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Abstract

We address the question of the measurement of social welfare and inequalities in the context of partially-ordered health variables. We propose a general framework based on the assumption that the distribution of well-being states forms an m-dimensional Boolean lattice. To this end, the distribution of well-being states is constructed based on the prevalence of a finite number of illnesses where each state represents the number of illnesses an individual may suffer from. The implementation of the framework involves breaking down the Boolean lattice into a set of linear extensions where all health states become fully ordered. The linear extensions account for all possible ordering of the health states based on the depth of health problems (i.e., *the severity of health conditions*). Having constructed these linear extensions, we then proceed on ranking distributions in terms of welfare by applying appropriate dominance criteria and employ aggregate metrics to provide a numerical representation of the social welfare and inequality associated with each distribution. An illustrative application of the methodology is provided.

JEL Classification: D63, I14, I15, O5

Keywords: Boolean lattice, Hammond dominance, Ordinal inequality, Partially-ordered variables, Stochastic dominance, Welfare function

1. Introduction

Welfare and inequality analysis relying on ordinal attributes has received a rising interest in the literature (e.g., Cuhadaroglu 2022; Gravel et al. 2021; Makdissi and Yazbeck 2017; Cowell and Flachaire 2017; Muller and Trannoy 2011; Erreygers and Van Ourti 2011; Abul Naga and Yalcin 2008; Allison and Foster 2004). Previous literature has hitherto focused on ordinal (categorical) variables (e.g., self-reported health status, life satisfaction and happiness), where well-being states can be fully ordered – i.e., all states are comparable. In this context, appropriate dominance criteria and inequality metrics have been advanced and applied to rank different distributions and measure the degree of inequality (e.g., Makdissi and Yazbeck 2014; Abul Naga and Yalcin 2008).

In population surveys, it is quite common to find information on individuals' wellbeing available in the form of a series of questions querying whether (or not) an individual has a given attribute (e.g., an illness). The resulting list of attributes allows to generate two distinct types of well-being variables: a *nominal variable* (e.g., types of illnesses) and a *fixed-scale variable* (e.g., the number of illnesses). In the context of a *nominal variable* such as the types of illnesses, Erreygers and Van Ourti (2011) and Makdissi and Yazbeck (2014) pointed out that one can only assign subjects to different groups without ranking them. Under such conditions, standard welfare and inequality metrics cannot be applied. However, in the case of a *fixed-scale variable*, standard measures are shown to be readily applied given the absolute (real) zero indicating the complete absence of illness (Erreygers and Van Ourti 2011). In the context of a multiple categorical variable, Makdissi and Yazbeck (2014) suggested to transform the available information on the *width* of health problems (e.g., vision, hearing, speech, …) into a *ratio-scale variable* – constructed by counting the number of attributes in which an individual is considered to have a health problem.

However, the use of the Alkire and Foster's (2011) *counting approach* to derive a measure of inequality based on the *breadth* of health problems involves two caveats. The first is emphasized by Makdissi and Yazbeck (2014) and relates to the loss of information on the depth of health problems (i.e., the severity of health conditions). The second stems from the fact that many survey data do not provide further information on the depth of health problems. Thus, by merely counting the number of attributes, or assuming one particular ordering of the health states, one may run the risk of *comparing the incomparable*.

This paper seeks to address the measurement problems that arise when well-being states are *a priori* incomparable. We propose a general framework based on the assumption that the distribution of well-being states forms an m-dimensional Boolean lattice. To this end, the distribution of well-being states is constructed based on the prevalence of a finite number of attributes where each state represents, for instance, the number of health problems an individual suffers from. The implementation of this framework involves breaking down the Boolean lattice into a set of linear extensions where all well-being states become fully ordered.

To illustrate, consider, for instance, an ordinal health variable $h = (h_1, \dots, h_K)$, where K is the number of health states. Let h_1 be the worst health state while h_K is the best health state. The variable h is totally-ordered if all health states can be compared such that $h_1 \preccurlyeq h_2 \preccurlyeq \cdots \preccurlyeq h_K$, where $\preccurlyeq$ is a binary order relation. The variable h is said to be partially-ordered if there is at least two health states $r, s \in \{1, \dots, K\}$ such that h_r and h_s are incomparable. For example, let $h = (h_1, h_2, h_3, h_4)$ where h_1 indicates the presence of a chronic and an infectious illness, h_2 and h_3 indicate the presence of a chronic illness and an infectious illness, respectively, while h_4 indicates no illness. It is obvious that having one illness, h_2 or h_3, is *not as bad as* having two illnesses, h_1, and *worse* than having no illness, h_4. However, in the absence of additional information on the *depth* of health problems, it may not be possible to conclude that h_2 is worse than h_3 or h_3 is worse than h_2. This kind of *partially-ordered sets* (POSET) can be visualised using a two-dimensional Boolean lattice (Fattore et al. 2014; Davey and Priestley 2002). We, therefore, apply the lattice theory that allows permuting all the possible combinations of health sates.

The lattice theory has recently been applied to study different socio-economic phenomena such as fuzzy multidimensional material deprivation (e.g., Fattore et al. 2011), and cooperative game theory (e.g., Alonso-Meijide et al. 2017; Caulier et al. 2015; Grabisch 2010). However, to the best of our knowledge, there has hitherto been no previous endeavours to apply this approach to social welfare and inequality analysis where health is the main attribute of individuals' well-being. This paper seeks therefore to introduce a general framework that is suitable for the analysis of partially-ordered variables which form a Boolean lattice.

The main contribution of this paper consists in converting the Boolean lattice of a distribution into a set of totally-ordered distributions. This is conducted by constructing all possible linear orderings of the lattice – the linear extensions – which are formed from the permutations of all incomparable elements in the lattice. In our example above, the set of all possible linear orderings contains only two elements $\{(h_1, h_2, h_3, h_4), (h_1, h_3, h_2, h_4)\}$, which result from the permutations of the incomparable health states, h_2 and h_3. Note that, the first element of the set is obtained by assuming that h_2 is more severe than h_3 while the second element is obtained by assuming that h_3 is more severe than h_2. Having constructed these linear extensions, one can proceed on ranking different distributions of health states using dominance criteria. We first illustrate and discuss the application of the first-order stochastic dominance and the Hammond dominance criteria in the context of partially-ordered sets. We then proceed by employing appropriate social welfare function (Gravel et al. 2021) and inequality index (Abul Naga and Yalcin 2008) to provide a summary measure of the level of social welfare and the degree of inequality associate with each distribution.

