence there is no possible deviation and this profile is a TNE. There is another TNE, where agent 1 informs regular agents 3 and 4, while player 2 informs regular agent 5: $\mathbf{s}_1^* = \{3, 4\}$ and $\mathbf{s}_2^* = \{5\}$. Payoffs are then $(23/150, 13/50)$. If player 1 stops informing regular agent 4, he gets $3/20 < 23/150$. If player 2 stops informing regular agent 5, he gets $1/4 < 13/50$. Note that the first TNE Pareto-dominates[11] the second one.

FIGURE 1. 2 tight Nash equilibriums (Arrows' boldness represents the share sizes)



■

In general, when a new agent, say agent k, gets informed, this can either strengthen or weaken the incentives for agent i to communicate. When the incoming share from

11. An action profile $\mathbf{S}'$ Pareto dominates $\mathbf{S}$ if $u_i(\mathbf{S}') \geq u_i(\mathbf{S})$, for all $i \in \mathcal{I}$.

agent k to player i is low enough, and as k gets informed, agent i's current utility is reduced by the fiercer competition to win the pie, which decreases the threshold on shares above which agent i communicates. In this case, player i's incentives to communicate are reinforced by the change in agent k's informational status. On the other hand, if agent k gives a sufficiently large share to player i, this can increase the threshold, i.e. it can decrease incentives to communicate.

An important property of the tight best response is that, for any player i, TBR_i is increasing in the following sense[12]:

PROPOSITION 1 (Monotonicity). *For any player i and any $\mathbf{S}_{-i}, \mathbf{S}'_{-i}$ such that $\mathbf{S}_{-i} \subseteq \mathbf{S}'_{-i}$, we have $TBR_i(\mathbf{S}_{-i}) \subseteq TBR_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$.*

Example 1 illustrates this monotonicity property: if player 1 finds it best to inform regular agent 3 when player 2 does not inform regular agent 5, she still prefers to inform regular agent 3, when regular agent 5 is informed by player 2. The reason why Proposition 1 holds is that, at the tight best response, the arrival of a new informed agent does not increase the current expected utility of the player. Indeed, the very fact that the new informed agent was not informed by player i means that her incoming share is lower than the average incoming share that player i receives from other informed agents; and thus informing that agent can only lower player i's average incoming share. One deep consequence of Proposition 1 is the existence of a minimum TNE.

THEOREM 1. *There exists a tight Nash equilibrium $\underline{\mathbf{S}}^*$ with the following property: for any TNE $\mathbf{S}^*$, we have $\underline{\mathbf{S}}^* \subseteq \mathbf{S}^*$. We call $\underline{\mathbf{S}}^*$ the* minimum TNE.[13]

12. Note that simultaneous best responses $\text{TBR} := (\text{TBR}_1, ..., \text{TBR}_i)$ may not be increasing: we might have $\mathbf{s}_i \subseteq \mathbf{s}'_i \;\; \forall i$, but $\text{TBR}(\mathbf{S}) \not\subseteq \text{TBR}(\mathbf{S}')$.

13. Formally, $\underline{\mathbf{S}}$ is not unique in terms of action profile. It is unique in terms of set of informed agents.

The proof is not trivial given that communication strategies are discrete and that the monotonicity property only holds over tight best responses. To prove Theorem 1 we introduce a sequential best-response map, and show that, starting from the empty strategy set, the iteration of the map converges to a minimum TNE, $\underline{\mathbf{S}}$. This result echoes supermodular games, through the monotonicity properties of tight best responses, although the game is not supermodular, because the payoffs are not supermodular on the partially ordered spaces of actions. Having shown the existence of a minimum TNE has a major welfare implication[14]:

PROPOSITION 2. *The minimum TNE strictly Pareto-dominates all other TNEs.*

Proposition 2 follows from this observation: for any equilibrium with a set of informed agents larger than $\underline{\mathbf{S}}^*$, the average incoming share from informed agents in the larger TNE who are not in set $\underline{\mathbf{S}}^*$ is lower than the average incoming share from agents in set $\underline{\mathbf{S}}^*$. Note that Pareto-dominance applies here on players only, and regular agents can be better off in larger equilibria.

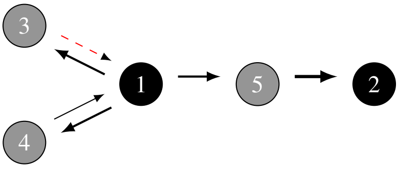
In terms of comparative statics, an immediate consequence of Lemma 1 is that the set of informed agents at the minimum equilibrium can only be enlarged when the set of players is enlarged (proof omitted). In terms of statics over the sharing network, it might at first seem as if communication at minimum equilibrium increases with shares. The following modification of example 1 shows that this intuition is not always true.

$$\mathbf{\Sigma}' := \begin{bmatrix} 3/10 & 0 & 7/30 & 7/30 & 7/30 \\ 0 & 1 & 0 & 0 & 0 \\ 1/6 & 0 & 5/6 & 0 & 0 \\ 1/6 & 0 & 0 & 5/6 & 0 \\ 0 & 3/10 & 0 & 0 & 7/10 \end{bmatrix}.$$

14. Actually we prove the more general statement that any TNE Pareto-dominates any TNE with a larger set of informed agents.

Compared to the sharing matrix $\mathbf{\Sigma}$ of Example 1, the incoming share from regular agent 3 to player 1 has been reduced. However there is now only one TNE, where $s_1^* = \{3,4\}$ and $s_2^* = \{5\}$ (because $1/6 > 3/20$, and $3/10 > 1/4$). Consequently the set of informed agents at the minimum TNE is larger than for $\mathbf{\Sigma}$, even though the shares are smaller than in $\mathbf{\Sigma}$.

FIGURE 2. Share from 3 to 1 is reduced: the minimum equilibrium of Example 1 disappears



However if sharing matrices are symmetric, this cannot occur:

PROPOSITION 3. *Consider two symmetric sharing matrices $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ such that $\sigma_{ij} \geq \sigma'_{ij}$ for all $i \neq j$. Then $TBR'_i(\mathbf{S}_{-i}) \subseteq TBR_i(\mathbf{S}_{-i})$, $\forall \mathbf{S}_{-i}$, and $\underline{\mathbf{S}}(\mathbf{\Sigma}') \subseteq \underline{\mathbf{S}}(\mathbf{\Sigma})$.*

By symmetry, the sum of received shares at a given equilibrium is equal to the sum of shares given to others, and the latter is not larger than one minus the agent's own share. Combined with row-stochasticity, this means that, considering the set of informed agents at any equilibrium under sharing matrix $\mathbf{\Sigma}$, the communication threshold increases under sharing matrix $\mathbf{\Sigma}'$.

We focus now on a specific set of sharing matrices, in which positive shares are homogeneous. These sharing rules are useful to isolate the pure network structure effects of sharing on communication. If, for any $i \neq j$, $\sigma_{ij} = \sigma_{ji} \in \{0, \lambda\}$, with $\lambda > 0$, we say that $\mathbf{\Sigma}$ is an *equi-sharing matrix*. Such a sharing rule is thus characterized by the pair $(\mathbf{G}, \lambda)$, where $\mathbf{G} = (g_{ij})_{i,j}$ is an adjacency matrix, with $g_{ij} \in \{0, 1\}$. We refer to the graph induced by $\mathbf{G}$ as the *equi-sharing network*, the neighborhood of agent i is

$\mathcal{N}_i := \{j \in \mathcal{N} : g_{ij} = 1\}$, and $\overline{d} := \max_{i \in \mathcal{N}} |\mathcal{N}_i|$. The associated equi-sharing matrix, $\mathbf{\Sigma}(\mathbf{G}, \lambda)$, is the row-stochastic matrix such that $\sigma_{ij} = \lambda g_{ij}$ for $i \neq j$.[15]

For a player $i \in \mathcal{I}$, let $\mathcal{R}_i := \{j \in \mathcal{J} : \; g_{ij} = 1\}$ be her set of regular neighbors. We also define $\mathcal{R}_{\mathcal{I}} := \cup_{i \in \mathcal{I}} \mathcal{R}_i$, the set of regular agents that are linked to at least one player. We then have:

$$\pi_i(\mathbf{s}_i, \mathbf{S}_{-i}) = \frac{1}{m(\mathbf{S})} \left(1 - \lambda(|\mathcal{R}_i| - |\mathbf{S} \cap \mathcal{R}_i|)\right) \tag{1}$$

For such sharing networks, there is a dichotomy on the set of informed agents at a TNE $\mathbf{S}^*$: either $\mathbf{S}^* = \emptyset$, in which case we say that it is a *no communication profile*; or $\mathbf{S}^* = \mathcal{R}_{\mathcal{I}}$, in which case we say that it is a *full communication profile*.

PROPOSITION 4. *In equi-sharing networks, a TNE is either a no communication profile or a full communication profile. Moreover:*

a) The no communication profile is a TNE if and only if $\lambda \leq \frac{1}{I + \max_{i \in \mathcal{I}} |\mathcal{R}_i|}$. *We then have* $\pi_i(\emptyset) = \frac{1 - \lambda |\mathcal{R}_i|}{I}$.

b) The full communication profile is a TNE if and only if $\lambda > \frac{1}{I + |\mathcal{R}_{\mathcal{I}}|}$. *We then have* $\pi_i^* = \frac{1}{I + |\mathcal{R}_{\mathcal{I}}|}$.

The intuition of Proposition 4 is simple. First, if one player finds it profitable to inform one neighbor, she should be better off informing all of them, because all neighbors giving a same incoming share receive the same treatment (this stems from uniqueness of tight best responses). Hence, a TNE contains only two classes of players: those who inform all neighbors, and those who do not inform any neighbors. Furthermore, all players who communicate have the same payoff. Second, if one player chooses to communicate, then every other player should do the same.

15. We assume throughout the section that agents do not give neighbors more than they keep for themselves: for any agent i, $1 - \lambda |\mathcal{N}_i| > \lambda$, i.e., $\lambda < \overline{\lambda} := \frac{1}{\overline{d}+1}$. Then $\sigma_{ii} = 1 - \lambda |\mathcal{N}_i| > \lambda$.

By Proposition 4, multiplicity arises in the interval $\left[\frac{1}{I+|\mathcal{R}_{\mathcal{I}}|}, \frac{1}{I+\max_{i\in\mathcal{I}}|\mathcal{R}_i|}\right]$. Note that, for $\lambda \leq \mu(\mathcal{I}) := \frac{1}{I+\max_{i\in\mathcal{I}}|\mathcal{R}_i|}$, the minimum TNE is the no communication profile, and above that threshold, the minimum TNE is the full communication profile.

3.2. Welfare

We now investigate social welfare issues, from the point of view of an inequality-averse observer, who evaluates aggregate utility through some concave non-decreasing function $U(\cdot)$, with $U(0) = 0$. More precisely, a welfare function $W(\mathbf{S}|\boldsymbol{\Sigma}, \mathcal{I})$ depends on both the sharing network and the set of players. Then, given a profile of informed agents $\mathcal{M}(\mathbf{S})$, the expected aggregate utility is then given by[16]

$$W(\mathbf{S}) = \frac{1}{m(\mathbf{S})} \sum_{j\in\mathcal{M}(\mathbf{S})} \sum_{i\in\mathcal{N}} U(\sigma_{ji})$$

There is always an efficient communication profile, i.e. one maximizing the welfare function. A simple algorithm allows to identify the efficient allocation in no more than $n - |\mathcal{I}|$ steps. In the case of equi-sharing networks for instance, the efficient communication profile can be partial, i.e. it can be neither the no communication nor the full communication profile.[17] And, according to the allocation of players, there are simple conditions under which the efficient communication is the no communication profile of the full communication profile.

We then discuss the efficiency level of communication equilibria. In Example 1, $W(\underline{\mathbf{S}}) = U(1)$, while $W(\overline{\mathbf{S}}) = \frac{1}{2}U(1) + \frac{1}{2}(U(2/5) + U(3/5))$. The utility is higher in the maximum equilibrium. This is not always the case: assume that the sharing

16. Abusing the notation, we denote $W(\mathbf{S})$ for convenience.

17. Consider a four-agent society organized in a line with agents 2 and 3 at the center of the line, an equi-sharing network, and the player set $\{1, 3\}$. For all concave utility functions, the efficient set of informed agents is $\hat{\mathcal{M}}(\mathbf{S}) = \{1, 2, 3\}$ for all shares. Indeed, informing agent 2 increases the average sharing of informed agents, while communicating to agent 4 can only decrease the average.

matrix is given by

$$\mathbf{\Sigma} = \begin{bmatrix} \beta & \alpha-\beta & 1-\alpha & 0 \\ \alpha-\beta & \beta & 0 & 1-\alpha \\ 1-\alpha & 0 & \alpha & 0 \\ 0 & 1-\alpha & 0 & \alpha \end{bmatrix},$$

where $\alpha > \beta > 1/2$, and $\mathcal{I} = \{1, 2\}$. We obviously have

$$U(1-\alpha) + U(\beta) + U(\alpha-\beta) > U(1-\alpha) + U(\alpha).$$

The profile $\underline{\mathbf{S}} = \emptyset$ is a TNE iif $U(\beta) + U(\alpha - \beta) \geq 2U(1-\alpha)$. On the other hand, $\overline{\mathbf{S}} = (\{3\}, \{4\})$ is a TNE iif $U(\alpha-\beta) + U(\beta) < 3U(1-\alpha)$. Therefore, if $2U(1-\alpha) \leq U(\beta) + U(\alpha-\beta) \leq 3U(1-\alpha)$,[18] then both $\underline{\mathbf{S}}$ and $\overline{\mathbf{S}}$ are TNEs, and $W(\underline{\mathbf{S}}) > W(\overline{\mathbf{S}})$.

