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WP 2022- Nr 23

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October 10, 2022

JEL Codes: D80, D81

Keywords: Ignorance, Ambiguity, Ranking Sets, Axioms, Ranks, Weights, Utility

Abstract

The paper provides an axiomatic characterization of a family of rank dependent weighted average utility criteria applicable to decisions under ignorance or objective ambiguity. A decision under ignorance is described by the finite set of its final consequences while a decision under objective ambiguity is described by a finite set of probability distributions over a set of final consequences. The criteria characterized are those that assign to every element in a set a *weight* that depends upon the rank of this element if it was available for sure (or non-ambiguously) and that compare sets on the basis of their weighted utility for some utility function. A specific subfamily of these criteria that requires the weights to be proportional to each other is also characterized.

1 Introduction

Consider the following decision problem, provided by Ahn (2008), of a cancer patient having to choose between two treatments. The first is a conventional and widely used chemotherapeutic treatment that is associated to a five-year survival rate of 0.5. The second treatment is a new targeted therapeutic treatment that has only been tried on two samples of patients of comparable sizes.

*Authors are listed alphabetically. They have contributed equally.

†The project leading to this publication has received funding from the French government under the "France 2030" investment plan managed by the French National Research Agency (reference: ANR-17-EURE-0020) and from Excellence Initiative of Aix-Marseille University - A*MIDEX.

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On one sample, 80% of the patients have been observed alive after 5 years but on the other sample, only 20% of the patients were alive after 5 years. This is an example of decision making under *objective ambiguity*. There is *ambiguity* because the probabilities (of survival) that enter in the description of the second treatment are not unique. The ambiguity is however *objective* because the probabilities, while multiple, are known to the decision maker and enter therefore in the descriptions of the decisions. This is in contrast to decision making under subjective ambiguity studied in papers such as Epstein and Zhang (2001), Ghirartado and Marinacci (2002), Ghirartado, Maccheroni, and Marinacci (2004) or Klibanoff, Marinacci, and Mukerji (2005) in which decisions are described as Savagian acts without any *a priori* probabilities. Another well-known example of an objectively ambiguous decisions is the sequence of two choices made in the Ellsberg (1961) experiment.

From a formal point of view, *ranking decisions* under objective ambiguity amounts to *ranking sets* of possible probability distributions over a set of final consequences. The description of decisions as sets of objects is also made in the literature on decision making under radical uncertainty or ignorance surveyed, for example, in Barberà, Bossert, and Pattanaik (2004), in which the elements of the sets are interpreted as the final consequences of the decisions rather than as probability distributions over those.

This paper contributes to the literature on decision making under ignorance or objective ambiguity under the additional assumption that the various possible decisions can be described as finite sets (of either final consequences or probability distributions). This approach therefore differs from that provided, for instance, in Ahn (2008) and Olszewski (2007) in which decisions are depicted as uncountable sets of objects. As argued in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018), we believe that the description of decisions as finite sets of consequences (probability distributions) is somewhat natural, and clearly in line with experimental contexts in which we may want to test these models. For sure the Ellsberg experiment or the choice faced by the cancer patient above concern finite sets.

A majority of the decision making criteria examined in the literature on ranking finite sets are based on the best and the worst consequences of the decisions or on associated lexicographic extensions. There are two obvious limitations of such "extremist" rankings. The first is that it is natural to believe (in line with various "expected utility" hypotheses) that decision makers are concerned with "averages" rather than "extremes". A second drawback of "extremist" rankings is that they do not allow for much diversity of attitudes toward ignorance across decision makers. In situations where decisions have only monetary consequences, all decision makers who prefer more money to less will rank all decisions in the same way under "extremist" rules such as maximin, maximax, leximin and so on. This is unsatisfactory since the fact for two decision makers to have the same preference over certain outcomes (or unambiguous decisions) should not imply that they have the same attitude toward ignorance or ambiguity. It is with the aim of obtaining less extreme rankings of finite sets that Gravel, Marchant, and Sen (2012) characterizes with three axioms the Uniform

Expected Utility (UEU) family of criteria for comparing finite sets of objects. Any criterion from that family results from assigning to every conceivable element of the universe a utility number and from comparing sets on the basis of the expectation of the utility of their elements under the (uniform) assumption that all elements in the sets are equally likely. Gravel, Marchant, and Sen (2018) generalizes the UEU family of rankings of finite sets to the Conditional Expected Utility (CEU) family. Any CEU ranking of finite sets assigns to every conceivable element in the universe both a utility number and a (strictly positive) likelihood, and compares sets on the basis of their expected utility, with expectations taken with respect to the relative likelihood of those elements conditional upon the fact that they are in the sets. A UEU ranking of sets can be viewed as a specific CEU ranking for which the likelihood function consider all conceivable elements as equally likely. CEU rankings can be viewed as the finite analogues of the ranking of atomless sets of objects characterized by Ahn (2008) and, before him, by Bolker (1966) and Jeffrey (1983).