The paper is organised as follows. Section 2 presents the methodology used to provide an ordering of distributions of partially-ordered health variables using stochastic dominance criteria and social welfare and inequality metrics. Section 3 illustrates our methodology in the context of a group of five MENA region countries. Section 4 illustrates the application of our methodology under some monotonicity restrictions. Section 5 concludes the paper.

2. Methodology

We consider a population of $n \geq 2$ individuals. Health status for each individual is measured using a set of $m \geq 2$ illnesses. Each illness is represented by a dichotomized variable that takes 1 if individual has the illness and 0 otherwise. Assume that these illnesses are denoted by $d_1, d_2, \ldots, d_m$. An individual may report no illness, ϕ, or having d illnesses, $0 < d \leq m$. The number of all possible combinations of health states is, thus, $K = 2^m$. Each health state is assigned an integer $k = 1, \ldots, K$. Let $h = (h_1, \ldots, h_K)$ denotes the vector of health states ranked from the worst (h_1) to the best (h_K). For simplicity, we consider $m = 3$, hence $K = 8$ and the set of all possible health states is

$$h = (h_1, \ldots, h_8) = (\{d_1, d_2, d_3\}, \{d_1, d_2\}, \{d_1, d_3\}, \{d_2, d_3\}, \{d_1\}, \{d_2\}, \{d_3\}, \phi)$$

where $h_1 = \{d_1, d_2, d_3\}$ is the worst health state and $h_8 = \phi$ is the best health state. The number of individuals in health state k is n_k. Let $f = (f_1, \ldots, f_8)$ be the probability density function (PDF) of health states where $f_k = n_k/n$, $k = 1, \ldots, 8$ is the share of individuals with health state k. The corresponding CDF is $F = (F_1, \ldots, F_8)$ where $F_k = \sum_{l=1}^{k} f_l$, $k = 1, \ldots, 8$. Let $\preccurlyeq$ be a binary relation on a set X that is reflexive, antisymmetric and transitive (Davey and Priestley 2002). Let x_i and x_j $(i \neq j)$ be two elements in the set X. Conventionally, the two elements are said to be comparable if either $x_i \preccurlyeq x_j$ or $x_j \preccurlyeq x_i$, otherwise, the two elements are incomparable – denoted by $x_i \parallel x_j$. If $x_i \preccurlyeq x_j$, then x_i is said to be worse than (or less preferred to) x_j. In what follows, X might be the set of health states or the set of vectors (distributions) of health states. In this section, we first provide a cursory description of the partially-ordered sets of health states. Then, we define the possible linear extensions of these sets and the dominance criteria that enable to order different distributions of partially-ordered sets of health states. Finally, we define a suitable social welfare function and inequality measure for the comparison of ordered health variables.

2.1 Partially-Ordered Health States

Let $\mathbb{D}$ be a set of K health states that are measured by the number of illnesses that an individual may report in a survey data. Assume that these illnesses are independent and can only be partially-ordered; i.e., some of the health states are *a priori* incomparable. The set of all possible health states, $\mathbb{D}$, consists of the following:

$$\mathbb{D}^m = \left\{\{d_1, d_2, \dots, d_m\}, P(\{d_i\}, m-1), P(\{d_i\}, m-2), \dots., P(\{d_i\}, 1), \phi\right\} \tag{1}$$

where $P(d_i, m-l)$, $l = 1, \dots, m-1$ is a permutation of the set of $m-l$ illnesses. If $m = 3$, then

$$\mathbb{D}^3 = \left\{\{d_1, d_2, d_3\}, P(\{d_i\}, 3-1), P(\{d_i\}, 3-2), \phi\right\}$$

$$\mathbb{D}^3 = \left\{\{d_1, d_2, d_3\}, P(\{d_i\}, 2), P(\{d_i\}, 1), \phi\right\}$$

$P(\{d_i\}, 2)$ is all possible permutations of the sets that contain two illnesses and $P(\{d_i\}, 1)$ is all possible permutations of the sets that contain one illness. Hence

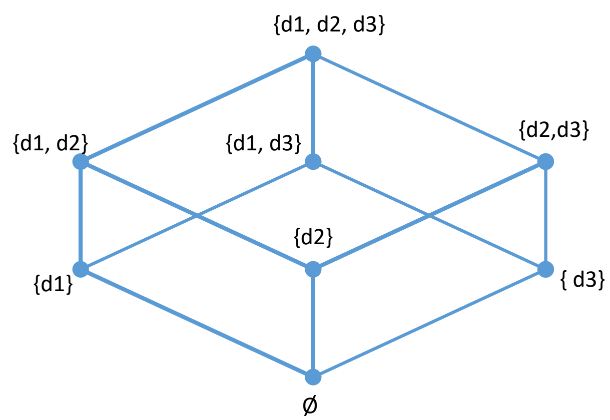
$$\mathbb{D}^3 = \left\{\{d_1, d_2, d_3\}, \{d_1, d_2\}, \{d_1, d_3\}, \{d_2, d_3\}, \{d_1\}, \{d_2\}, \{d_3\}, \phi\right\}$$

where $\{d_g\}, \{d_g, d_r\} \preccurlyeq \phi$ and $\{d_1, d_2, d_3\} \preccurlyeq \{d_g\}, \{d_g, d_r\}$, $g, r = 1,2,3$. The state of having one illness, $\{d_g\}$, or two illnesses $\{d_g, d_r\}$ are clearly worse than the state ϕ and better than the state $\{d_1, d_2, d_3\}$. However, the states within the subsets of one illness $\left\{\{d_g\},\ g = 1,2,3\right\}$ and two illnesses $\left\{\{d_g, d_r\},\ g, r = 1,2,3\right\}$ are *incomparable*. For instance, $\{d_1\} \parallel \{d_2\} \parallel \{d_3\}$. Furthermore, not all elements in the different subsets are comparable: while $\{d_g, d_r\}$ is worse than $\{d_g\}$, $\forall r \neq g$, $\{d_g\}$ is incomparable to $\{d_s, d_r\}$ $\forall\, s, r \neq g$; that is either $\{d_g\} \preccurlyeq \{d_s, d_r\}$ or $\{d_s, d_r\} \preccurlyeq \{d_g\}$. Given incomparability between some elements in the relation defined over the set in Eq. 1, the relation $\preccurlyeq$ is not a *unique complete order*. In this case, $\preccurlyeq$ is said to be a *partially-ordered relation*, hence $\mathbb{D}$ is a partially-ordered set ($POSET$) – denoted as $P = (\mathbb{D}, \preccurlyeq)$.

The $POSET$, P, can be depicted using the Hass Diagram (also known as *Boolean lattice ordering*) of dimension $m = 3$, denoted as B_3 (Figure 1).