Let now $\mathbf{\Sigma}(\mathbf{G}, \lambda)$ be an equi-sharing matrix, and $\mathcal{A}, \mathcal{B}$ be two subset of $\mathcal{N}$. We say that $\mathcal{A}$ degree-dominates $\mathcal{B}$ if, the degree distribution of $\mathcal{B}^A$ first-order dominates the degree distribution of $\mathcal{A}^B$.[19]

PROPOSITION 5. *Consider an equi-sharing network $\mathbf{\Sigma}(\mathbf{G}, \lambda)$ such that both equilibria exist*[20]. *Then*

- *if $\mathcal{R}_\mathcal{I}$ degree-dominates $\mathcal{I}$, then all concave utilitarian evaluators agree that social welfare is larger at the full Communication equilibrium;*
- *if $\mathcal{I}$ degree-dominates $\mathcal{R}_\mathcal{I}$, then all concave utilitarian evaluators agree that social welfare is larger at the no Communication equilibrium;*

18. For instance, choosing $\alpha = 8/10$ and $\beta = 6/10$, the condition amounts to having $U(0.6) \leq 2U(0.2)$.

19. $\mathcal{B}^A$ is equal to the set $\mathcal{B}$ duplicated A times. Thus if the degree distribution of $\mathcal{A}$ is $(1, 1, 2)$ and the degree distribution of $\mathcal{B}$ is $(1, 3)$ then the degree distribution of $\mathcal{B}^A$ is $(1, 1, 1, 3, 3, 3)$, while the degree distribution of $\mathcal{A}^B$ is $(1, 1, 1, 1, 2, 2)$. Hence $\mathcal{B}$ degree-dominates $\mathcal{A}$.

20. i.e. $\lambda \in \left[\frac{1}{I+|\mathcal{R}_\mathcal{I}|}, \frac{1}{I+\max_{i\in\mathcal{I}}|\mathcal{R}_i|}\right]$

The average degree is not the right statistics to look at. Suppose that the degree distribution of $\mathcal{R}_{\mathcal{I}}$ is $(4, 1)$, while the degree distribution of $\mathcal{I}$ is $(2, 2)$. Then, if $U(x) = x$ for $x \in [0, 1 - 2\lambda]$ and $U(x) = 1 - 2\lambda$ for $x \geq 1 - 2\lambda$ we have $W(\emptyset) = 2(1 - 2\lambda) + 4\lambda = 2$ while $W(\mathcal{R}_{\mathcal{I}}) = 2 - \lambda$. Hence although the average degree is strictly higher in $\mathcal{R}_{\mathcal{I}}$, the social welfare is higher at the no communication equilibrium.

4. Robustness and limits

The main results in this section are driven by the monotonicity property established in Lemma 1. We undertake some robustness analysis by discussing three variations or extensions of the model in that perspective: we analyse a variation of the model in which only informers are rewarded, then we consider the situation in which the probability of winning is an increasing function of the number of informed agents in the society, and finally we introduce endogenous effort to win the contest. A fourth extension examines how endogenous investment to be an initial player affects the interplay between sharing and communication.

4.1. Rewarding information transmission only

In the model, the winner of the contest gives transfers to others according to the sharing matrix. This means that the transfer is made whatever the status of the receiver of the transfer; and in particular, whether the receiver of the transfer has informed the sender of the transfer or not is of no matter here. Suppose rather that transfers are only given in exchange of information. The payoff of a player would then by given by the sum of incoming shares originated from those she informs over the sum of informed agents. E.g., under equi-sharing, λ might represent a price of information.

In this context, there is no impact of a newly informed agent say j by a third party on a given player i's best-response communication strategy by construction. Indeed, those who are informed by other players don't reward player i, meaning that her communication threshold is not affected by the informational status of agent j. We deduce

RESULT 3. *When players only reward those agents they inform, the monotonicity property holds.*

Therefore, by Result 3, all our previous results, including the existence of a minimum equilibrium in terms of set of informed agents, and the fact that the minimum equilibrium Pareto dominates all other equilibria, directly extend to that context.

4.2. Probability to win the contest as an increasing function of the number of informed agents

In the benchmark model, we assume that the probability to win the contest does not depend on the number of informed agents. Suppose rather that the probability to win the contest is an increasing function of the number of informed agents in the society (in a setting where, still, every informed agent has the same probability to win).[21] This can arise, for instance, in teams, where collective searching can boost the probability to win, or when the pressure of competition increases incentives.

Formally, given m informed agents, assume that the probability to have a winner is $a(m) < 1$, where $a(\cdot)$ is a non-decreasing function. Then the probability that a given informed agent wins is $\frac{a(m)}{m}$. $a(m) = 1$ for all m corresponds to our benchmark. The shape of function $a()$ matters. Indeed:

RESULT 4. *When function $a()$ is increasing and convex, the monotonicity property holds.*

Again, by Result 4, there exists a minimum equilibrium that Pareto dominates all other equilibria. In opposite, the monotonicity property can fail under sufficiently

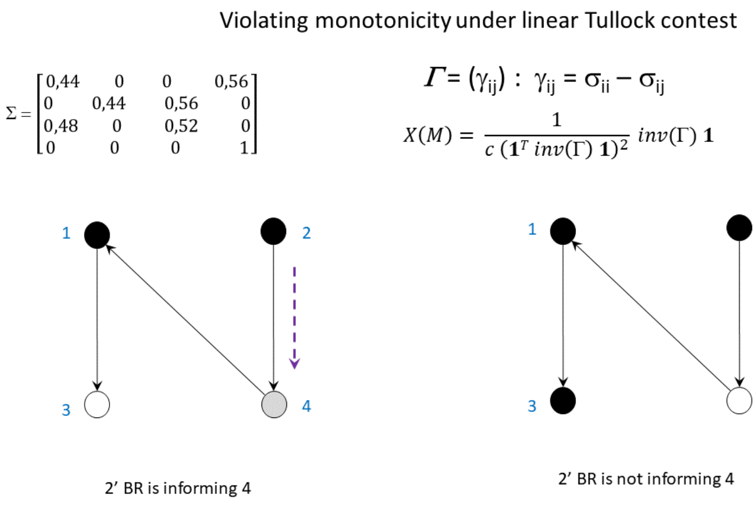
21. An alternative interpretation is that the value of the pie increases with the number of informed agents.

concave function $a(\cdot)$, since there are now decreasing returns to investment as the number of informed agents grows. To illustrate, consider the extreme case of an increasing function until an upper bound. Take for instance $\mathcal{N} = \{i, j, l, k\}$, $\sigma_{ji} = \sigma_{li} = 1/4, \sigma_{ki} = 1/7$, and take function $a()$ such that $a(2)/a(3) = 3/4, a(4) = a(3)$. First, player i prefers informing agent k when only j, i are informed if $\sigma_{ki} > \left(\frac{3}{2}\frac{a(2)}{a(3)} - 1\right)\sigma_{ji}$ (own shares are null in this stylized example), which holds as $1/7 > 1/32$. Second, player i prefers not informing agent k when i, j, l are informed if $\sigma_{ki} < \frac{1}{3}(\sigma_{ji} + \sigma_{li})$, which holds as $1/7 < 1/6$. Hence, monotonicity property fails in this example, in which there are significant decreasing returns to investment.

4.3. Endogenous contest

In the model, the probability to win the contest is exogenous, meaning that agents do not exert effort to win the contest. Consider rather that the result of the contest depends on individual costly efforts to win the contest. This brings a new mechanism shaping incentives to communicate, on top of the sharing-the-pie mechanism. This new motive is is due to the interaction between the efforts of the participants to the contest: agents may want to communicate in the purpose of influencing - typically decreasing - the efforts of competitors. This mechanism can lead to the failure of the monotonicity property.

For instance, consider the case of a linear Tullock contest (more details are presented in Appendix C). Figure 4.3 depicts a 4-player 3-link example in which the monotonicity property fails: In that example, agent 2 informs agent 4 only because this fosters a reduction of agent 1's effort (indeed, note that agent 4 does not give a share to agent 2!). Now if agent 3 is informed, the impact of agent 4 on that of agent 1 is reduced, and this is then no longer interesting for agent 2 to inform agent 4.



4.4. *Endogenous investment to be a player*

The model takes the set of initially informed agents (the players) as exogenous. Suppose rather that, prior to communication, agents can make a costly investment to identify the opportunity (see Appendix D for more details).

We assume that investors communicate about the opportunity in accordance with the minimum TNE. This approach is supported by the fact that the minimum TNE Pareto-dominates other equilibrium (for players). Hence, when the set of investors changes, the set of informed agents evolves in accordance with the minimum equilibrium asssociated with the new set of investors. We focus on un-directed equi-sharing networks for tractability: in this case we indeed have a full characterization of the minimum equilibrium, which is either the no communication configuration, or the full communication configuration (see Proposition 4). Importantly, the communication threshold above which communication emerges at the minimum equilibrium is endogenous to the investment decisions; It depends on both the structure of the sharing network and the allocation of investors.

This game has interesting features. In particular, a Nash equilibrium of the investment game can fail to exist for intermediate values of shares. Furthermore, introducing endogenous investment can be detrimental to communication, but this depends on the investment cost.

Under small investment cost, endogenous investment is not detrimental to communication. At first glance we might believe that, at equilibrium, some agents could end up being uninformed. Indeed, an agent linked to the neighbor of an investor receives an excpected share if this neighbor is informed by the investor, and thus might possibly be better-off with uninformed status. However the next theorem forbids that possibility, showing that information about the opportunity always fully disseminates in the society:

THEOREM 2. *When the investment cost is sufficiently low, any Nash equilibrium of the investment game is a dominating set, meaning that all agents are informed in the society. Moreover: if $\lambda \leq \frac{1}{n}$, there is a unique Nash equilibrium in which all agents in the society invest; if $\lambda > \frac{1}{n}$, every Nash equilibrium induces communication.*

(see Appendix D for more details) Theorem 2 says that, at any equilibrium, any non-investor is necessarily linked to an investor. Whether λ is larger or smaller than $\frac{1}{n}$ determines if communication emerges at equilibrium.

Under large investment cost however, endogenous investment can be detrimental to communication. This tension between incentives to invest and to communicate can be analyzed with a comparative statics on shares. Indeed, in contrast to the benchmark model, it can be that an unambiguous (bilaterally symmetric) increase of sharing reduces the number of informed agents at equilibrium. There are two possibles channels, one related to free riding, and the other related to the communication generated by investment decisions, that we briefly illustrate by the means of simple examples:

Investing increases free riding. In the example depicted in figure 4.4, adding a sharing link reduces the number of informed agents at equilibrium. The reason is that link addition increases the incentives to free ride on the other investment. The consequence can be a decrease of the number of informed agents at equilibrium.

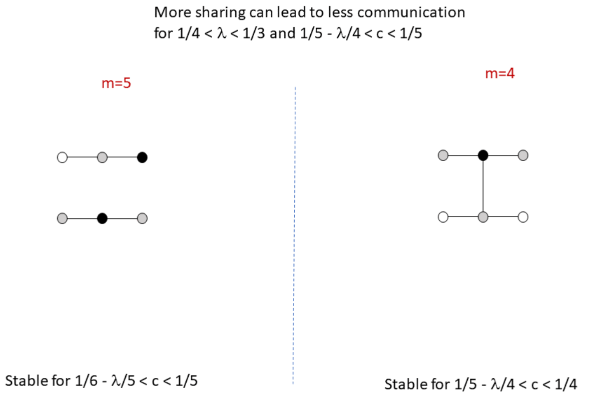


FIGURE 3. Under endogenous investment, free riding can lead societies with more sharing to communicate less: Both Left and Right configurations represent an equilibrium with minimum number of informed agents for $\lambda \in (\frac{1}{4}, \frac{1}{3})$ and $c \in (\frac{1}{5} - \frac{\lambda}{4}, \frac{1}{5})$. The Right network represents a society with more sharing, but the minimal number of informed agents among all equilibria (4) is smaller than for the Left network (5).

Note that, in both configurations there are multiple equilibria, and the two equilibria depicted here minimize the number of informed agents over all equilibria in each network. In that respect, more sharing induces less information diffusion.

The communication induced by investment deters incentives. In the example presented in Figure 4.4, there is no communication in the Left configuration with a low value of the share; In the Right configuration with increased share, agent i's investment generates communication, and this deters investment. Given that the investment cost is substantial, investing is less attractive, which reduces the set of investors and ultimately the number of informed agents at equilibrium.

5. Conclusion

In this paper, we studied incentives to inform others about the existence a rival opportunity, when the value generated by the opportunity is shared through an exogenous, possibly heterogeneous, sharing rule. In this environment, incentives to

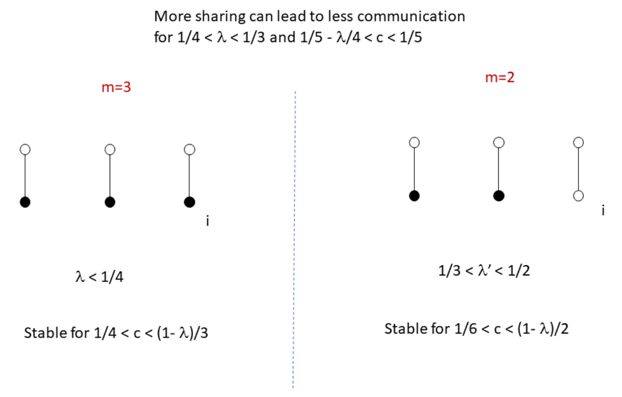


FIGURE 4. Under endogenous investment, the threat of increased communication can lead societies with more sharing to communicate less for $c \in (\frac{1}{4}, \frac{1-\lambda}{3})$, when $\lambda < \frac{1}{4}$ in the Left configuration and $\lambda \in (\frac{1}{3}, \frac{1}{2})$ in the right configuration.

communicate increase with the amount of profitable communication on the network, which generates Pareto-ranked equilibria over the subgroup of initially informed agents. One salient message is that more sharing does not necessarily leads to more communication. Yet, under bilaterally symmetric shares, a society with more sharing communicates more. This conclusion might not hold when agents can make investment into information prior to communication for large investment cost.