While UEU and CEU criteria provide simple and reasonably plausible rankings of decisions under ignorance or objective ambiguity, they both satisfy an axiom that may be at odd with actual decision making behavior. This axiom, called Averaging in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018) (and also Fishburn (1972)) and disjoint set betweenness in Ahn (2008)[1], requires the ranking (weak or strict) of two disjoint sets to be equivalent to the requirement that their union be ranked between the two sets. To see why this axiom may not always provide an accurate depiction of actual decision making under objective ambiguity, consider again the choice of a cancer treatment described above. Imagine that, in addition to the second targeted therapeutic experimental treatment, the patient be proposed a third treatment tested this time on three samples of sizes comparable to those of the second treatment, and with 5-year survival rates of 0.2, 0.5 and 0.8. One can then represent the three treatments by the sets $\{0.5\}$, $\{0.2, 0.8\}$ and $\{0.2, 0.5, 0.8\}$ respectively. It is plausible that a patient facing this (horrible) decision could prefer the traditional $\{0.5\}$ treatment to the new $\{0.2, 0.8\}$ treatment because of a (pessimistic) fear that the sample on which the new treatment has performed poorly provides a better assessment of the true effectiveness of the new treatment than the sample on which it has performed well. The Averaging axiom would then imply that the patient should also prefer the third treatment $\{0.2, 0.5, 0.8\}$ (which is nothing else than the union of $\{0.5\}$ and $\{0.2, 0.8\}$) to $\{0.2, 0.8\}$. Should he really? This may not be clear. Indeed, one could argue that the results of the experimentation of the third treatment $\{0.2, 0.5, 0.8\}$ are noisier than those of the second in terms of the information that it provides about the treatment's effectiveness. Hence, a pessimistic patient who gives more weights to the samples where the treatment performs poorly to those where it performs well could very well choose the third treatment over the second even though he has chosen the first treatment over the second. In the only instance we know where

[1]Weaker variants of this axiom are also satisfied by the ranking examined in Olszewski (2007) and, in another context, in Gul and Pesendorfer (2001).

the averaging axiom has been tested in an experimental context (Vridags and Marchant, 2015), it has been rejected by a vast majority of subjects.

In this paper, we accordingly characterize a family of decision models that keeps the smoothness associated to the evaluation of decisions as per their expected utility, while dispensing with the Averaging axiom in any of the forms considered in the literature. The criteria analyzed can be viewed as variants of the rank-dependent expected utility family originally proposed by Quiggin (1982) (see also Quiggin (1993)) that are suitable to the considered finite set theoretic framework. These criteria were hinted at in Vridags and Marchant (2015) where they were referred to as Uniform Rank Dependent Utility criteria because they are equivalent to a RDU model applied to a uniform probability distribution. However, in this paper, we use the more explicit name of Rank Dependent Weighted Average Utility (RDWAU) to designate these criteria. A RDWAU criterion compares two decisions (finite sets) on the basis of their weighted average utility, for some utility function defined over all elements of the universe and some set-specific weight function—summing to one over all elements of the set—that depends upon the ranking of those elements if they were certain. While our general theorem characterizes the family of all such RDWAU rankings of finite sets with arbitrary rank-dependent weighting of the elements of the sets, we also provide the characterization of three subfamilies of those rankings that may be of interest. One of these families consists in RDWAU rankings that are neither extremely optimistic nor extremely pessimistic. Another family consists in RDWAU rankings that are mildly optimistic or pessimistic and whose rank-dependent weights are, accordingly, weakly increasing or decreasing with respect to the rank. We also provide a characterization of all RDWAU rankings of finite sets that satisfy the (strong) condition that the ratio between any two adjacent weights is constant.

The organization of the remaining of the paper is as follows. The next section introduces the framework, notation and main definition. Section 3 provides the results and section 4 concludes.