Insert Figure 1 here

Figure 1: Boolean Lattice of Dimension *m=3*



As can be seen from Figure 1, B_3 has the following sets of comparable and incomparable pairs – denoted respectively as $Comp(B_3)$ and $Inc(B_3)$:

$$Comp(B_3) = \left\{ \begin{matrix} (\phi, \{d_g\}), (\phi, \{d_g, d_r\}), (\phi, \{d_1, d_2, d_3\}), (d_g, \{d_g, d_r\}), \\ (d_g, \{d_1, d_2, d_3\}), (\{d_g, d_r\}, \{d_1, d_2, d_3\}) \end{matrix} \right\} \quad (2)$$

$$\forall g, r = 1,2,3 \text{ and } g \neq r$$

$$Inc(B_3) = \{(d_g, d_r), (\{d_g\}, \{d_r, d_s\}), (\{d_g, d_r\}, \{d_g, d_s\})\}$$
$$\forall\ g, r, s = 1,2,3 \text{ and } g \neq r \neq s \tag{3}$$

In general, the total number of comparable and incomparable pairs of a set $\mathbb{D}$ of dimension k is

$$C_2^K = \binom{K}{2} = \frac{K!}{(K-2)!\,2!} = \frac{K(K-1)(K-2)!}{(K-2)!\,2} = \frac{K(K-1)}{2} \tag{4}$$

In our case, where $K = 8$, the number of all pairs is $8(8-1)/2 = 28$ of which 19 pairs are comparable, which equals to the size of the set $Comp(B_3)$: $|Comp(B_3)| = 19$, and 9 pairs are incomparable, which equals to the size of the set $Inc(B_3)$: $|Inc(B_3)| = 9$.

A set that includes comparable elements is referred to as $\preccurlyeq$-*chain*, while a set that includes incomparable elements is referred to as $\preccurlyeq$-*antichain*. The set of chains in Figure 1 includes: $C_1 = \{\phi, \{d_3\}, \{d_2, d_3\}, \{d_1, d_2, d_3\}\}$, and $C_2 = \{\phi, \{d_3\}, \{d_1, d_3\}\}$. The set of *antichains* in Figure 1 includes: $A_1 = \{\{d_1\}, \{d_2\}, \{d_3\}\}$, and $A_2 = \{\{d_2\}, \{d_1, d_3\}\}$. The size of the largest chain is the height of P and the size of the largest antichain is the width of P. In our case, the height is four and the width is three.

2.2 Complete (Linear) Order Extensions

A social welfare function shall provide an order extension of B_m, which produces a complete ranking of health states. A linear extension is a total (linear) order of the POSET, P, which does not contain incomparable elements. A linear extension can be obtained by imposing some restrictions on the incomparable health states. For instance, by assuming that $d_1 \preccurlyeq d_2 \preccurlyeq d_3$ (assumption 1), the set of incomparable pairs in Eq. 3 reduces to:

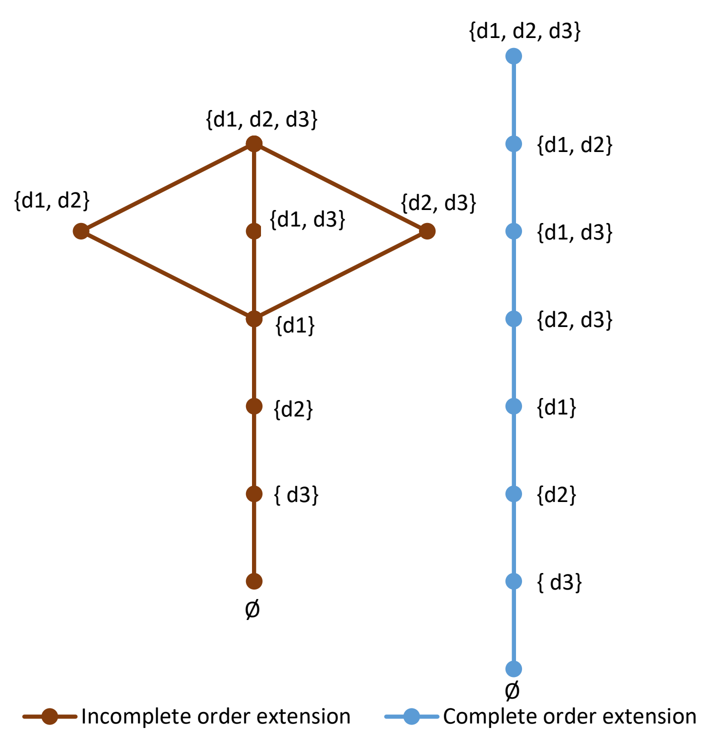
$$Inc(B_3) = \{(\{d_g\}, \{d_r, d_s\})\},\ \ \forall g, r, s = 1,2,3 \text{ and } g \neq r \neq s \tag{5}$$

Since the elements of two illnesses are still incomparable, such restriction yields the non-linear (incomplete order) extension of B_3 given in Figure 2-A. Another plausible restriction is to assume further that $\{d_1, d_2\} \preccurlyeq \{d_1, d_3\} \preccurlyeq \{d_2, d_3\}$ (assumption 2), then the extension of B_3 given in Figure 2-B is a linear (complete order) extension. This linear extension can be expressed as:

$$E = \{\{d_1, d_2, d_3\}, \{d_1, d_2\}, \{d_1, d_3\}, \{d_2, d_3\}, \{d_1\}, \{d_2\}, \{d_3\}, \phi\} \quad (6)$$

Insert Figure 2 here

Figure 2: Incomplete vs. complete order extensions of B3



Assumption 1 and 2 yield an ordering of the subset of one illness and the subset of two illnesses, respectively. Each of these two subsets can be ordered in 3! manners. For example, the elements $\{d_1\}, \{d_2\}, \{d_3\}$ can be ordered in the following manners: $d_1 \preccurlyeq d_2 \preccurlyeq d_3$, $d_1 \preccurlyeq d_3 \preccurlyeq d_2$, $d_2 \preccurlyeq d_1 \preccurlyeq d_3$, $d_2 \preccurlyeq d_3 \preccurlyeq d_1$, $d_3 \preccurlyeq d_1 \preccurlyeq d_2$, or $d_3 \preccurlyeq d_2 \preccurlyeq d_1$. Thus, we will obtain from this ordering $3! \times \binom{3}{2}! = 36$ linear extensions. Furthermore, the element d_g is incomparable to $\{d_r, d_s\}\ \forall r, s \neq g$. This means that, for example, in the linear extension in Figure 2, $d_3 \parallel \{d_1, d_2\}$. Thus, another linear extension can be obtained if we assume that $\{d_3\} \preccurlyeq \{d_1, d_3\}$. The respective linear extension is:

$$E' = \{\{d_1, d_2, d_3\}, \{d_1, d_2\}, \{d_3\}, \{d_1, d_3\}, \{d_2, d_3\}, \{d_1\}, \{d_2\}, \phi\} \tag{7}$$

There are another 12 linear extensions resulting from the permutations of d_g with $\{d_r, d_s\}$. The total number of linear extensions, $E_i, i = 1, \dots, e$ is, thus, $e = 48$.