Even if this model is extremely stylized, it would be interesting to understand better the policy implications of of this model, especially in contexts where public policy aims at maximizing communication into the society. Furthermore, it would be challenging to explore the formation of sharing networks in presence of strategic communication about economic opportunities.

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Appendix A: Proofs

Proof of Result 1. Let $\mathbf{s}_i \in BR_i(\mathbf{S}_{-i})$. If $j \in \mathbf{s}_i \setminus \mathbf{S}_{-i}$ and $\sigma_{ji} < \pi_i(\mathbf{s}_i, \mathbf{S}_{-i})$ then $\pi_i(\mathbf{s}_i \setminus \{j\}, \mathbf{S}_{-i}) > \pi_i(\mathbf{s}_i, \mathbf{S}_{-i})$, a contradiction. Now, if $j \notin \mathbf{S}$ and $\sigma_{ji} > \pi_i(\mathbf{s}_i, \mathbf{S}_{-i})$ then $\pi_i(\mathbf{s}_i \cup \{j\}, \mathbf{S}_{-i}) > \pi_i(\mathbf{s}_i, \mathbf{S}_{-i})$, also a contradiction.

Suppose now that both inequalities hold for $\mathbf{s}_i$. It is straightforward to check that it is a best response. □

Proof of Result 2. Uniqueness directly follows from the fact that $\text{TBR}_i(\mathbf{S}_{-i})$ is the intersection of all elements of $Br_i(\mathbf{S}_{-i})$.

We now prove the second statement. Note that there cannot be $1 \leq l < l' \leq L$ such that $l' \in \text{TBR}_i(\mathbf{S}_{-i})$ while $l \notin \text{TBR}_i(\mathbf{S}_{-i})$, because deviating to informing l instead of l' would yield an equal or higher payoff, contradicting the fact that $\text{TBR}_i(\mathbf{S}_{-i})$ is the tight best response. Thus there exists $l^* \geq 0$ such that $\text{TBR}_i(\mathbf{S}_{-i}) = \{j_1, ...j_{l^*}\}$.[22] Let now

$$f(l) := Mean\left\{(\sigma_{ki})_{k \in \mathcal{I} \cup \mathbf{S}_{-i}}, \sigma_{j_1 i}, ..., \sigma_{j_l i}\right\}$$

for $l = 0, ..., L$. Note that

$$f(l) \geq f(l+1) \Leftrightarrow f(l) \geq \sigma_{j_{l+1} i} \Rightarrow f(l+1) \geq \sigma_{j_{l+2} i} \Leftrightarrow f(l+1) \geq f(l+2).$$

As a consequence the map $f(\cdot)$ is quasi-concave in the sense that

$$f(l) \geq f(l+1) \Rightarrow f(l+1) \geq f(l+2).$$

Hence l^* is the only integer in $\{0, ..., L\}$ such that $f(l^* - 1) < f(l^*)$, and $f(l^*) \geq f(l^* + 1)$[23]. This proves the second statement. □

Proof of Proposition 1. Write $\mathcal{J} \setminus \mathbf{S}'_{-i} = \{j_1, ..., j_L\}$, where $\sigma_{j_1 i} \geq \sigma_{j_2 i} \geq ...\sigma_{j_L i}$. Then $\text{TBR}_i(\mathbf{S}_{-i}) \setminus \mathbf{S}'_{-i} = \{j_1, ..., j_l\}$ for some $l \leq L$. By definition of j_l belonging to the tight best response to $\mathbf{S}_{-i}$ we must have that $\sigma_{j_l i}$ is strictly greater than $Mean(A)$, where

$$\mathbf{A} := \{\sigma_{ji} : \ j \in \mathbf{S}_{-i} \cup \mathcal{I}\} \cup \{\sigma_{j_1 i}, ...\sigma_{j_l i}\} \cup \{\sigma_{ji} : \ j \in \mathbf{S}'_{-i}, \sigma_{ji} > \sigma_{j_l i}\}$$

We want to prove that j_l belongs to $\text{TBR}_i(\mathbf{S}'_{-i})$. Let $\mathbf{A}' := \{\sigma_{ji} : \ j \in \mathbf{S}'_{-i} \cup \mathcal{I}\} \cup \{\sigma_{j_1 i}, ...\sigma_{j_l i}\}$. Then

$$\mathbf{A}' = \mathbf{A} \cup \{\sigma_{ji} : j \in \mathbf{S}'_{-i}, \ \sigma_{ji} \leq \sigma_{j_l i}\}$$

22. Note that fact that the ordering is not uniquely defined does not contradict that the tight best response is unique.

23. With the convention that $f(-1) < f(0)$ and $f(L+1) \leq f(L)$

Hence, since $\sigma_{j_l,i} > Mean(\mathbf{A})$, we necessarily also have that $\sigma_{j_l,i} > Mean(\mathbf{A}')$, because every element in $\mathbf{A}' \setminus \mathbf{A}$ is smaller or equal than $\sigma_{j_l i}$. Thus $j_l \in$ TBR$_i(\mathbf{S}'_{-i})$. □

A profile $\mathbf{S}$ is *under-informed* if $\mathbf{s}_i \subseteq$ TBR$_i(\mathbf{S}_{-i})$ for any $i \in \mathcal{I}$. We call $\mathcal{S}_u$ the set of under-informed profiles. Furthermore, for any $i \in \mathcal{I}$, let $\mathbf{B}_i$ be given by

$$\mathbf{B}_i : (\mathbf{s}_i, \mathbf{S}_{-i}) \mapsto (\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}), \quad \text{and } B_i(\mathbf{s}_i, \mathbf{S}_{-i}) = \text{TBR}_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i}.$$

DEFINITION A.1. The *sequential best-response map* is constructed as follows. Let $\mathbf{S} = (\mathbf{s}_i)_{i \in \mathcal{I}}$ be an action profile. Then $\mathbf{B} : \mathcal{S} \to \mathcal{S}$ is defined as[24]

$$\mathbf{B}(\mathbf{S}) := \mathbf{B}_I \circ \mathbf{B}_{I-1} \circ ... \circ \mathbf{B}_1(\mathbf{S})$$

We write $B(\mathbf{S}) = \cup_i (\mathbf{B}(\mathbf{S}))_i$.

We first prove the following claim:

LEMMA A.A.1. *If $\mathbf{S}$ and $\mathbf{S}'$ are such that $\mathbf{S} \subseteq \mathbf{S}'$ and $\mathbf{S}'$ is under-informed then, for any player i, we have $B_i(\mathbf{S}) \subseteq B_i(\mathbf{S}')$. More importantly, $\mathbf{B}(\mathbf{S}')$ is under-informed and $B(\mathbf{S}) \subseteq B(\mathbf{S}')$.*

Proof. By assumption, $\mathbf{S}'$ is such that $\mathbf{s}'_i \subseteq$ TBR$_i(\mathbf{S}'_{-i})$. Hence $\mathbf{S}_{-i} \subseteq \mathbf{S}' \subseteq B_i(\mathbf{S}') =$ TBR$_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$. Consequently we only need to prove that TBR$_i(\mathbf{S}_{-i}) \subseteq$ TBR$_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$. Without loss of generality, we can write $\mathcal{J} \setminus \mathbf{S}_{-i} = \{j_1, ..., j_P\} \cup (\mathbf{S}' \setminus \mathbf{S}_{-i})$ where $\{j_1, ..., j_P\} = \mathcal{J} \setminus \mathbf{S}'$ and $\sigma_{j_1 i} \geq ... \geq \sigma_{j_P i}$.
The set TBR$_i(\mathbf{S}_{-i})$ can then be written $B \cup \{j_1, ..., j_p\}$ (where $B \subseteq \mathbf{S}' \setminus \mathbf{S}_{-i}$), while TBR$_i(\mathbf{S}'_{-i}) = \mathbf{s}'_i \cup \{j_1, ..., j_{p'}\}$. We need to prove that $j_p \in$ TBR$_i(\mathbf{S}'_{-i})$. Since

24. Note that map $\mathbf{B}$ depends on the order of players. However, as we will see the important objects do not depend on the order chosen. Note also that $\mathbf{B}_i(\mathbf{S})$ and $(\mathbf{B}(\mathbf{S}))_i$ are different objects; the map $\mathbf{B}$ is not monotonic in the classical sense, as there are simple examples where $\mathbf{S}_i \subseteq \mathbf{S}'_i$ for all i does not imply that $(\mathbf{B}(\mathbf{S}))_i \subseteq (\mathbf{B}(\mathbf{S}'))_i$.

$j_p \in \text{TBR}_i(\mathbf{S}_{-i})$, we have

$$\sigma_{j_p i} > Mean\{\sigma_{ji} : \ j \in \mathcal{I} \cup \mathbf{S}_{-i} \cup B \cup \{j_1, ..., j_{p-1}\}\}$$

Thus we have

$$\sigma_{j_p i} > Mean\left(\sigma_{ji} : \ j \in \mathcal{I} \cup \mathbf{S}_{-i} \cup (\mathbf{S}' \setminus \mathbf{S}_{-i}) \cup \{j_1, ..., j_{p-1}\}\right),$$

because B consists of the elements of the elements of $\mathbf{S}' \setminus \mathbf{S}_{-i}$ who give the largest share to i. This proves that $j_p \in \text{TBR}_i(\mathbf{S}'_{-i})$, and therefore that $B_i(\mathbf{S}) \subseteq B_i(\mathbf{S}')$.

Let us now prove that $B(\mathbf{S}) \subseteq B(\mathbf{S}')$. By a recursive argument, it is enough to show that $\mathbf{B}_i(\mathbf{S}')$ is under-informed, to be able to repeatedly apply the first point of the lemma. Let $j \neq i$. We must prove that $(\mathbf{B}_i(\mathbf{S}'))_j \subseteq \text{TBR}_j((\mathbf{B}_i(\mathbf{S}'))_{-j})$. Since $(\mathbf{B}_i(\mathbf{S}'))_j = \mathbf{s}'_j$, it amounts to proving that $\mathbf{s}'_j \subseteq \text{TBR}_j((B_i(\mathbf{S}'))_{-j})$. Note that $\mathbf{s}'_j \cap (\mathbf{B}_i(\mathbf{S}'))_{-j} = \emptyset$. Hence

$$\mathbf{s}'_j \subseteq \text{TBR}_j(\mathbf{S}'_{-j}) \setminus (\mathbf{B}_i(\mathbf{S}'))_{-j} \subseteq \text{TBR}_j((\mathbf{B}_i(\mathbf{S}'))_{-j}),$$

because $\mathbf{S}'_{-j} \subseteq (\mathbf{B}_i(\mathbf{S}'))_{-j}$, and applying Lemma 1.

We now turn to the proof of Theorem 1, by first proving some useful lemmas.

LEMMA A.A.2. *Let $\mathbf{S} \in \mathcal{S}_u$. Then $\mathbf{s}_i \subseteq (\mathbf{B}(\mathbf{S}))_i$ for any $i \in \mathcal{I}$.*

Proof. We have

$$(\mathbf{B}(\mathbf{S}))_i = \text{TBR}_i((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{s}_{i+1}, ..., \mathbf{s}_I), \text{ for } i = 1, ..., I.$$

We show the proposition by induction on i. By definition of $\mathbf{S} \in \mathcal{S}_u$ we have $\mathbf{s}_1 \subseteq \text{TBR}_1(\mathbf{S}_{-1}) = (\mathbf{B}(\mathbf{S}))_1$. Assume that $\mathbf{s}_j \subseteq (\mathbf{B}(\mathbf{S}))_j$ for $j = 1, ..., i-1$. Then

$$\mathbf{S}_{-i} \subseteq ((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{s}_{i+1}, ..., \mathbf{s}_I)$$

and $\mathbf{s}_i \cap (\mathbf{B}(\mathbf{S}))_1 \cup ... \cup (\mathbf{B}(\mathbf{S}))_{i-1} \cup \mathbf{s}_{i+1} \cup ... \cup \mathbf{s}_I)$ by construction. Hence

$$\begin{aligned} \mathbf{s}_i &\subset Br_i(\mathbf{S}_{-i}) \setminus ((\mathbf{B}(S))_1, ..., \mathbf{B}(S))_{i-1}, \mathbf{s}_{i+1}, ..., \mathbf{s}_I) \\ &\subset Br_i(((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{s}_{i+1}, ..., \mathbf{s}_I)) \\ &= (\mathbf{B}(\mathbf{S}))_i \end{aligned}$$

by Lemma 1. ■

LEMMA A.A.3. *If* $\mathbf{s}_i \subseteq (\mathbf{B}(\mathbf{S}))_i \ \forall i$ *then* $\mathbf{B}^k(\mathbf{S})$ *is non-decreasing. In particular if* $\mathbf{S}$ *is under-informed then* $\mathbf{B}^k(\mathbf{S})$ *is non-decreasing.*

Proof. Suppose that $\mathbf{s}_i \subseteq (\mathbf{B}(\mathbf{S}))_i$ for any $i \in \mathcal{I}$. We only need to prove that $(\mathbf{B}(\mathbf{S}))_i \circ (\mathbf{B} \circ \mathbf{B}(\mathbf{S}))_i$ and the result follows by induction. We can write the terms of $\mathbf{B}(\mathbf{S})$ recursively:

$$(\mathbf{B}(\mathbf{S}))_i = \text{TBR}_i((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{s}_{i+1}, ..., \mathbf{s}_I), \text{ for } i = 1, ..., I.$$