2 Formal Framework

2.1 Notation and definitions

We let X be some universe of *outcomes* that we interpret either as possible *consequences* of decisions (decision making under ignorance) or as *probability distributions* over a more fundamental set of final consequences (decision making under objective ambiguity). We shall nonetheless use the generic term of "outcomes" to designate these elements of X. Many results stated and proved in this paper will actually also ride on the assumption that X is a connected topological space[2]. Relevant examples of a set X could be monetary (possibly negative) consequences ($X = \mathbb{R}$), non-negative commodity bundles ($X = \mathbb{R}^l_+$

[2] A set A is connected for the (relevant) topology if it cannot be written as a finite union of pairwise disjoint open sets.

for some integer l) or, in an ambiguity context, the $l-1$ dimensional simplex interpreted as the set of all probability distributions over l consequences. A *decision* is a finite non-empty subset D of X. We denote by $\mathcal{P}(X)$ the set of all such decisions. Decisions made of a single outcome (singletons) are naturally interpreted as *certain* or *non-ambiguous*. For all integers m and n such that $m \leq n$, the set $\{m, m+1, \ldots, n\}$ is denoted by $[m, n]$. When $m = 1$, we write simply $[n]$ instead of $[1, n]$. A set of the form $\{m, m+1, \ldots\}$ is denoted by $[m, \cdot]$. The set $[1, \cdot]$ is also denoted by $\mathbb{N}$.

Decisions are compared by an ordering[3] $\succsim$ on $\mathcal{P}(X)$ with the usual interpretation that $D \succsim D'$ if and only if the decision maker weakly prefers decision D to decision D'. The asymmetric (strict preference) and symmetric (indifference) factors of $\succsim$ are denoted respectively by $\succ$ and $\sim$. For reasons that will soon become clear, whenever we write a decision D in $\mathcal{P}(X)$ with n possible outcomes in the form $D = \{d_1, \ldots, d_n\}$, we *label* the outcomes of D in such a way that $\{d_1\} \precsim \ldots \precsim \{d_n\}$. There may of course be several such labellings if there are indifferences between some singleton subsets of D. For every set for which such indifferences happen, we choose once and for all any of the several labellings that could do. For any decision $D \in \mathcal{P}(X)$ labeled in this way and any $x \in D$, we denote by $r_x^D \in [\#D]$ the rank of x in D defined by $r_x^D = i \iff x = d_i$ for $D = \{d_1, \ldots, d_{\#D}\}$.

This paper is specifically interested in Rank Dependent Weighted Average Utility (RDWAU) orderings of $\mathcal{P}(X)$ for which there exist a continuous function $u : X \to \mathbb{R}$ and, for any $n \in \mathbb{N}$, n strictly positive real numbers w_i^n satisfying $\sum_{i \in [n]} w_i^n = 1$ such that, for all $A = \{a_1, \ldots, a_{\#A}\}$ and $B = \{b_1, \ldots, b_{\#B}\}$:

$$A \succsim B \iff \sum_{i \in [\#A]} w_i^{\#A} u(a_i) \geq \sum_{i \in [\#B]} w_i^{\#B} u(b_i). \tag{1}$$

Hence, an RDWAU ordering of decisions *can be* thought of as resulting from the comparisons of a weighted average of the utility of the possible outcomes of those decisions for some utility function, and for some weights depending upon the ranking of the outcomes in the decisions if these outcomes were obtained for sure. There are obviously many RDWAU orderings, as many in fact as there are conceivable ways of assigning utility levels to outcomes and weights to their ranks in the decisions.

To illustrate how a RDWAU ordering compares decisions, reconsider the introductory example of the cancer patient. In this setting, $X = [0, 1]$, interpreted as the various conceivable five-year probabilities of survival ordered in the obvious way if they were known non-ambiguously. The three decisions faced by the patient would then be $\{1/2\}$, $\{1/5, 4/5\}$ and $\{1/5, 1/2, 4/5\}$ and an RDWAU ordering of the decisions *could be* based on the utility function $u(p) = p^2$ for every $p \in [0, 1]$ and on the weights $w_1^1 = 1$, $w_1^2 = 3/4$, $w_2^2 = 1/4$, $w_1^3 = 2/3$, $w_2^3 = 2/9$ and $w_3^3 = 1/9$. In this case, we would have $\{1/2\} \succ \{1/5, 4/5\}$ because:

$$u(1/2) = \frac{1}{4} > w_1^2 u(1/5) + w_2^2 u(4/5) = \frac{19}{100}$$

[3] An ordering is a reflexive, complete and transitive binary relation.