2.3 Stochastic Dominance of Partially-Ordered Sets

For totally-(linearly-) ordered continuous health variables, comparisons of distributions in terms of welfare can be achieved using the FOSD criterion (Hammond et al. 2014; Yalonetzky 2013; Muller and Trannoy 2011). Consider two CDFs of health distributions, F^1 and F^2, where health states are ranked from the worst to the best health state. The distribution F^2 is said to first-order stochastically dominants F^1 – written as $F^1 \preccurlyeq_{FOSD} F^2$ – if $F_k^2 \leq F_k^1\ \ \forall k = 1, \dots, 8$ where F_k^1 is the k^{th} coordinate of the distribution F^1. In this case, F^2 is said to exhibit higher welfare than F^1. This section constructs the conditions for ordering a pair of distributions in terms of social welfare for partially-ordered health variables. In this case, there will be e linear extensions of each health distribution. Let $f^{1,i}$ and $f^{2,i}$ be two PDFs of the i^{th} linear extension of the first and second health distributions, respectively, defined as follows:

$$\begin{aligned} f^{1,i} &= \left(f_1^{1,i}, \dots, f_8^{1,i}\right) \\ f^{2,i} &= \left(f_1^{2,i}, \dots, f_8^{2,i}\right) \end{aligned} \tag{8}$$

where $f_k^{j,i}$ is the share of the population with health state, $k = 1, \dots, K = 8$, in distribution $j = 1, 2$ and linear extension $i = 1, \dots, e = 48$. The corresponding CDFs are

$$F^{1,i} = \left(F_1^{1,i}, \dots, F_8^{1,i}\right)$$
$$F^{2,i} = \left(F_1^{2,i}, \dots, F_8^{2,i}\right) \tag{9}$$

where $F_k^{j,i} = \sum_{l=1}^{k} f_l^{j,i}$, $\forall\, j = 1, 2,\;\; k, l = 1, \dots, 8$ and $i = 1, \dots, 48$. For each linear extension, E_i, there are three possible cases of FOSD between $F^{1,i}$ and $F^{2,i}$: $F^{1,i} \preccurlyeq_{FOSD} F^{2,i}$ or $F^{2,i} \preccurlyeq_{FOSD} F^{1,i}$ or $F^{1,i} \parallel_{FOSD} F^{2,i}$. The dominance criteria can be established as in **Definition 1**.

Definition 1: Let $\mathbb{D}$ be a POSET of K health states. For any two health distributions F^1 and $F^2 \in \mathbb{D}$, F^1 first-order stochastically dominates F^2 – written as $F^2 \preccurlyeq_{FOSD} F^1$ – if

$$F_k^{1,i} \leq F_k^{2,i} \;\; \forall\, k = 1, \dots, K,\;\; i = 1, \dots, I \tag{10}$$

where I is the number of all possible linear extensions. Eq. 10 suggests that F^1 exhibits higher social welfare than F^2 if $F^2 \preccurlyeq_{FOSD} F^1$ for all linear extensions. As will be shown below (in Section 3), the CDFs of any two health distributions may cross for some linear extensions. Thus, the relation $\preccurlyeq_{FOSD}$ which is defined over the set $\mathbb{D} \times \mathbb{D}$ does not allow for comparisons of the two distributions. A weaker dominance criterion than the FOSD is, thus, in order. A possible order extension which preserves the original order relation $\preccurlyeq_{FOSD}$ but allows for a larger subset of linear extensions to be ordered is the *Hammond dominance criterion* – denoted as $\preccurlyeq_H$ (Gravel et al 2021). Let $F \in \mathbb{D}$ be a distribution of health states. Define the k-dimensional Hammond distribution function (HDF) $H: \mathbb{D} \rightarrow [0, 2^{k-1}]^k$ as

$$H(F) = \left(H_1(F), \dots, H_k(F)\right) \tag{11}$$

where the k^{th} coordinate of the Hammond function, $H_k(F): \mathbb{D} \rightarrow [0, 2^{k-1}]$ is defined by

$$H_k(F) = \sum_{l=1}^{k} 2^{k-l} f_l \tag{12}$$

The values of the first and last coordinates of the HDF are, respectively, $H_1(F) = \sum_{l=1}^{k=1} 2^{1-l} f_l = f_1 = F_1$ and $H_8(F) = 64F_1 + 32F_2 + 16F_3 + 8F_4 + 4F_5 + 2F_6 + F_7 + F_8$. For any two distributions $F^1, F^2 \in \mathbb{D}$, the Hammond dominance criteria for partially-ordered health variables can be established as in **Definition 2**.

Definition 2: Let $\mathbb{D}$ be a POSET of K health states. For any two health distributions F^1 and $F^2 \in \mathbb{D}$, F^1 Hammond dominates F^2 – written as $F^2 \preccurlyeq_H F^1$ – if

$$H_k^{1,i}(F^1) \leq H_k^{2,i}(F^2) \; \forall \; k = 1, \dots, K, \;\; i = 1, \dots, I \tag{13}$$

Similar to the relation $\preccurlyeq_{FOSD}$, Eq. 13 suggests that F^1 exhibits higher social welfare than F^2 if $F^2 \preccurlyeq_H F^1$ for all linear extensions. In general, the Hammond dominance criterion enables to extend the number of comparable extensions for different distributions. For any two distributions F^1 and F^2, if there is at least one extension where F^1 Hammond dominates F^2, then either F^1 Hammond dominates F^2 for some/all extensions or F^1 and F^2 are incomparable for some extensions. This result is summarized in Proposition 1.

Proposition 1: Let $\mathbb{D}$ be a POSET of K health states. For any two health distributions F^1 and $F^2 \in \mathbb{D}$, if there is a linear extension $i' = 1, \dots, I$ such that $H_1^{1,i'}(F^1) \leq H_1^{2,i'}(F^2)$ then

(i) either $F^{2,i'} \preccurlyeq_H F^{1,i'}$

(ii) or $F^{2,i'} \parallel_H F^{1,i'}$.