Also

$$(\mathbf{B}^2(\mathbf{S}))_i = \text{TBR}_i \left((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_{i-1}, (\mathbf{B}(\mathbf{S}))_{i+1}, ...(\mathbf{B}(\mathbf{S}))_I\right)$$

By assumption we have $\mathbf{S}_{-1} \subseteq (\mathbf{B}(\mathbf{S}))_{-1}$. Moreover $\text{TBR}_1(\mathbf{S}_{-1}) \cap \mathbf{B}(\mathbf{S})_{-1} = \emptyset$. As a consequence

$$\text{TBR}_1(\mathbf{S}_{-1}) \subseteq \text{TBR}_1((\mathbf{B}(\mathbf{S}))_{-1}).$$

Suppose we proved that $(\mathbf{B}(\mathbf{S}))_j \subseteq (\mathbf{B}^2(\mathbf{S}))_j$ for $j = 1, ..., i$ ($i < n$). We now prove that $(\mathbf{B}(\mathbf{S}))_{i+1} \subseteq (\mathbf{B}^2(\mathbf{S}))_{i+1}$, and it will conclude the proof. We have

$$((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_i, \mathbf{s}_{i+2}, ..., \mathbf{s}_I) \subseteq ((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_i, (\mathbf{B}(\mathbf{S}))_{i+2}, ..., (\mathbf{B}(\mathbf{S}))_I)$$

and $\text{TBR}_{i+1}\left(((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_i, \mathbf{s}_{i+2}, ..., \mathbf{s}_I)\right)$ does not intersect the set $\mathbf{B}^2(\mathbf{S}))_1 \cup ... \cup \mathbf{B}^2(\mathbf{S}))_i \cup (\mathbf{B}(\mathbf{S}))_{i+2} \cup ... \cup (\mathbf{B}(\mathbf{S}))_I$. Consequently it is contained in

$$Br_{i+1}\left((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_i, (\mathbf{B}(\mathbf{S}))_{i+2}, ..., (\mathbf{B}(\mathbf{S}))_I\right).$$

In other terms $(\mathbf{B}(\mathbf{S}))_{i+1} \subseteq (\mathbf{B}^2(\mathbf{S}))_{i+1}$, and the proof is complete. When $\mathbf{S} \in \mathcal{S}_u$ this follows from Lemma A.A.2. ■

Proof of Theorem 1. The sequence $(\mathbf{B}^k(\emptyset))_k$ is non-decreasing and bounded above in a finite set. Thus there exist $\underline{\mathbf{S}}^*$ and an integer K such that $\mathbf{B}^K(\emptyset) = \underline{\mathbf{S}}^*$. Let $\mathbf{S}^*$ be a tight Nash equilibrium. We need to show that $\underline{\mathbf{S}}^* \subseteq \mathbf{S}^*$ and the proof will be complete. Both $\emptyset$ and $\mathbf{S}^*$ are under-informed. Thus $\mathbf{B}^k(\emptyset) \subseteq \mathbf{B}^k(\mathbf{S}^*) = \mathbf{S}^*$ for any k by Lemma A.A.1. This concludes the proof. □

Proof of Proposition 2. We show that if $\mathbf{S}^* \in TNE$ and $\mathbf{S}^* \subseteq \overline{\mathbf{S}}$ then $\pi_i(\mathbf{S}^*) \geq \pi_i(\overline{\mathbf{S}})$. Therefore, any TNE Pareto-dominates any TNE with a larger set of informed agents. Let $\mathbf{D} = \overline{\mathbf{S}} \setminus \mathbf{S}^*$. We have

$$\pi_i(\overline{\mathbf{S}}) = \frac{m(\mathbf{S}^*)}{m(\overline{\mathbf{S}})}\pi_i(\mathbf{S}^*) + \frac{m(\mathbf{D})}{m(\overline{\mathbf{S}})} \sum_{d \in \mathbf{D}} \sigma_{d,i}.$$

However $\sigma_{d,i} \leq \pi_i(\mathbf{S}^*),\ \forall d \in \mathbf{D}$ because $\mathbf{S}^*$ is tight. Hence $\pi(\overline{\mathbf{S}}) \leq \pi(\mathbf{S}^*)$. □

Proof of Proposition 3. We first show that, for any $\mathbf{S}_{-i}$, we have $\text{TBR}'_i(\mathbf{S}_{-i}) \subseteq \text{TBR}_i(\mathbf{S}_{-i})$. Let π'_i denote the payoff function of player i in the game with sharing matrix $\mathbf{\Sigma}'$. Since $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are symmetric, we have

$$\pi_i(\mathbf{s}_i, \mathbf{S}_{-i}) = \frac{1}{m(\mathbf{S})}\left(1 - \sum_{j \notin \mathcal{M}(\mathbf{S})} \sigma_{ji}\right) \quad \pi'_i(\mathbf{s}_i, \mathbf{S}_{-i}) = \frac{1}{m(\mathbf{S})}\left(1 - \sum_{j \notin \mathcal{M}(\mathbf{S})} \sigma'_{ji}\right).$$

Note that the characterization of TBRs presented in Result 2 implies that, if $j \in \text{TBR}_i(\mathbf{S}_{-i})$, then

$$\sigma_{ji} > \pi_i(\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}), \tag{A.1}$$

meaning that the value of the shares of every informed neighbor strictly exceeds the agent's current payoff. Now, by (A.1), It is sufficient to show that

$$\pi_i'(\mathrm{TBR}_i'(\mathbf{S}_{-i}), \mathbf{S}_{-i}) \geq \pi_i(\mathrm{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}).$$

Let $\mathcal{M} := \mathcal{M}(\mathrm{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i})$ and $m := |\mathcal{M}|$. Then

$$\begin{aligned}
\pi_i'(\mathrm{TBR}_i'(\mathbf{S}_{-i}), \mathbf{S}_{-i}) &\geq \pi_i'(\mathrm{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}) \\
&= \frac{1}{m}\left(1 - \sum_{j \notin \mathcal{M}} \sigma_{ji}'\right) \\
&\geq \frac{1}{m}\left(1 - \sum_{j \notin \mathcal{M}} \sigma_{ji}\right) \\
&= \pi_i\left(\mathrm{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}\right)
\end{aligned}$$

which concludes the proof of the first point. We now prove the last point. First note that, if $\mathbf{S}'_{-i} \subseteq \mathbf{S}_{-i}$ then

$$\mathrm{TBR}_i'(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i} \subseteq \mathrm{TBR}_i'(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i} \subseteq \mathrm{TBR}_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i}.$$

Consequently, for any $k \in \mathbb{N}^*$, we have that $(\mathbf{B}')^k(\emptyset) \subseteq \mathbf{B}^k(\emptyset)$, which proves that $\underline{\mathbf{S}}(\mathbf{\Sigma}') \subseteq \underline{\mathbf{S}}(\mathbf{\Sigma})$.

□

Proof of Proposition 4. Suppose $\mathbf{S}^*$ is a nonempty TNE. Then there exists $i \in \mathcal{I}$ and $j \in \mathcal{R}_{\mathcal{I}}$ such that $j \in \mathbf{s}_i^*$. By Result 2, $\mathcal{R}_i \subseteq \mathbf{S}^*$ because $\sigma_{ji} = \lambda$ for any $j \in \mathcal{R}_i$. Also $\lambda = \lambda_{ji} > \pi_i^* = \frac{1}{m(\mathbf{S}^*)} \geq \pi_j^*$ for any $j \in \mathcal{I}$ (see identity (1)). Let $j' \in \mathcal{J}$ and $i' \in \mathcal{I}$ be such that $j' \in \mathcal{R}_{i'}$. We necessarily have $\sigma_{j'i'} = \lambda > \frac{1}{m(\mathbf{S}^*)} \geq \pi_{i'}(\mathbf{S}^*)$, which implies that j' must belong to $\mathcal{M}(\mathbf{S}^*)$. Finally $\mathbf{S}^* = \mathcal{R}_{\mathcal{I}}$ and $\mathbf{S}^*$ is a full communication profile.

Note that $\pi_i(\emptyset) = \frac{1-\lambda|\mathcal{R}_i|}{I}$. Thus point $a)$ follows from the fact that the empty profile is a tight Nash equilibrium iif $\lambda \leq \min_{i\in\mathcal{I}} \pi_i(\emptyset) = \frac{1-\lambda \max_{i\in\mathcal{I}}|\mathcal{R}_i|}{I}$, which is equivalent to having $\lambda \leq \frac{1}{I+\max_{i\in\mathcal{I}}|\mathcal{R}_i|}$. At last, point $b)$ simply follows from the fact that, at any full communication profile, the payoff of any player i is equal to $\frac{1}{I+\mathcal{R}_\mathcal{I}}$. Thus any full communication profile is a TNE if and only if $\lambda > 1/(I+\mathcal{R}_\mathcal{I})$.

□

Proof of Proposition 5. Pick any concave function on $[0,1]$, such that $U(0)=0$, and suppose that $\mathcal{R}_\mathcal{I}$ degree-dominates $\mathcal{I}$. Then

$$I\sum_{j\in\mathcal{R}_\mathcal{I}}\left(U(1-\lambda|\mathcal{N}_j|)+|\mathcal{N}_j|U(\lambda)\right)\geq|\mathcal{R}_\mathcal{I}|\sum_{i\in\mathcal{I}}\left(U(1-\lambda|\mathcal{N}_i|)+|\mathcal{N}_i|U(\lambda)\right),$$

because $U(1-d'\lambda)+d'U(\lambda)\geq U(1-d\lambda)+dU(\lambda)$ for $d'\geq d$. Hence

$$\frac{1}{|\mathcal{R}_\mathcal{I}|}\sum_{j\in\mathcal{R}_\mathcal{I}}\left(U(1-\lambda|\mathcal{N}_j|)+|\mathcal{N}_j|U(\lambda)\right)\geq\frac{1}{I}\sum_{i\in\mathcal{I}}\left(U(1-\lambda|\mathcal{N}_i|)+|\mathcal{N}_i|U(\lambda)\right),$$

which implies that $W(\emptyset)\leq W(\mathcal{R}_\mathcal{I})$. The second point is proved similarly.

□

Proof of Result 4. Consider a player i finding profitable to inform agent k but not agent l is not informed. We want to be sure that this still holds once agent l gets informed by a third party. Let us denote:

$$\begin{cases}\psi(m)=\frac{m}{m-1}\frac{a(m-1)}{a(m)}\\ \varphi_i=(\psi(m)-1)\sum\sigma_{ji}\\ \varphi_i'=(\psi(m+1)-1)(\sum\sigma_{ji}+\sigma_{li})\end{cases}$$

The conditions write

$$\begin{cases}\max(\varphi_i,\varphi_i')<\sigma_{ki}\\ \sigma_{li}\leq\varphi_i'\end{cases}$$

The former condition says that player i wants to inform agent k whether agent l is informed or not, the latter says that i does not want to inform l once he informs k. Then, a sufficient condition for monotonicity to hold is that $\varphi_i'\leq\varphi_i$ (i.e. the communication threshold decreases once l gets informed). Few calculus shows that this condition writes

$$\Big(\psi(m+1)-1\Big)\sigma_{li} \leq \Big(\psi(m)-\psi(m+1)\Big)\sum \sigma_{ji}$$

Now, since $\sigma_{li} \leq \Big(\psi(m+1)-1\Big)\sum \sigma_{ji}$, a sufficient condition is therefore given by

$$\Big(\psi(m+1)-1\Big)^2 \leq \psi(m)-\psi(m+1)$$

This latter inequality is equivalently written

$$\left(\frac{m^2-1}{m^2}\right)\left(\frac{a(m)}{a(m+1)}\right)^2 \leq \frac{m\,a(m-1)-(m-1)\,a(m)}{(m+1)\,a(m)-m\,a(m+1)}$$

The LHS is clearly less than unity. Now, the RHS is larger than unity if and only if function $a()$ is convex: indeed, this amounts to have $ma(m-1)-(m-1)a(m) \geq (m+1)a(m)-ma(m+1)$, i.e. $a(m+1)-a(m) \geq a(m)-a(m-1)$.

□

Appendix B: Subgame perfect equilibrium and minimum TNE

Consider the extensive form game Γ, whose set of players is $\mathcal{I}$ and whose associated tree $\mathcal{T}$ is defined by the set of nodes $\{(t,i,\mathbf{S})\}_{t\leq T, i\in\mathcal{I}, \mathbf{S}\subseteq\mathcal{J}}$ - where $T=J+1$ - with the following structure:

- the *root* is $(1,1,\emptyset)$
- for $t \leq T-1$, $\mathbf{S} \subseteq \mathcal{J}$, $i < I$, the set of successors of node $(t,i,\mathbf{S})$ is $\{(t,i+1,\mathbf{S}'):\ \mathbf{S}\subseteq\mathbf{S}'\}$
- for $t \leq T-1$ and $\mathbf{S} \subseteq \mathcal{J}$, the set of successors of node $(t,I,\mathbf{S})$ is$\{(t+1,1,\mathbf{S}'):\ \mathbf{S}\subseteq\mathbf{S}'\subseteq\mathcal{N}\setminus\mathcal{J}\}$;
- $(T,1,\mathbf{S})$ is a terminal node with payoff $(\pi_i(\mathbf{s}))_{i\in\mathcal{N}}$.[25]

25. In order to avoid unnecessary cumbersomeness in the definition of the extensive form game, we also implicitly assume that we reach a terminal node with this payoff if all I players decide to leave the set of

PROPOSITION B.B.1. *The sub-game perfect equilibriums* [26] *of the extensive form game* Γ *are the TNE with the minimum set of informed agents.*

Proof. Recall that $\underline{S}$ denotes the set of informed agents associated with the minimum equilibrium. The statement of the proposition can be rephrased as follows: *an action profile is a sub-game perfect equilibrium if and only if the associated set of informed agents is* $\underline{\mathbf{S}}$.