Proof: Suppose that there is an extension i'' such that $F^{1,i''} \preccurlyeq_H F^{2,i''}$. This implies that $H_1^{1,i'}(F^1) > H_1^{2,i'}(F^2)$ which is a contradiction since $H_1^{1,i'}(F^1) \leq H_1^{2,i'}(F^2)$.

2.4 Constructing a Social Welfare Function for POSETs

A social welfare function (SWF) is conventionally used in the case of totally-ordered variables to provide an aggregate numerical representation that allows to compare different distributions (e.g., Robert 2018; Asheim et al. 2016). In the case of partially-ordered variables, such a SWF will be evaluated for each possible linear extension. Given that the health variable considered in this paper is ordinal, then a suitable SWF is in order. An appropriate SWF for ordinal variables, accounts for the number of individuals in each health state, is proposed by Gravel et al (2021). This SWF satisfies the Hammond progressive transfers, which is equivalent to the Pigou-Dalton principle of transfers in the case of ordinal variables. The Hammond progressive transfers implies that an increase in an individual's welfare associated with a decrease in another individual's welfare – holding their positions unchanged – improves the overall social welfare (Hammond 1976). For a population with K health states, the Hammond SWF can be defined as

$$W_H = \frac{1}{n} \sum_{k=1}^{K} n_k \alpha_k \tag{14}$$

where n_k is the size of the population with health state k, and α_k is a scale (*weight*). These weights are constructed such that they are increasing at a decreasing rate with the best health state being attached the highest weight. To illustrate the intuition underlying this choice of the vector $\alpha = (\alpha_1, \dots, \alpha_K)$, we define the best and the worst health PDFs, $\overline{f}$ and $\underline{f}$, respectively, as follows

$$\begin{aligned} \overline{f} &= \left(0, \dots, 0, \frac{n}{n}\right) \\ \underline{f} &= \left(\frac{n}{n}, \dots, 0, 0\right) \end{aligned} \tag{15}$$

For the best distribution, $\overline{f}$, all the population reports no illness while for the worst distribution, $\underline{f}$, all the population reports the three illnesses. Using Eq. 14, the corresponding SWF of the distributions $\overline{f}$ and $\underline{f}$ can be defined, respectively, as

$$W_H(\overline{f}) = \frac{1}{n} \left(\sum_{l=1}^{K-1} 0 + n\, \alpha_K \right) = \alpha_K \tag{16}$$

$$W_H(\underline{f}) = \frac{1}{n}\left(n\,\alpha_1 + \sum_{l=2}^{K} 0\right) = \alpha_1$$

The monotonicity property of the SWF implies that the best distribution shall exhibit higher welfare than the worst distribution. Accordingly, $\alpha_1 < \alpha_K$. In general, let f^k and $f^{k'}$ be two PDFs such that all the population reports the health state k and k', respectively, where $k < k'$. The corresponding SWFs are $W_H(f^k) = \alpha_k$ and $W_H(f^{k'}) = \alpha_{k'}$. An increasing SWF implies that $W_H(f^k) < W_H(f^{k'})$, hence $\alpha_k < \alpha_{k'}$. Moreover, the strict concavity property implies that the SWF is increasing at a decreasing rate. Let f^k, $f^{k'}$ and $f^{k''}$ be three PDFs such that all the population reports the health state k, k' and k'', respectively, where $k < k' < k''$. As shown, an increasing SWF implies that $W_H(f^k) < W_H(f^{k'}) < W_H(f^{k''})$. Strict concavity implies that $W_H(f^{k'}) - W_H(f^k) > W_H(f^{k''}) - W_H(f^{k'})$. Accordingly, $\alpha_{k'} - \alpha_k > \alpha_{k''} - \alpha_{k'}$. A possible choice of the vector of weights α that will be used in Section 3 is

$$\alpha_k = \sum_{l=1}^{k-1}\left(\frac{1}{2}\right)^l \equiv \begin{cases} \left(\frac{1}{2}\right)^0 = 1, & k = 1 \\ \alpha_{k-1} + \left(\frac{1}{2}\right)^{k-1}, & k > 1 \end{cases} \tag{17}$$

The choice of the weights is arbitrary except that these weights shall be increasing at a decreasing rate. Eq. 17 shows that the differences between the weights attached to different health states decrease with health states. The vector of differences between health states is

$$\alpha_k - \alpha_{k-1} = \left(\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4, \dots, \left(\frac{1}{2}\right)^7\right)$$

In order to bound the values of the SWF in the interval [0,1], the SWF defined in Eq. 14 can be normalized as follows

$$W_N(f) = \frac{W_H(f) - W_H\left(\underline{f}\right)}{W_H(\overline{f}) - W_H\left(\underline{f}\right)} \tag{18}$$

Using Eq. 14 and Eq. 17, Eq. 18 can be expressed as

$$W_N(f) = \frac{\frac{1}{n}\sum_{k=1}^{K} n_k \alpha_k - \alpha_1}{\alpha_K - \alpha_1} \tag{19}$$

Eqs. 18 and 19 show how the welfare value of a distribution is relatively far from that of the worst distribution. Since the range of the values of the SWF will vary based on the choice of the weights, α_k, the normalization of the SWF brings the values of W into the fixed range [0,1]. Some intuition of using a normalized version of the SWF is to allow comparison of different distributions if different weights are assigned to health states in each distribution. For any two PDFs f^1 and f^2, the dominance criteria of social welfare for partially-ordered health variables defined in Eq. 19 can be established as in **Definition 3**.

Definition 3: Let $\mathbb{D}$ be a POSET of K health states. For any two health distributions F^1 and $F^2 \in \mathbb{D}$, F^1 exhibits higher social welfare than F^2 – written as $F^2 \preccurlyeq_W F^1$ – if

$$W_N\left(f^{1,i}\right) \geq W_N\left(f^{2,i}\right) \,\forall\;\; i = 1, \dots, I \tag{20}$$

where $f^{j,i}$ is the PDF of extension i of distribution j.

2.5 Measuring Inequality for POSETs Distributions

Inequality metrics can be used to provide a single-valued measure to compare different distributions. The standard measures of inequality such as the Gini index or the Atkinson index are inappropriate for ordinal variables (Abul Naga and Yalcin 2008). One approach to compare distributions of an ordinal variable with finite categories is the method developed by Abul Naga and Yalcin (2008). This inequality measure can be applied to compare distributions of ordinal health data which have the same median health state. As is shown in Section 3, most of individuals in survey data report no illness, thus, the median health state is the last (best) state. Therefore, we can employ the inequality index proposed by Abul Naga and Yalcin (2008) for ordered response health data. In our context, this inequality index can be defined as

$$I(F) = 1 - \left(\frac{2\sum_{k=1}^{K} |F_k - 0.5| - 1}{K - 1} \right) \tag{21}$$

As shown in Eq. 21, this inequality index is based on the CDF only, thus, it is insensitive to values assigned to the different health states.

Definition 4: Let $\mathbb{D}$ be a POSET of K health states. For any two health distributions F^1 and $F^2 \in \mathbb{D}$, F^1 exhibits lower inequality than F^2 – written as $F^2 \preccurlyeq_I F^1$ – if

$$I\left(F^{1,i}\right) \leq I\left(F^{2,i}\right) \forall \quad i = 1, \ldots, I \tag{22}$$

Similar to the dominance criteria, Definition 4 states that the distribution F^1 exhibits lower inequality than the distribution F^2 if the degree of inequality is lower in distribution F^1 than in distribution F^2 for all possible linear extensions.

3. An Empirical Application

To illustrate the proposed methodology, we use data on three main morbidities from the available World Health Surveys (WHS 2000-2001) that have been conducted in five MENA countries: Egypt, Iran, Lebanon, Syria and Turkey. The WHS offer a detailed list of illnesses (up to 15 illnesses and health problems) declared by adult respondents (18 years old and above). For the purpose of this application, we consider the distributions of three major illnesses: $d_1 = cardiovascular\ diseases$; $d_2 = respiratory\ diseases$, and $d_3 = diabetes$. The first illness includes heart diseases, high blood pressure and stroke while the second includes in addition to pulmonary diseases, chronic bronchitis. Table 1 summarizes the frequency distributions of these illnesses in the five countries under consideration.

Insert Table 1 here

Table 1: Descriptive Statistics of the five health distributions for the five MENA countries					
	Egypt	Iran	Lebanon	Syria	Turkey
ϕ	3466	7612	2414	7605	4412
$\{d_1\}$	465	1031	368	762	318
$\{d_2\}$	239	490	202	435	292
$\{d_3\}$	115	131	52	140	53
$\{d_1, d_2\}$	85	265	94	199	73
$\{d_1, d_3\}$	99	149	94	151	36
$\{d_2, d_3\}$	9	7	8	15	9
$\{d_1, d_2, d_3\}$	12	32	13	37	10
n	4490	9717	3245	9344	5203

Table 1 shows that the number of individual reporting three illnesses is the lowest in Turkey (with a proportion of population of 0.19%), whilst the highest proportion of individuals with three illnesses is observed in Lebanon (0.40%).

FOSD and Hammond Dominance

We first illustrate, in Table 2.a and Figure 3, the construction of the CDF and HDF for one of the possible 48 linear extensions that corresponds to the following linear extension

$$E_1 = \big\{\{d_1, d_2, d_3\}, \{d_1, d_2\}, \{d_1, d_3\}, \{d_2, d_3\}, \{d_1\}, \{d_2\}, \{d_3\}, \phi\big\} \qquad (23)$$

Insert Table 2.a here

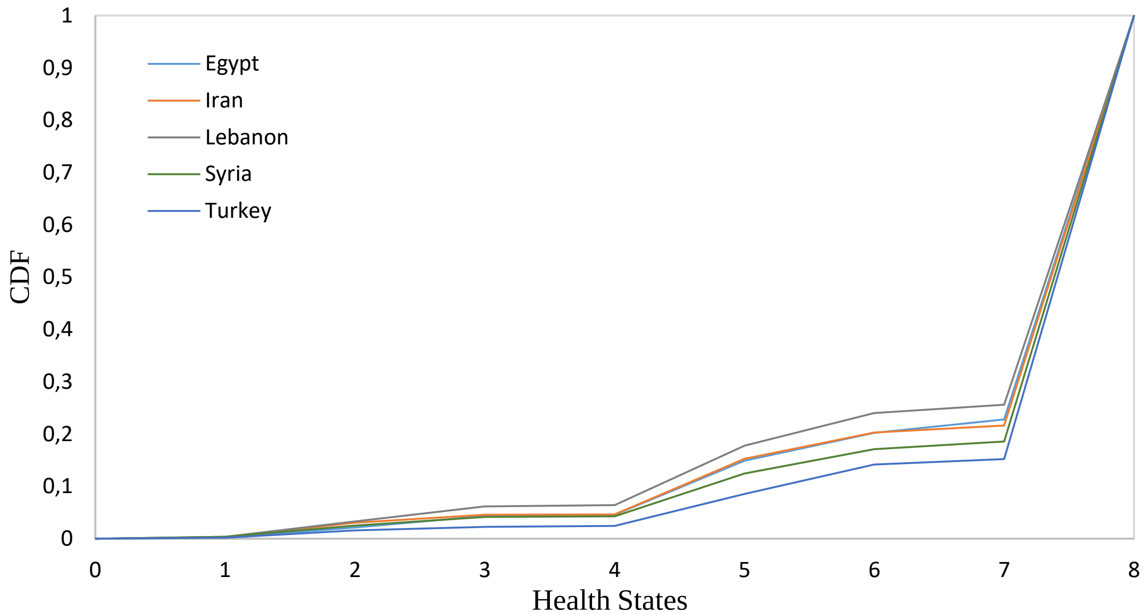
Table 2.a: The CDF and HDF of five health distributions for a possible linear extension

	CDF					HDF				
	Egypt	Iran	Lebanon	Syria	Turkey	Egypt	Iran	Lebanon	Syria	Turkey
E_i	F_i^E	F_i^I	F_i^L	F_i^S	F_i^T	H_i^E	H_i^I	H_i^L	H_i^S	H_i^T
$\{d_1, d_2, d_3\}$	0.003	0.003	0.004	0.004	0.002	0.003	0.003	0.004	0.004	0.002
$\{d_1, d_2\}$	0.022	0.031	0.033	0.025	0.016	0.024	0.034	0.037	0.029	0.018
$\{d_1, d_3\}$	0.044	0.046	0.062	0.041	0.023	0.071	0.083	0.103	0.075	0.043
$\{d_2, d_3\}$	0.046	0.047	0.064	0.043	0.025	0.143	0.167	0.208	0.151	0.087
$\{d_1\}$	0.149	0.153	0.178	0.125	0.086	0.390	0.440	0.530	0.383	0.235
$\{d_2\}$	0.202	0.203	0.240	0.171	0.142	0.833	0.930	1.122	0.813	0.527
$\{d_3\}$	0.228	0.217	0.256	0.186	0.152	1.692	1.873	2.261	1.641	1.063
ϕ	1	1	1	1	1	4.156	4.530	5.265	4.095	2.975

Table 2.a clearly show that $\forall\ k = 1, \dots, 8$, $F_k^T \leq F_k^{-T}$, $-T = \{E, I, L, S\}$, thus $F^T \leq_{FOSD} F^{-T}$. This suggests that Turkey has the least ill-health distribution with the corresponding CDF lying below the CDFs of all other countries (see Figure 3). In this case, Turkey appears to have higher health welfare compared with all other countries.