Since any profile associated to $\underline{\mathbf{S}}$ Pareto-dominates any other Nash equilibrium of the communication game, it is immediate to conclude that any profile associated to $\underline{\mathbf{S}}$ is subgame-perfect, since we reach a terminal node only after all players decide not changing the set of informed agent.

Consider an action profile such that $\mathbf{S}$ is not contained in $\underline{\mathbf{S}}$, and let $(\hat{t}, \hat{\imath}, \hat{\mathbf{s}})$ be the first node in the path with the property that $\hat{\mathbf{S}} \not\subseteq \underline{\mathbf{S}}$. In the sub-game associated with this initial node, the induced action profile associated with $\mathbf{S}$ can not correspond to a Nash equilibrium since any Nash equilibrium of the normal-form game is Pareto dominated by the minimum TNE. ■

Appendix C: Tullock contest

In the paper, the contest was modelled by assuming that the probability to win the contest was exogenous and uniformly distributed across agents.

Consider rather that informed agents can invest an amount $x_i \in [0, +\infty)$ to win the contest. We consider a linear Tullock contest for simplicity. That is, for a given profile of effort $\mathbf{x} = (x_i)_{i \in \mathcal{M}}$, and denoting $x = \mathbf{1}_m^T \mathbf{x}$ (where here $\mathbf{1}_m$ is the profile

---

informed agents as it is. In order to write this formally, we would need to characterize the nodes of the tree as the whole sequence of moves which led to the current state, instead of only labeling the nodes as the current state. Hence our choice of T makes sure that, after T rounds, either everyone is informed, or nobody decided to inform any additional agent in the last round.

26. A sub-game perfect is such that, in any sub-game, the induced profile is a TNE of the reduced game.

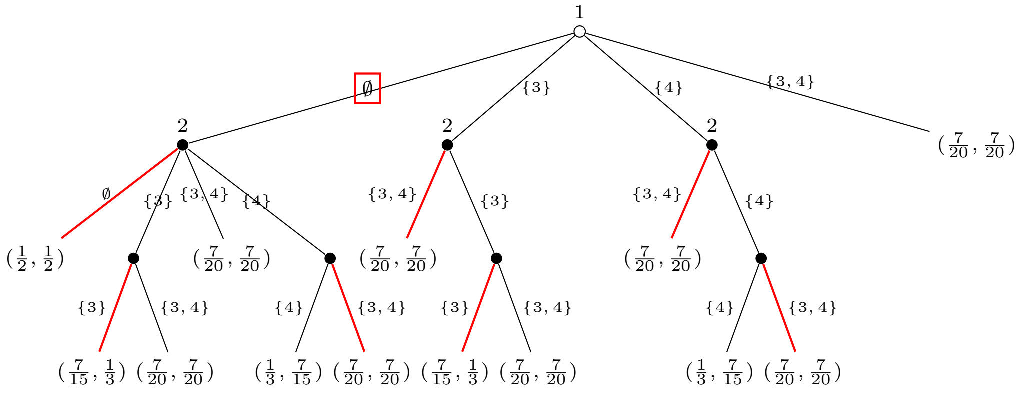


FIGURE B.1. extensive-form game associated with Example 1: the sub-game perfect equilibrium corresponds to the “empty path", with payoff $(1/2, 1/2)$.

of ones if dimension m), each informed agent wins the contest with probability $\frac{x_i}{\mathbf{x}}$ if $\mathbf{x} > 0$, whereas $\mathbf{x} = 0$ entails a uniform probability to win $\frac{1}{m}$. Assume also that producing effort x_i generates a linear cost cx_i, with $c > 0$.

Then, agent $i \in \mathcal{M}$ gets the expected payoff:

$$\pi_i(\mathbf{x}) = \sum_{j \in \mathcal{M}} \sigma_{ji} \frac{x_j}{x} - cx_i$$

The set of first order conditions (for an interior equilibrium) gives

$$cx^2 = \sum_{j \in \mathcal{M}} (\sigma_{ii} - \sigma_{ji}) x_j \quad \forall i \in \mathcal{M} \tag{C.1}$$

The system of FOCs generates a unique interior equilibrium which satisfies:

$$\mathbf{\Gamma x} = cx^2 \mathbf{1}_m$$

where $\mathbf{\Gamma} = (\gamma_{ij})$ is a the matrix with null diagonal entries such that $\gamma_{ij} = \sigma_{ii} - \sigma_{ji}$ for all i, j. Hence, whenever $\mathbf{\Gamma}$ is invertible, the equilibrium writes

$$\mathbf{x} = cx^2 \mathbf{\Gamma}^{-1} \mathbf{1}_m$$

Summing all entries of vector $\mathbf{x}$, we get

$$x = \frac{1}{c\,\mathbf{1}_m^T \mathbf{\Gamma}^{-1} \mathbf{1}_m} \tag{C.2}$$

We deduce the characterization of the equilibrium, denoted $\mathbf{x}^*(\mathcal{M})$:

$$\mathbf{x}^*(\mathcal{M}) = \frac{1}{c\,(\mathbf{1}_m^T \mathbf{\Gamma}^{-1} \mathbf{1}_m)^2}\,\mathbf{\Gamma}^{-1}\mathbf{1}_m \tag{C.3}$$

We can distinguish which relationships are strategic substitutes or strategic complements at equilibrium. Precisely:

RESULT C.1. *Consider a sharing matrix $\mathbf{\Sigma}$ and a set of informed agents $\mathcal{M}$. At any interior equilibrium effort profile $\mathbf{x}$, there is strategic complementarity from j's effort to i's effort if and only if*

$$\sigma_{ii} - \sigma_{ji} > \frac{2}{\mathbf{1}_\mathbf{m}^\mathbf{T} \mathbf{\Gamma}^{-1} \mathbf{1}_m}$$

.

Proof of Result C.1. For all $i \in \mathcal{M}$, $cx^2 = \sum\limits_{k\in\mathcal{M}\setminus\{i\}} \gamma_{ik} x_k$. That is,

$$x_i = \sqrt{\sum_{k\in\mathcal{M}\setminus\{i\}} \frac{\gamma_{ik}}{c} x_k} - \sum_{k\in\mathcal{M}\setminus\{i\}} x_k$$

Hence,

$$\frac{\partial x_i}{\partial x_j} = \frac{\frac{\gamma_{ij}}{c}}{2\sqrt{\sum_{k\in\mathcal{M}\setminus\{i\}} \frac{\gamma_{ik}}{c} x_k}} - 1$$

That is, noting that $\sum_{k\in\mathcal{M}\setminus\{i\}} \frac{\gamma_{ik}}{c} x_k = cx^2$,

$$\frac{\partial xi}{\partial x_j} > 0 \text{ iff } \gamma_{ij} > 2cx$$

Plugging (C.2), we get the result. □

We then observe that, under low enough sharing, the nature of strategic interaction is not ambiguous:

RESULT C.2. *Consider a sharing matrix $\mathbf{\Sigma}$ with off-diagonal entries sufficiently close to* 0. *Then, efforts are strategic substitutes.*

Proof of Result C.2. Suppose that $\mathbf{\Sigma}$ is sufficiently close to the identity matrix. Thus, for a profile with q interior efforts, the q-square matrix $\mathbf{\Gamma}$ is close to the matrix with all off-diagonal entries equal to 1 and with null diagonal; the sum of entries of its inverse matrix is $q/(q-1)$ (as $\mathbf{\Gamma}^{-1} = 1/(q-1).(\mathbf{J}_q - 2.\mathbf{I}_q)$, where $\mathbf{J}_q$ is the q-square matrix with all entries equal to 1 and $\mathbf{I}_q$ is the q-square identity matrix). Thus, when $\gamma_{ij} = 1-\varepsilon$ for $\varepsilon > 0$ around 0, we get strategic complementarities when $1-\varepsilon > \frac{2(q-1)}{q}$; for $\varepsilon = 0$, this means $2 > q$, which holds. □

We give condition for profitable communication for equi-sharing undirected matrices. Let $\mathbf{x}, \mathbf{x}'$ denote the respective profiles of effort in $\mathcal{M}$ and $\mathcal{M} \cup \{j\}$:

$$\frac{(1-\lambda|\mathcal{N}_i|)x'_i + \lambda \sum_{k\in\mathcal{M}} x'_k}{x' + x'_j} = \pi_i(\mathcal{M}\cup\{j\}, \mathbf{x}') > \pi_i(\mathcal{M}, \mathbf{x}) = \frac{(1-\lambda|\mathcal{N}_i|)x_i + \lambda \sum_{k\in\mathcal{M}} x_k}{x}$$

if and only if

$$\lambda > \underbrace{\frac{A}{x}}_{=\pi_i(\mathcal{M},\mathbf{x})} + \frac{x'_{-j}A - xA'}{xx'_j}$$

where $A = (1-\lambda|\mathcal{N}_i|)x_i + \lambda \sum_{k\in\mathcal{M}} x_k$ and $A' = (1-\lambda|\mathcal{N}_i|)x'_i + \lambda \sum_{k\in\mathcal{M}} x'_k$

Note that when $\mathbf{x}'_{-j} = \mathbf{x}$, the above condition boils down to that of the benchmark model whatever the effort choice of agent j:

$$\pi_i(\mathcal{M}\cup\{j\}, \mathbf{x}, x'_j) > \pi_i(\mathcal{M}, \mathbf{x}) \Leftrightarrow \lambda > \pi_i(\mathcal{M}, \mathbf{x})$$

Appendix D: The investment game

In this Appendix, we supplement the communication game with a stage in which, prior to communication, agents can invest in the ability to identify opportunities. Our aim is to understand how the structure of the sharing network affects investment decisions, given the interplay between investment and communication. We focus on un-directed equi-sharing networks for tractability.

Our main findings are as follows. Under low investment costs, at every equilibrium of the investment game, the set of investors is a dominating set, meaning that the whole society is informed after the communication stage - This is not straightforward, because the neighbor of an agent who is informed through communication gets a positive payoff. At an equilibrium with communication, an investor is called *communication-critical* if removing him from the set of investors cuts communication, and the analysis stresses that communication-critical investors are key in shaping incentives to invest. By contrast, under large investment cost, the whole society does not necessarily get informed, and on a given network, there can be different equilibria with communication, each with distinct number of informed agents. All proofs are deferred to the end of this appendix.

D.1. The investment game in equi-sharing networks

We consider a set $\mathcal{N} := \{1, ..., n\}$ of agents, facing the binary choice of investing or not in a technology allowing them to know the existence of an opportunity. Agent i chooses $x_i \in \{0, 1\}$, $x_i = 1$ meaning that she invests, in which case she pays a cost $c > 0$. Given an action profile $\mathbf{x}$, let

$$\mathcal{I}(\mathbf{x}) := \{i \in \mathcal{N} : \ x_i = 1\}; \ \mathcal{J}(\mathbf{x}) := \{i \in \mathcal{N} : \ x_i = 0\}$$

be respectively the set of investors and non-investors. Let $I(\mathbf{x}) := |\mathcal{I}(\mathbf{x})|$. We call $\underline{\mathbf{S}}(\mathbf{x})$ the set minimum TNE in the communication game associated with the set of investors $\mathcal{I}(\mathbf{x})$.

As seen in the preceding section, given a set of investors, there can be multiple TNEs in communication. We assume that investors communicate about the opportunity in accordance with the minimum TNE.[27] This approach is supported by the fact that the minimum TNE Pareto-dominates other equilibrium, and corresponds

27. In other terms, we do not consider the game where strategies consist in choosing both investment and communication. Communication is induced by the set of investors resulting in investing strategy profiles.

to the sub-game perfect equilibrium of the associated extensive-form game (see the Appendix, Section B). Note that, if the set of investor changes, then the set of informed agents evolves in accordance with the minimum equilibrium.

We focus on un-directed equi-sharing networks for tractability. Indeed, as seen before, in this case only two configurations can be equilibria, the no communication and the full communication configuration, and the value of the share above which communication emerges at the minimum equilibrium is given by[28]

$$\mu(\mathbf{x}) := \mu\left(\mathcal{I}(\mathbf{x})\right) = \frac{1}{I(\mathbf{x}) + \max_{i\in\mathcal{I}(\mathbf{x})} |\mathcal{R}_i(\mathbf{x})|}$$

By Proposition 4 there is either no communication or full communication during the communication stage:

1) when $\lambda \leq \mu(\mathbf{x})$, then $\mathcal{M}(\mathbf{x}) = \mathcal{I}(\mathbf{x})$ and

$$U_i(x_i, \mathbf{x}_{-i}) = \begin{cases} -c + \frac{1-\lambda|\mathcal{R}_i(\mathbf{x})|}{I(\mathbf{x})} & \text{if} \quad i \in \mathcal{I}(\mathbf{x}); \\ \frac{\lambda(|\mathcal{N}_i|-|\mathcal{R}_i(\mathbf{x})|)}{I(\mathbf{x})} & \text{if} \quad i \notin \mathcal{I}(\mathbf{x}). \end{cases} \tag{D.1}$$

2) when $\lambda > \mu(\mathbf{x})$, then $\mathcal{M}(\mathbf{x}) = \mathcal{I}(\mathbf{x}) \cup \mathcal{R}(\mathbf{x})$[29], and

$$U_i(x_i, \mathbf{x}_{-i}) = \begin{cases} -cx_i + \frac{1-\lambda|\mathcal{R}_i(\mathbf{x})|}{m(\mathbf{x})} & \text{if} \quad i \in \mathcal{M}(\mathbf{x}); \\ \frac{\lambda(|\mathcal{N}_i|-|\mathcal{R}_i(\mathbf{x})|)}{m(\mathbf{x})} & \text{if} \quad i \notin \mathcal{M}(\mathbf{x}). \end{cases} \tag{D.2}$$

Importantly, the communication threshold $\mu(\mathbf{x})$ is endogenous to the investment decisions; It depends on both the structure of the sharing network and the allocation of investors.