Insert Figure 3 here

Figure 3: FOSD of five health distributions for one linear extension



Given that the ill-health distributions of some countries cross, we can extent the dominance analysis to apply the Hammond dominance criteria. Figure 4 draws the HDFs for the distributions pertaining to E_1. As shown, while Egypt and Iran are incomparable under the FOSC criteria, they turn out to be comparable under the Hammond dominance criteria, $H_k^E \leq H_k^I\ \forall\ k = 1, \dots, 8$.

Insert Figure 4 here

Figure 4: Hammond dominance of five health distributions for one linear extension

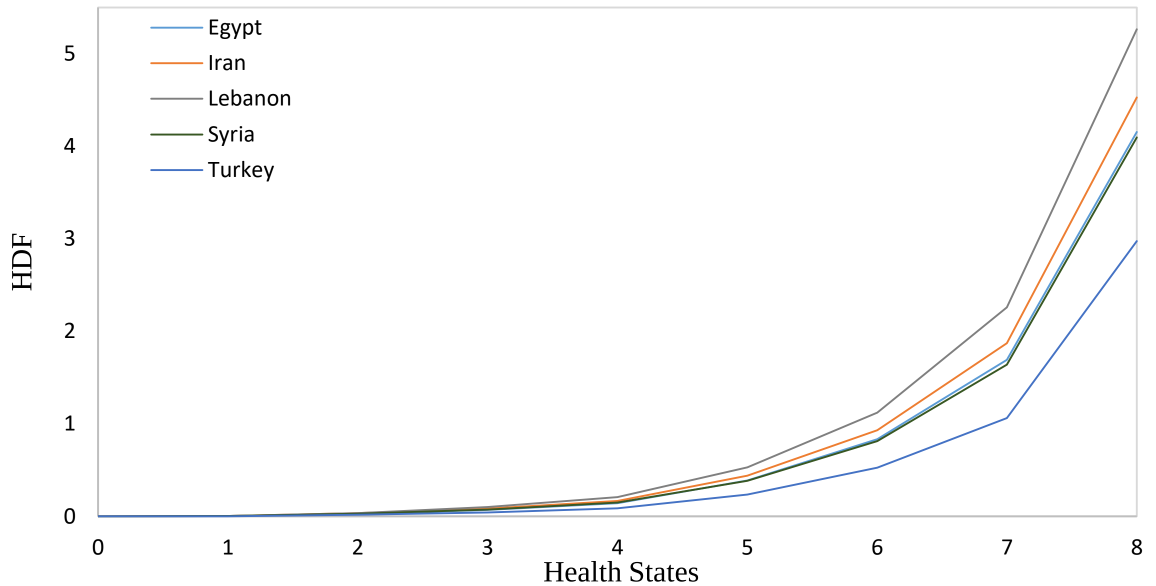


Table 2.a also shows that $F_k^E \leq F_k^L\ \forall\ k = 1, \ldots, 8$ suggesting that Egypt dominates Lebanon under this linear extension. However, this is not always the case: changing the order of incomparable health states results in a change in the rank of distributions. For instance, consider the following linear extension:

$$E_2 = \big\{\{d_1, d_2, d_3\}, \{d_2, d_3\}, \{d_1, d_3\}, \{d_3\}, \{d_1, d_2\}, \{d_1\}, \{d_2\}, \phi\big\} \qquad (24)$$

where the health state $\{d_3\}$ is supposed to be worse than the health state $\{d_1, d_2\}$. Under such extension, $F_k^E < F_k^L\ \forall\ k \neq 4$ but $F_4^E \geq F_4^L$. This suggests that the two distribution are incomparable in the sense of the FOSC, i.e., $F^E \parallel_{FOSD} F^L$. However, as soon as Hammond dominance criteria is applied, the two distributions pertaining to these two countries under the extension E_2 become now comparable with $F^E \leq_H F^L$.

Insert Table 2.b here

Table 2.b: The CDF and HDF of five health distributions for another possible linear extension

	CDF					HDF				
	Egypt	Iran	Lebanon	Syria	Turkey	Egypt	Iran	Lebanon	Syria	Turkey

E_i	F_i^E	F_i^I	F_i^L	F_i^S	F_i^T	H_i^E	H_i^I	H_i^L	H_i^S	H_i^T
$\{d_1, d_2, d_3\}$	0.003	0.003	0.004	0.004	0.002	0.003	0.003	0.004	0.004	0.002
$\{d_2, d_3\}$	0.005	0.004	0.006	0.006	0.004	0.007	0.007	0.010	0.010	0.006
$\{d_1, d_3\}$	0.027	0.019	0.035	0.022	0.011	0.037	0.030	0.050	0.035	0.018
$\{d_3\}$	0.052	0.033	0.051	0.037	0.021	0.099	0.073	0.116	0.085	0.046
$\{d_1, d_2\}$	0.071	0.060	0.080	0.058	0.035	0.217	0.174	0.261	0.192	0.107
$\{d_1\}$	0.175	0.166	0.194	0.140	0.096	0.538	0.454	0.635	0.466	0.274
$\{d_2\}$	0.228	0.217	0.256	0.186	0.152	1.129	0.959	1.332	0.978	0.605
ϕ	1	1	1	1	1	3.030	2.701	3.408	2.770	2.058

Table 3 presents the dominance results for the 48 linear extensions. The value in each cell represents the share (out of the 48) of linear extensions where distribution j (column) dominates distribution j' (row). A positive (*negative*) value means that distribution j dominates (dominated by) distribution j'. As shown in Table 3, the health distributions of Egypt and Iran are incomparable under the 45 extensions using the FOSD criteria. Interestingly, the Hammond stochastic dominance criteria allows, as expected, to extend the $FOSD$ as reflected by higher proportions of dominance under HDF (with Egypt dominating Iran in about 33.3% of the 48 linear extensions). Also of note, Lebanon appears to be dominated by all other countries under all linear extensions in the Hammond sense (as is captured by the value of -1 in Table 3). To sum up, Table 3 shows that the health distribution of Lebanon is always the worst distribution while the health distribution of Turkey is always the best under the Hammond dominance criteria.