For the sake of clarity, we first assume that the investment cost is *small*, in the sense that any positive payoff difference between two strategy profiles is necessarily

28. Analogously to the definitions given above, for $i \in \mathcal{I}(\mathbf{x})$ we define $\mathcal{R}_i(\mathbf{x}) := \{j \in \mathcal{J}(\mathbf{x}) : \ \sigma_{ji} = \lambda\}$ and $\mathcal{R}(\mathbf{x}) := \cup_{i\in\mathcal{I}(\mathbf{x})}\mathcal{R}_i(\mathbf{x})$.

29. Note that we have the implication $\lambda > \mu(\mathbf{x}) \Rightarrow \lambda > \frac{1}{m(\mathbf{x})}$

strictly greater than c.[30] This assumption guarantees that no agent has a zero payoff at equilibrium, because investing entails a positive payoff. We study large cost thereafter.

D.2. Cover-criticality and communication-criticality

Some definitions will prove useful in the analysis. A *dominating set* is a set of agents such that every non member of the set is linked to at least one member of the set. A dominating set is *minimal* when by excluding any member of the dominating set, the resulting set is no longer a dominating set. The smallest cardinal of all dominating sets is called the *domination number* of the graph. An agent $i \in \mathcal{S}$ said *cover-critical* with respect to dominating set $\mathcal{S}$ whenever the set $\mathcal{S} \setminus \{i\}$ is no longer a dominating set. An *independent set* is a set in which there is no link between any pair of members, a *maximal independent set* (called MIS thereafter) is an independent dominating set (it is thus a particular case of minimal dominating set).

DEFINITION D.1. An investor $i \in \mathcal{I}$ is *communication-critical* with respect to the set of investors $\mathcal{I}$ if

$$\mu(\mathcal{I}) < \lambda \leq \mu\left(\mathcal{I} \setminus \{i\}\right) \tag{D.3}$$

In words, the set of investors $\mathcal{I}$ induces communication, while $\mathcal{I} \setminus \{i\}$ doesn't. Communication-criticality depends on the value of λ. Indeed, for communication to be triggered by the investment decision of an investor, it must be that λ crosses the threshold with the new set of investors.[31] To fix ideas on the role of communication-critical and cover-critical agents, we start with a simple example:

EXAMPLE D.1 (*A kite*). In the sharing network depicted by Figure D.1, $\mathcal{I}(\mathbf{x}) =$

30. This definition makes sense, since the game is *finite*.

31. Identifying communication-critical investors is thus a difficult task. Note that, for a given network structure and a given investor set, those investors whose investment decision does not affect the communication threshold cannot be communication-critical.

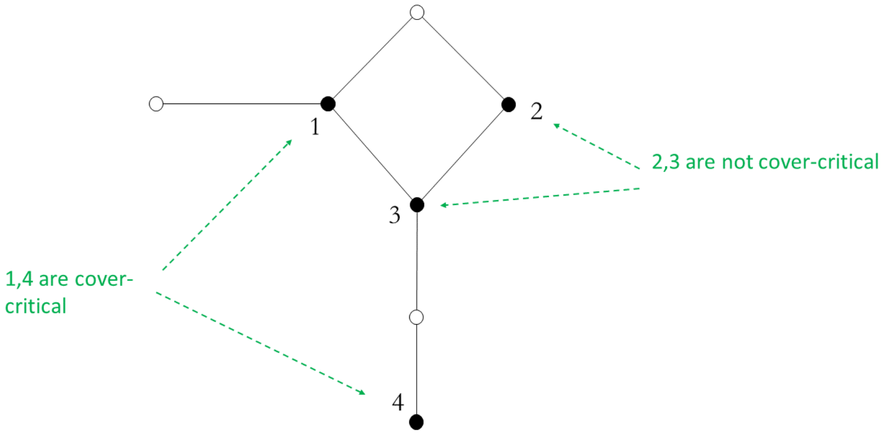


FIGURE D.1. Black nodes represent investors, who form a dominating set.

$\{1, 2, 3, 4\}$, and only agents 1 and 4 are cover-critical. We have $\mu(\mathbf{x}) = 1/6$.
Suppose first that $\lambda > 1/5$. then none of the agents is communication-critical, because adding a non-investor can only increase μ from $1/6$ to $1/5$.
Suppose now that $\lambda \in]1/5, 1/6[$. Then there is communication and agents $1, 2$ and 4 are communication-critical, while agent 3 is not (note that, if agent 3 stops investing, the quantity μ does not change).
Finally, if $\lambda < 1/6$, then this profile does not induce communication. Hence no agent can be communication-critical. ■

D.3. Equilibrium characterization of the investment game

At first glance we might believe that, at equilibrium, some agents could end up being uninformed. Indeed, an agent linked to neighbors of investors receives shares, and thus might possibly be better-off free riding. However the next theorem forbids that possibility, showing that information about the opportunity always fully disseminates in the society:

THEOREM D.D.1. *If $\mathbf{x}^*$ is a Nash equilibrium then $\mathcal{I}(\mathbf{x}^*)$ is a dominating set. Moreover, the following statements hold:*

(i) *If $\lambda \leq \frac{1}{n}$ there is a unique Nash equilibrium $\mathbf{x}^*$, with $\mathcal{I}(\mathbf{x}^*) = \mathcal{N}$.*

(ii) *If $\lambda > \frac{1}{n}$*[32]*, every Nash equilibrium $\mathbf{x}^*$ induces communication: $\lambda > \mu(\mathbf{x}^*)$, and every investor who is not communication-critical is cover-critical.*

Theorem 2 says that, at any equilibrium, any non-investor is necessarily linked to an investor. Whether λ is larger or smaller than $\frac{1}{n}$ determines if communication emerges at equilibrium. An important property of the set of investors at equilibrium is that any strictly larger set of investors induces communication (see Lemma D.D.3]. When shares are smaller than $\frac{1}{n}$, what an agent gets in expectation from investing outweights the expected benefit from free riding. Hence everyone investing is necessarily an equilibrium. On the other hand, any smaller set of investor violates the property. On the other hand, when shares are larger than $1/n$, point (ii) states that communication occurs at equilibrium, if any. The logic behind this result is that, if there were a Nash equilibrium without communication, the configuration would be such that any new investor would trigger communication. Combined with the fact that λ is strictly greater than $1/n$, this brings a contradiction. Importantly, among agents who are not communication-critical, cover-criticality is the right criterion to rank incentives, in the sense that agents who prefer investing are precisely cover-critical agents. However, cover-criticality is not informative about incentives to invest for communication-critical agents.

Let us go back to Example D.1. First, if $\lambda > 1/5$, no agent is communication-critical. Hence both agents 2 and 3 have an incentive to stop investing, while agents 1 and 4 are cover-critical and therefore do not want to stop investing. This is what we have in mind when saying that, among agents who are not communication-critical,

32. If $\overline{d} = n - 1$ (we then say that *there exist central agents*), we have $\overline{\lambda} = \frac{1}{n}$, and condition (ii) never holds. However, as soon as the maximal degree is strictly smaller than $n - 1$ (which we call *networks without central agents*), λ can be larger than $1/n$, and communication can emerge.

cover-criticality is the right criterion to look at. We can check that $\mathcal{I}^* = \{1, 2, 4\}$ is a Nash equilibrium then. If $\lambda \in]1/6, 1/5[$ the configuration is not a Nash equilibrium because agent 3 is neither communication-critical nor cover-critical. Agents $1, 2$ and 4 are all communication-critical, and establishing whether or not they are better off investing can only be done by comparing their payoffs. We can easily check that $\mathcal{I}^* = \{1, 2, 4\}$ is a Nash equilibrium. If $\lambda < 1/6$ then there is no communication, because $\mu(\mathbf{x}) = 1/6$. Then one can check that $\mathcal{I}^* = \{1, 2, 4, 5\}$ is a Nash equilibrium.

As we just saw in the last case of previous example, a Nash equilibrium with communication is not necessarily a minimal dominating set. However, the members of an important subclass of minimal dominating set - the maximal independent sets - are natural candidates to stability. As it turns out, they are systematically equilibria for large enough values of the share λ, because every agent is then cover-critical.

PROPOSITION D.D.1. *If $\mathcal{I}(\mathbf{x})$ is a maximal independent set with communication, then $\mathbf{x}$ is a Nash equilibrium. Moreover, if there exists a minimum dominating set $\mathcal{D}$ inducing communication then there exists a Nash equilibrium such that investors form a maximal independent set.*

EXAMPLE D.2. In the 13-agents network depicted in Figure D.2, the black nodes constitute a minimum dominating set, with $\mu(\mathbf{x}) = 1/7$. Suppose that $\lambda \in]2/13, 1/6]$. Then $\mathbf{x}$ is not a Nash equilibrium because, if agent i deviates, $\mu(\mathbf{x}') = 1/6 \geq \lambda$, communication is shut, and thus she obtains $\lambda/2 > 1/13$. We can easily build a MIS containing the black node on the right, with 7 active agents (μ is then equal to $1/11$).

■

D.4. Existence

We consider here the interesting case $\lambda > \frac{1}{n}$. If a cover-critical agent always prefer investing, a communication-critical investor that is not cover-critical can be better off

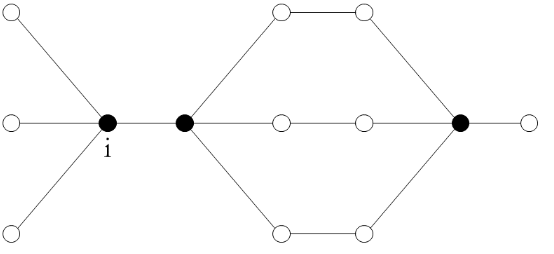


FIGURE D.2. Black nodes are investors

not investing. We will see how this particularity of the game can deter the existence of a Nash equilibrium. Then we will give conditions for existence.

First of all, we observe that the existence of an equilibrium with communication is guaranteed under low or high values of shares:

PROPOSITION D.D.2. *Suppose that the network has no central agent. When λ is sufficiently close to $\frac{1}{n}$ from above or when λ is sufficiently close to $\overline{\lambda}$ from below, there exists a Nash equilibrium of the investment game.*

When the share λ tends to $\frac{1}{n}$ from above, starting from a configuration such that all agents are informed under communication, communication-critical agents don't find it profitable to free ride; too low shares deter such profitable deviations. And when the share λ tends to $\frac{1}{\overline{d}+1}$ from below, there always exists a stable MIS with communication; in fact, any MIS containing an agent of maximal degree is stable for shares of value just below the upper bound, and the non-emptiness of such interval is guaranteed by the absence of central agents.

However, existence is an issue for intermediate values of λ, as illustrated in the next 14-agent example

EXAMPLE D.3 (*Non-existence of Nash equilibrium*). Suppose that $n = 14$ and the network is the union of two complete components C and C', with $|C| = |C'| = 7$. Then $\overline{d} = 6$ and the interval of values of λ we are interested in is $]1/14, 1/7]$. Given a Nash equilibrium $\mathbf{x}^*$, $\mathcal{I}(\mathbf{x}^*)$ is necessarily the union of one agent in one of the component and a subset of agents in the other. Indeed at any profile with at least two investors in each component, investors are neither communication critical, nor cover critical. Thus such a profile cannot be a Nash equilibrium.

Suppose without loss of generality that the isolated investor is in component C, and there are $p \geq 0$ investors in component C'. We then have $\mu(\mathbf{x}^*) = \frac{1}{7+p}$. There is no communication under the deviation of one of the investors in C' if and only if $\frac{1}{7+p} < \lambda \leq \frac{1}{6+p}$. Moreover, the deviation is profitable whenever $\frac{p-1}{p}\lambda > \frac{1}{14}$, i.e. $\lambda > \frac{p}{14(p-1)}$. Thus, this is possible whenever $\frac{p}{14(p-1)} \in \left[\frac{1}{7+p}, \frac{1}{6+p}\right]$, which holds for $p \in \{3, 4, 5\}$. Finally, there is no equilibrium for $\lambda \in \left]\frac{5}{56}, \frac{1}{11}\right] \cup \left]\frac{2}{21}, \frac{1}{10}\right] \cup \left]\frac{3}{28}, \frac{1}{9}\right]$.

Figure D.3 represents a best-response cycle[33] (actually there are two cycles, one of which being embedded in the other one). The key deviation is that of the communication-critical investor[34] who prefers free riding and interrupting communication rather than investing and triggering communication (on the figure, this is the deviation from the bottom-left configuration to the up-left configuration). This deviation is profitable because, by interrupting communication, there is a sufficiently large number of expected shares to free ride on. ■

As illustrated by the above example, communication-critical investors play a key role in non-existence. Next lemma shows that, if there is no Nash equilibrium, then there must exist a best-response cycle along which communication is lost, meaning that the cycle contains a communication-critical investor.

33. We say that $(\mathbf{x}^0, ..., \mathbf{x}^T)$ is a *best-response path* if, for $t = 0, .., T-1$, there exists $i_t \in \mathcal{N}$ such that $\mathbf{x}^{t+1} = \left(Br_{i_t}(\mathbf{x}^t), \mathbf{x}^t_{-i}\right)$. It is a *best-response cycle* if we also have $\mathbf{x}^0 = \mathbf{x}^T$.

34. This investor is not cover-critical; otherwise she would necessary prefer investing.

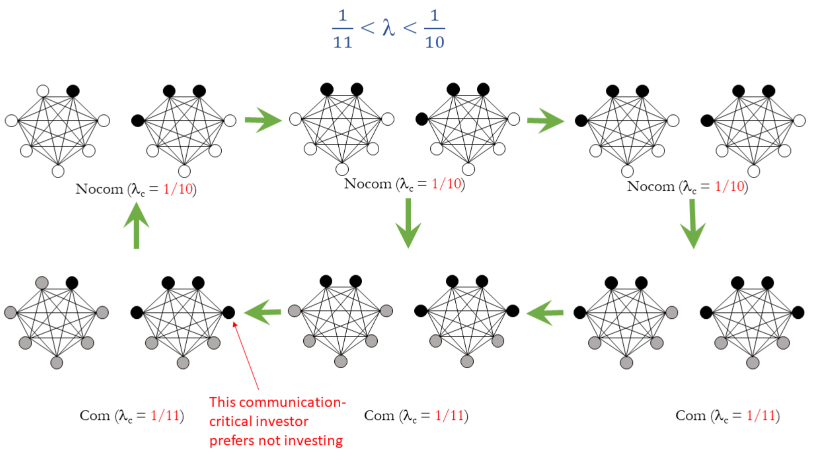


FIGURE D.3. A best-response cycle for $\lambda \in \left]\frac{1}{11}, \frac{1}{10}\right]$; Investors are in black, agents in grey are informed through communication, agents in white are not informed.

LEMMA D.D.1. *Let $\mathbf{x}$ be such that $\mu(\mathbf{x}) < \lambda$ and $\mathcal{I}(\mathbf{x})$ is a dominating set. If $\mathbf{x}' := (Br_i(\mathbf{x}_{-i}), \mathbf{x}_{-i})$ for some $i \in \mathcal{N}$ then either $\mu(\mathbf{x}') \geq \lambda$ or $\mathcal{I}(\mathbf{x}')$ is a dominating set. As a consequence, if there is no Nash equilibrium then there exists a best response cycle $(\mathbf{x}^0, ..., \mathbf{x}^T)$ along which communication is lost: $\mu(\mathbf{x}^{t-1}) < \lambda \leq \mu(\mathbf{x}^t)$, for some $t \in \{0, ..., T\}$.*

Existence is not guaranteed under intermediate values of the share. We now use Lemma D.D.1 to give conditions on the network structure guaranteeing the existence of a Nash equilibrium, regardless of the value of λ. Let $\gamma(\mathbf{G})$ be the domination number of network $\mathbf{G}$:

PROPOSITION D.D.3. *If the domination number $\gamma(\mathbf{G}) \geq \frac{1}{2}\left(-1 + \sqrt{1 + 4\bar{d}n}\right)$[35] then there exists a Nash equilibrium for any λ.*

The crucial point of the proof is that, if there is non-existence, there must be a cycle along which communication is lost. When this happens, there must be a

35. A simple sufficient condition being that $\gamma(\mathbf{G}) \geq \sqrt{\bar{d}n}$.

profitable deviation for a communication-critical investor. This condition guarantees that a communication-critical investor is never better off free riding.

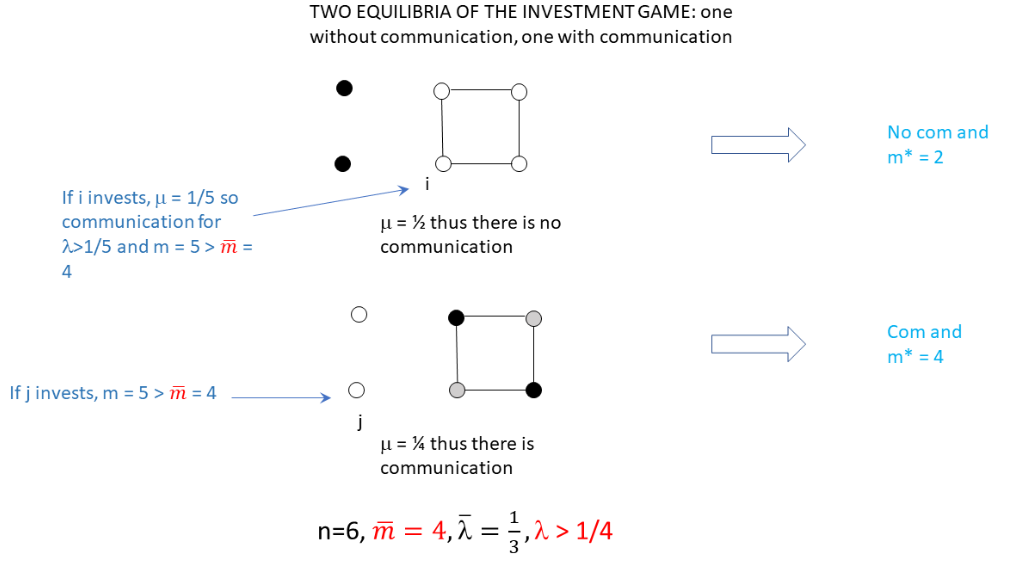
Lower bounds on the domination number are particularly useful when having this condition in mind.[36] For instance, take a circle network with n an even number. Then $\overline{d} = 2$, and $\gamma(\mathbf{G}) = \left[\frac{n}{3}\right]$; therefore there is an equilibrium if $n \geq 15$.

D.5. Beyond small costs

We suppose now that the cost of investment c is large. Generally speaking, for any value $c > 0$ there is an integer $\bar{m}(c)$ such that $\frac{1}{\bar{m}(c)+1} \leq c < \frac{1}{\bar{m}(c)}$. Let us abuse the notation and denote $\bar{m}$ for convenience. Investment cost is said to be large whenever $\bar{m} < n$. Generally speaking, larger costs reduce non-existence concern since less agents (not the whole society) get informed. However, it is interesting to note that some key properties established under low investment cost do not hold any more under large costs. We discuss the coexistence of equilibriums with and without communication, and the coexistence of equilibriums with distinct numbers of informed agents.

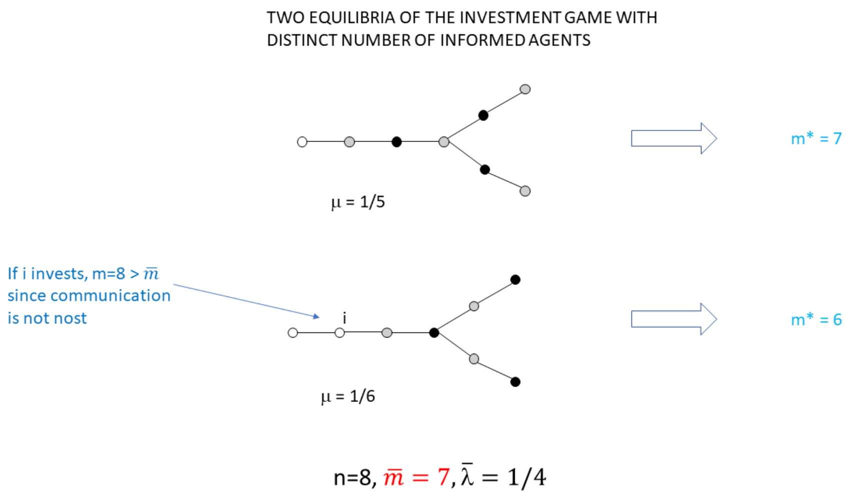
Coexistence of equilibriums with and without communication. Under low cost, there is a sharp separation between low share and large share. Under low share ($\lambda \leq \frac{1}{n}$), there is a single equilibrium without communication, whereas all equilibria entail communication for large shares ($\lambda > \frac{1}{n}$). This separation is no longer effective under large costs, as depicted in Figure D.5. In this example, the no communication equilibrium emerges because a communication-critical agent cannot invest without crossing the upper bound on informed agents $\bar{m}$; and the full communication

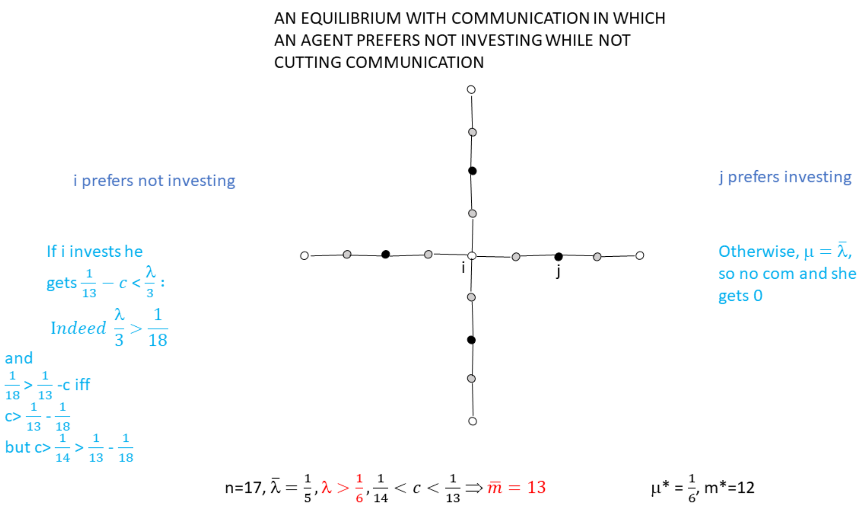
36. Several lower bounds on the domination number of connected graphs have been proved in the related literature in mathematics (see DeLaViña and Pepper (2010) for instance). In particular, the domination number is at least two thirds of the radius of the graph, three times the domination number is at least two more than the number of cut-vertices in the graph, and the domination number is at least two more than the number of cut-vertices in the graph, and the domination number of a tree is at least as large as the minimum order of a maximal matching.



equilibrium arises because the upper bound is already attained so that any additional investment would lead to a negative payoff for investors.

Equilibriums with distinct numbers of informed agents. Under low investment cost, the whole society gets informed in all equilibriums with communication, meaning that all equilibriums with communication have the same number of informed agents. This is no longer true under large costs, for at least two reasons. First, the equilibrium number of investors can be strictly lower than $\bar{m}$ in the case where any new investment generates a number of informed agents larger than the upper bound. This can generate multiple equilibriums with distinct numbers of informed agents, as shown in Figure D.5. Second, there is another, perhaps more subtle, channel. Under low investment cost, an agent who does not cut communication when switching to free riding always prefers investing; this property leads to the fact that the whole society gets informed in any equilibrium. This property no longer holds under large costs, as presented in Figure D.5. In this example, agent i does not cut communication when switching to free riding, and prefers not investing. Hence, there are various channels by which large costs limit incentives to invest.





D.6. *Proofs of the results on the investment game*

Given $i \in \mathcal{N}$ and a profile $\mathbf{x}$, let $\mathcal{N}_i^E(\mathbf{x}) := \{j \in \mathcal{J}(\mathbf{x}),\ \mathcal{N}_j \cap \mathcal{I}(\mathbf{x}) \subseteq \{i\}\}$ be the set of non-investor agents which are exclusively linked to i.[37]

37. Note that this set can be defined, whether $i \in \mathcal{I}(\mathbf{x})$ or not. If $i \in \mathcal{I}(\mathbf{x})$ then the inclusion is an equality.

LEMMA D.D.2. *Suppose that $\mathbf{x}$ is a profile with communication and $i \in \mathcal{I}(\mathbf{x})$ is not communication-critical. Then $U(x_i, \mathbf{x}_{-i}) < U(0, \mathbf{x}_{-i})$ if and only if $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}) \neq \emptyset$ and $\mathcal{N}_i^E(\mathbf{x}) = \emptyset$.*

Proof of Lemma D.D.2. We have $U_i(\mathbf{x}) = -c + \frac{1}{m(\mathbf{x})}$. Suppose first that $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}) = \emptyset$. Then $m(\mathbf{x}) \geq N_i + 1$, which directly implies that $(|\mathcal{N}_i| + 1)(m(\mathbf{x}) - 1 - |\mathcal{N}_i^E(\mathbf{x})|) \geq (|\mathcal{N}_i| - |\mathcal{N}_i^E(\mathbf{x})|)m(\mathbf{x})$. Hence

$$U_i(0, \mathbf{x}_{-i}) = \frac{(|\mathcal{N}_i| - |\mathcal{N}_i^E(\mathbf{x})|)\lambda}{m(\mathbf{x}) - 1 - |\mathcal{N}_i^E(\mathbf{x})|} < \frac{(|\mathcal{N}_i| - |\mathcal{N}_i^E(\mathbf{x})|)}{(|\mathcal{N}_i| + 1)(m(\mathbf{x}) - 1 - |\mathcal{N}_i^E(\mathbf{x})|)} \leq \frac{1}{m(\mathbf{x})},$$

Since c is small, this implies that $U_i(0, \mathbf{x}_{-i}) < U_i(x_i, \mathbf{x}_{-i})$. Suppose now that $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}) \neq \emptyset$ and $\mathcal{N}_i^E(\mathbf{x}) \neq \emptyset$. Then

$$U_i(0, \mathbf{x}_{-i}) = \frac{1 - |\mathcal{N}_i^E(\mathbf{x})|\lambda}{m(\mathbf{x}) - |\mathcal{N}_i^E(\mathbf{x})|}.$$

Suppose that $\mathcal{N}_i^E(\mathbf{x}) \neq \emptyset$. Then $|\mathcal{N}_i^E(\mathbf{x})|(\lambda m(\mathbf{x}) - 1) > 0$, because $\lambda > \frac{1}{m(\mathbf{x})}$. Consequently $m(\mathbf{x})(1 - |\mathcal{N}_i^E(\mathbf{x})|\lambda) < m(\mathbf{x}) - |\mathcal{N}_i^E|(\mathbf{x})$, and $U_i(0, \mathbf{x}_{-i}) < U_i(x_i, \mathbf{x}_{-i})$. We proved that, if $U(x_i, \mathbf{x}_{-i}) < U(0, \mathbf{x}_{-i})$ then $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}) \neq \emptyset$ and $\mathcal{N}_i^E(\mathbf{x}) = \emptyset$.