Insert Table 3 here

Table 3: The FOSD and Hammond dominance for the 48 linear extensions

	CDF					HDF				
$j \backslash j'$	Egypt	Iran	Lebanon	Syria	Turkey	Egypt	Iran	Lebanon	Syria	Turkey
Egypt	.	0	-0.917	0	1	.	-0.333	-1	-0.125	1
Iran	0	.	-1	0	1	0.333	.	-1	-0.125	1
Lebanon	0.917	1	.	1	1	1	1	.	1	1
Syria	0	0	-1	.	0.917	0.125	0.125	-1	.	1
Turkey	-1	-1	1	-0.917	.	-1	-1	-1	-1	.

Social Welfare and Inequality for Partially-Ordered Health Distributions

Table 4 presents the results on social welfare and inequality indices for the 48 linear extensions. Each cell in row, j, and column, j', reports the proportion of linear extensions where country j has higher social welfare or inequality as compared to country j'.

Insert Table 4 here

Table 4: The SWF and inequality index for the 48 linear extensions

	SWF					Inequality index				
$j \backslash j'$	Egypt	Iran	Lebanon	Syria	Turkey	Egypt	Iran	Lebanon	Syria	Turkey
Egypt	.	0.5	1	0.292	0	.	0.813	0	1	1
Iran	0.5	.	1	0.208	0	0.188	.	0	1	1
Lebanon	0	0	.	0	0	1	1	.	1	1
Syria	0.708	0.792	1	.	0	0	0	0	.	1
Turkey	1	1	1	1	.	0	0	0	0	.

Table 4 shows that, for instance, Syria appears to have higher social welfare and lower inequality as compared to Egypt in about 70.8% and 100% of the 48 linear extensions, respectively. Overall, results confirm the trends reported in Table 3 on the FOSD and Hammond dominance where Turkey appears to have the highest welfare and lowest inequality as compared to all other countries under all the 48 linear extensions, while Lebanon appears to have the lowest welfare and the highest inequality as compared to all other countries.

4. Some Restrictions

As shown in Section 3, there are some cases where the ranking of distributions according to the four criteria introduced in this paper is still ambiguous. In order to obtain a clearer ranking, it is possible to reduce the number of linear extensions by imposing some restrictions on the health states ordering. In general, from a normative point of view, ordering illnesses according to, for example, their degree of severity might be problematic. This is because the severity of an illness might not be easily measurable and it can be expressed differently from different points of view.

However, there have been several attempts to measure the severity of illness using different indicators such as the mortality rate. Suppose that d_1 and d_2 can be ranked in terms of the rate of mortality where, for example, d_1 is less severe than d_2 – denoted as $d_1 \preccurlyeq d_2$ (restriction 1). This restriction assumption implies that $\{d_1, d_3\} \preccurlyeq \{d_2, d_3\}$. This restriction assumption reduces the number of linear extensions illustrated in Section 3 to 14. Suppose further that $d_1 \preccurlyeq d_2 \preccurlyeq d_3$ (restriction 2). This implies that $\{d_1, d_2\} \preccurlyeq \{d_1, d_3\} \preccurlyeq \{d_2, d_3\}$, hence reduces the number of linear extensions to the following two extensions.

$$E_a = \{\{d_1, d_2, d_3\}, \{d_2, d_3\}, \{d_1, d_3\}, \{d_1, d_2\}, \{d_3\}, \{d_2\}, \{d_1\}, \phi\}$$
$$E_b = \{\{d_1, d_2, d_3\}, \{d_2, d_3\}, \{d_1, d_3\}, \{d_3\}, \{d_1, d_2\}, \{d_2\}, \{d_1\}, \phi\} \quad (25)$$

Results pertaining to the dominance analysis under (restriction 1 and restriction 2) are illustrated in Table 5. The two monotonicity restrictions yield a complete order of the five distributions as follows: $L \preccurlyeq I \preccurlyeq E \preccurlyeq S \preccurlyeq T$.

Insert Table 5 here

Table 5: The SWF and inequality index under restriction 1 and restriction 2

	SWF					Inequality index				
$j \backslash j'$	Egypt	Iran	Lebanon	Syria	Turkey	Egypt	Iran	Lebanon	Syria	Turkey
Egypt	.	1	1	0	0	.	0	0	1	1
Iran	0	.	1	0	0	1	.	0	1	1
Lebanon	0	0	.	0	0	1	1	.	1	1
Syria	1	1	1	.	0	0	0	0	.	1
Turkey	1	1	1	1	.	0	0	0	0	.

5. Conclusion

This paper presents a general framework for the analysis of social welfare when well-being states are *a priori* incomparable. In such context, the distribution of well-being states forms an m-dimensional Boolean lattice. The basic idea of the present framework is to breakdown the Boolean lattice into a set of linear extensions where all well-being states are totally ordered. Then, the different distributions can be evaluated in term of social welfare using appropriate dominance criteria as well as inequality metrics. Accordingly, the ranking of two distributions is the result of the simple ranking of all possible linear extensions of the original lattice. When some monotonicity restrictions are imposed, the number of linear extensions of each distribution would decrease. Accordingly, a full ranking of all distributions can be established. Furthermore, under such restrictions, the Hammond SWF and the inequality index provide the same ranking of the distributions under consideration.

The method proposed in this paper can be generalized to the case of m illnesses where the number of health states is 2^m. If we assume, for simplicity, that one illness is as bad as two illnesses and two illnesses are as bad as four illnesses, etc., then the number of possible linear extensions is $e = m!\binom{m}{2}!\ldots\binom{m}{m-1}!$. For the case where $m = 4$, there will be hundreds of thousands linear extensions which may render the tractability and comparison of linear extensions burdensome. A possible solution to this problem is to reduce the number of linear extensions. This can be done by either imposing some monotonicity restrictions as is shown in Section 4 or by aggregating illnesses into three categories, hence reducing the problem to the case presented in this paper.

The framework introduced in this paper can be also applied to the analysis of social welfare in terms of alternative attributes. One example is when the social welfare is measured in terms of amenities available for each household. In this case, given that the availability of an amenity is a good outcome, then the worst well-being state is having no amenity while the best well-being is the availability of all amenities under consideration. Another example includes the analysis of multidimensional well-being as measured by richness, healthiness and happiness or any other potential welfare attribute. In order to apply the current framework, each of these attribute is assumed to be a binary variable where an individual is said to be rich, happy and health based on a certain threshold for each dimension. Then, well-being states can be generated in terms of the number of deprivations an individual may suffer from with the worst well-being state being that when an individual is poor, unhappy and unhealthy. The well-being states where individuals are deprived in only one dimension (poor or unhappy or unhealthy) are incomparable. Of course, a variety of possible applications can be analyzed using the framework presented in this paper.

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