Suppose now that $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}) \neq \emptyset$ and $\mathcal{N}_i^E(\mathbf{x}) = \emptyset$, we then have $U_i(0, \mathbf{x}_{-i}) = \frac{1}{m(\mathbf{x})} > U_i(x_i, \mathbf{x}_{-i})$. □

LEMMA D.D.3. *Let $\mathbf{x}^*$ be a Nash equilibrium. Then, for any $\hat{\mathcal{I}}$ which stricly contains $\mathcal{I}(\mathbf{x}^*)$, we have $\mu(\hat{\mathcal{I}}) < \lambda$,*

Proof. Suppose by contradiction that there exists $i \notin \mathcal{I}(\mathbf{x}^*)$ such that $\mu(\mathbf{x}) \geq \lambda$, where $\mathbf{x} := (1, \mathbf{x}^*_{-i})$. We have

$$U_i(\mathbf{x}^*) = \frac{1}{I(\mathbf{x}^*)}|\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^*)|\lambda.$$

while, using the fact that $(1-|\mathcal{N}_i|\lambda)I(\mathbf{x}^*) > |\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^*)|\lambda$,

$$\begin{aligned} U_i(\mathbf{x}) &= \frac{1}{I(\mathbf{x}^*)+1}\left(|\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^*)|\lambda + 1 - |\mathcal{N}_i|\lambda\right) \\ &> \frac{|\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^*)|\lambda}{I(\mathbf{x}^*)+1}\left(1+\frac{1}{I(\mathbf{x}^*)}\right) \\ &= U_i(\mathbf{x}^*). \end{aligned}$$

Hence $\mathbf{x}^*$ is not a Nash equilibrium □

Proof of Theorem 2. Suppose that $\mathbf{x}^*$ is a Nash equilibrium and that $\mathcal{I}(\mathbf{x}^*)$ is not a dominating set. Then there exists $i \in \mathcal{J}(\mathbf{x}^*)$ such that $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^*) = \emptyset$. Consequently we necessarily have $\lambda > \mu(\mathbf{x}^*)$, because otherwise agent i would get a null payoff, which cannot happen at equilibrium. Let $\mathbf{x} := (1, \mathbf{x}^*_{-i})$. Then $\mathbf{x}$ is a profile with communication and $i \in \mathcal{I}(\mathbf{x})$ is not communication-critical. Hence, by Lemma D.D.2, we have that $U_i(\mathbf{x}) > U_i(0, \mathbf{x}_{-i}) = U_i(x_i^*, \mathbf{x}^*_{-i})$, which is a contradiction. We now prove the two statements:

(i) If $\lambda \leq 1/n$ then any profile $\mathbf{x}^*$ such that $\mathcal{I}(\mathbf{x}^*) \neq \mathcal{N}$ is not a Nash equilibrium, using Lemma D.D.3 with $\hat{\mathcal{I}} = \mathcal{N}$. On the other hand $\mathbf{x}^* := (1, ...1)$ is a Nash equilibrium since $U_i(0, \mathbf{x}^*_{-i}) = \frac{1}{n-1}\lambda|\mathcal{N}_i| \leq \frac{n-2}{n-1}\lambda \leq \frac{n-2}{n(n-1)}$ when there is no central agent. If there is a central agent then $\lambda < 1/n$ (by assumption) and the result still holds.

(ii) Let $\lambda > 1/n$, and suppose by contradiction that $\mathbf{x}^*$ is a Nash equilibrium, such that $\lambda \leq \mu(\mathbf{x}^*)$. For any $j \notin \mathcal{I}(\mathbf{x}^*)$, we necessarily have

$$\frac{|\mathcal{N}_j \cap \mathcal{I}(\mathbf{x}^*)|\lambda}{I(\mathbf{x}^*)} \geq \frac{1}{n}, \quad \text{i.e. } |\mathcal{N}_j \cap \mathcal{I}(\mathbf{x}^*)| \geq \frac{I(\mathbf{x}^*)}{\lambda n},$$

by Lemma D.D.3.

Since the number of links between agent j and $\mathcal{I}(\mathbf{x}^*)$ is equal to $|\mathcal{N}_j \cap I(\mathbf{x}^*)|$, the total number of links between $\mathcal{I}(\mathbf{x}^*)$ and $\mathcal{J}(\mathbf{x}^*)$ is equal to $\sum_{j \in \mathcal{J}(\mathbf{x}^*)} |\mathcal{N}_j \cap I(\mathbf{x}^*)|$, and we have

$$\max_{i \in \mathcal{I}(\mathbf{x}^*)} |\mathcal{R}_i(\mathbf{x}^*)| \geq \frac{1}{I(\mathbf{x}^*)} \sum_{j \in \mathcal{J}(\mathbf{x}^*)} |\mathcal{N}_j \cap I(\mathbf{x}^*)| = \frac{n - I(\mathbf{x})}{I(\mathbf{x}^*)} \frac{I(\mathbf{x}^*)}{\lambda n} = \frac{n - I(\mathbf{x}^*)}{\lambda n}.$$

Now since $\lambda \leq \mu(\mathbf{x}^*)$, we have

$$\frac{1}{\lambda} \geq I(\mathbf{x}^*) + \max_{i \in \mathcal{I}(\mathbf{x}^*)} |\mathcal{R}_i(\mathbf{x}^*)| \geq I(\mathbf{x}^*) + \frac{n - I(\mathbf{x}^*)}{\lambda n},$$

which implies that $I(\mathbf{x}^*)(\lambda n - 1) \leq 0$, a contradiction.

The last statement of point (ii) directly follows from the fact that an investor who is neither communication-critical nor cover-critical would save c by deviating. □

Proof of Proposition D.D.1. Suppose that $\mathcal{I}(\mathbf{x}^*)$ is a maximal independent set with $\lambda > \mu(\mathbf{x}^*)$, and assume that agent $i \in \mathcal{J}(\mathbf{x}^*)$ can profitably deviate:

$$U_i(1, \mathbf{x}^*_{-i}) > U_i(0, \mathbf{x}^*_{-i}).$$

Let $\mathbf{x} := (1, \mathbf{x}_{-i})$. Since $\mathcal{I}(\mathbf{x}^*)$ is a maximal independent set, we necessarily have $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}) \neq \emptyset$ and $\mathcal{N}_i^E(\mathbf{x}) = \emptyset$. Hence, by Lemma D.D.2 applied to profile $\mathbf{x}$ and agent i, we have $U_i(1, \mathbf{x}_{-i}) < U_i(0, \mathbf{x}_{-i})$, a contradiction.

Suppose now that $\mathcal{D}$ is a minimum dominating set, with $\lambda > \mu(\mathcal{D})$. If $\mathcal{D}$ is independent there is nothing to prove, because $\mathcal{D}$ itself supports a Nash equilibrium. Now, if $\mathcal{D}$ is not independent, pick $i_{max} \in \arg\max_{i \in \mathcal{D}} |\mathcal{R}_i|$ and choose any maximal independent set $\mathcal{I}$ containing i_{max}. We then have $|\mathcal{I}| + \max_{i \in \mathcal{I}} |\mathcal{R}_i| \geq |\mathcal{D}| + \max_{i \in \mathcal{D}} |\mathcal{R}_i|$, and therefore $\mu(\mathcal{I}) \leq \mu(\mathcal{D}) < \lambda$. □

Proof of Proposition D.D.2. Recall that $\overline{d} < n - 1$ by assumption. Suppose that $\lambda \in]\frac{1}{\overline{d}+2}, \frac{1}{\overline{d}+1}[$. Choose $\mathbf{x}^*$ such that $\mathcal{I}(\mathbf{x}^*)$ is a maximal independent set including an agent with maximal degree. We then have $I(\mathbf{x}^*) \geq 2$ and

$$I(\mathbf{x}^*) + \max_{i \in \mathcal{I}(\mathbf{x}^*)} |\mathcal{R}_i(\mathbf{x}^*)| = I(\mathbf{x}^*) + \overline{d} \geq \overline{d} + 2.$$

Consequently $\mu(\mathbf{x}^*) < \lambda$ and $\mathbf{x}^*$ is a Nash equilibrium, by Corollary D.D.1.

Let now i_{max} be an agent with maximal degree. We will show that the profile $\mathbf{x}^*$ with $\mathcal{I}(\mathbf{x}^*) = \mathcal{N} \setminus \mathcal{N}_{i_{max}}$ is a Nash equilibrium for $\lambda \in]\frac{1}{n}, \frac{1}{n-1}[$. By construction, $\mu(\mathbf{x}^*) = \frac{1}{n}$. Hence $\mathbf{x}^*$ is a communication profile. Given any $i \in \mathcal{I}(\mathbf{x}^*)$, let $\mathbf{x} :=$

$(0, \mathbf{x}^*_{-i})$. Then $\mu(\mathbf{x}) \geq \frac{1}{n-1} > \lambda$. Thus there is no communication at profile $\mathbf{x}$. We now show that no agent can profitably deviate. First, agent i_{max} cannot profitably deviate, because he would get a null payoff by doing so. Let now $i \in \mathcal{I}(\mathbf{x}) \setminus \{i_{max}\}$. Since $i_{max} \in \mathcal{I}(\mathbf{x}) \setminus \mathcal{N}_i$, we have $|\mathcal{N}_i \cap \mathcal{I}(\mathbf{x})| < I(\mathbf{x}) \leq n-2$. Hence, we have

$$U_i(\mathbf{x}) = \frac{|\mathcal{N}_i \cap \mathcal{I}(\mathbf{x})|\lambda}{I(\mathbf{x})} < \frac{n-3}{n-2}\frac{1}{n-1} < \frac{1}{n}$$

Hence, c being small, we have $U_i(\mathbf{x}) < U_i(\mathbf{x}^*)$. □

Proof of Lemma D.D.1. Let us prove the first point, by assuming that $\mu(\mathbf{x}') < \lambda$. Without loss of generality we may suppose that $\mathbf{x} = (1, \mathbf{x}_{-i})$ and $\mathbf{x}' = (0, \mathbf{x}_{-i})$, for some $i \in \mathcal{I}(\mathbf{x})$. Since it is a best response for agent i to stop investing, it means, by Lemma D.D.2 ((i) and (ii)), that agent i has an active neighbour and no exclusive inactive neighbor in configuration $\mathbf{x}$. Thus $\mathcal{I}(\mathbf{x}')$ is a dominating set. We now turn to the second statement. If there is no Nash equilibrium then, starting from any configuration $\mathbf{x}^0$ such that $\mathcal{I}(\mathbf{x}^0)$ is a dominating set, there exists a best-response cycle $(\mathbf{x}^0, ..., \mathbf{x}^T)$. By the first statement, either communication is lost along this cycle, or communication holds, and $\mathcal{I}(\mathbf{x}^t)$ is a dominating set for $t = 0, ..., T-1$. Suppose by contradiction that communication holds along the cycle. Let i be such that $x_i^0 = 0$ and $x_i^1 = 1$. It implies that agent i cannot profitably deviate in configuration $\mathbf{x}^1$, meaning that we are either in case (i) or (ii) of Lemma D.D.2: namely either $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^1) = \emptyset$ or $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^1) \neq \emptyset$ and $\mathcal{N}_i^E(\mathbf{x}^1) = \emptyset$. In the first case, agent $\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}_0) = \emptyset$ so that $\mathcal{I}(\mathbf{x}^0)$ is not a dominating set, a contradiction. In the second case, $\mathcal{N}_j \cap \mathcal{I}(\mathbf{x}_0) = \emptyset$, for any $j \in \mathcal{N}_i^E(\mathbf{x}^1)$, again a contradiction. The second point is proved. □

Proof of Proposition D.D.3. We prove that, if there is no Nash equilibrium then

$$\gamma(\mathbf{G}) \leq \frac{1}{2}\left(-1 + \sqrt{1 + 4\bar{d}n}\right). \tag{D.4}$$

If there is no Nash equilibrium then there exists a best-response cycle $(\mathbf{x}^0, ...\mathbf{x}^T)$, along which communication is shut. More precisely, and without loss of generality, we may assume that

- $\mathbf{x}^0$ is such that $\mathcal{I}(\mathbf{x}^0)$ is a dominating set, with $\mu(\mathbf{x}^0) < \lambda$,
- $\mathbf{x}^1 = (0, \mathbf{x}^0_{-i})$, for some agent $i \in \mathcal{I}(\mathbf{x}^0)$, and $\mu(\mathbf{x}^1) \geq \lambda$.
- $U_i(\mathbf{x}^1) > U_i(\mathbf{x}^0)$, namely

$$\frac{\lambda |\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^0)|}{I(\mathbf{x}^0) - 1} > \frac{1}{n}$$

Combining all this, we get

$$\frac{1}{I(\mathbf{x}^0) + \max_{i \in \mathcal{I}(\mathbf{x}^0)} |\mathcal{R}_i(\mathbf{x}^0)| - 1} \geq \lambda > \frac{I(\mathbf{x}^0) - 1}{|\mathcal{N}_i \cap \mathcal{I}(\mathbf{x}^0)| n} \geq \frac{I(\mathbf{x}^0) - 1}{\overline{d} n}$$

Since there is no Nash equilibrium, $\mathcal{I}(\mathbf{x}^0)$ cannot be a minimum dominating set (by Proposition D.D.1). Hence $I(\mathbf{x}^0) \geq \gamma(\mathbf{G}) + 1$. As a result, since $\max_{i \in \mathcal{I}(\mathbf{x}^0)} |\mathcal{R}_i(\mathbf{x}^0)| \geq 1$,

$$\gamma(\mathbf{G})(\gamma(\mathbf{G}) + 1) \leq \overline{d} n,$$

which gives (D.4). □