and we would have $\{1/5, 4/5\} \succ \{1/5, 1/2, 4/5\}$ because:

$$w_1^2 u(1/5) + w_2^2 u(4/5) = \frac{19}{100} > w_1^3 u(1/5) + w_2^3 u(1/2) + w_3^3 u(4/5) = \frac{23}{150}$$

As mentioned earlier, this ranking of the three decisions violates the averaging axiom used in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018) (and also in Ahn (2008) and, in some weakened forms, Olszewski (2007) and Gul and Pesendorfer (2001)) according to which $D \succsim D' \Leftrightarrow D \succsim D \cup D' \succsim D'$ for any two disjoint decisions D and D' (like $\{1/2\}$ and $\{1/5, 4/5\}$ are). Hence, this ranking would not be agreed upon by Uniform Expected Utility criteria characterized in Gravel, Marchant, and Sen (2012) or Conditional Expected Utility criteria characterized in Gravel, Marchant, and Sen (2018) (and Ahn (2008) in a setting where decisions consists in atomless sets). This ranking could not even be produced by the convex combination of the (utility of the) best and the worst outcomes of a decision characterized by Olszewski (2007) (again in a setting where decisions are described by uncountable sets of outcomes).

We observe that the class of RDWAU orderings contains the class of UEU orderings, who are nothing else than RDWAU orderings for which the weights w_i^k are equal to $1/k$ for every $i = 1, \ldots, k$. However, there is no inclusion relation between the classes CEU and RDWAU. The example just given shows that there are RDWAU orderings that are not CEU orderings. The following example provides a CEU ordering that is not a RDWAU ordering.

Example 1 *Let $X = [0, 1]$ (interpreted again as the five-year survival probabilities ordered in the obvious way), and define $\succsim$ on $\mathcal{P}(X)$ by*

$$D \succsim D' \Longleftrightarrow \frac{\sum\limits_{p \in D} \rho(p) u(p)}{\sum\limits_{p \in D} \rho(p)} \geq \frac{\sum\limits_{p' \in D'} \rho(p') u(p')}{\sum\limits_{p' \in D'} \rho(p')},$$

for the functions ρ and u defined (on $[0, 1]$) by

$$\rho(p) = 1 + p - p^2 \quad \text{and} \quad u(p) = p.$$

Observe that this CEU ordering would rank decision $\{1/80, 1/2, 19/20\}$ above decision $\{1/15, 1/2, 9/10\}$ because

$$\frac{(1 + 1/80 - 1/6400)1/80 + (1 + 1/2 - 1/4)1/2 + (1 + 19/20 - 361/400)19/20}{1 + 1/80 - 1/6400 + 1 + 1/2 - 1/4 + 1 + 19/20 - 361/400} = 0.49331 >$$

$$0.49286 = \frac{(1 + 1/15 - 1/225)1/15 + (1 + 1/2 - 1/4)1/2 + (1 + 9/10 - 81/100)9/10}{1 + 1/15 - 1/225 + 1 + 1/2 - 1/4 + 1 + 9/10 - 81/100}.$$

This CEU ordering would also rank decision $\{1/80, 1/5, 19/20\}$ below decision $\{1/15, 1/5, 9/10\}$ because

$$\frac{(1 + 1/80 - 1/6400)1/80 + (1 + 1/5 - 1/25)1/5 + (1 + 19/20 - 361/400)19/20}{1 + 1/80 - 1/6400 + 1 + 1/5 - 1/25 + 1 + 19/20 - 361/400} = 0.38504 <$$

$$\frac{(1+1/15-1/225)1/15+(1+1/5-1/25)1/5+(1+9/10-81/100)9/10}{1+1/15-1/225+1+1/5-1/25+1+9/10-81/100}=0.38760.$$

These two rankings however cannot result from an RDWAU ordering. Indeed, if they were, the first ranking would imply, for some numbers w_i^3 ($i \in [3]$) and utilities $u(1/80)$, $u(1/2)$ and $u(19/20)$,

$$\begin{aligned} w_1^3 u(1/80) + w_2^3 u(1/2) + w_3^3 u(19/20) &> w_1^3 u(1/15) + w_2^3 u(1/2) + w_3^3 u(9/10) \\ &\Leftrightarrow \\ w_1^3 u(1/80) + w_3^3 u(19/20) &> w_1^3 u(1/15) + w_3^3 u(9/10) \end{aligned}$$

while the second ranking would imply the reverse inequality.

Before presenting the axioms that characterize the RDWAU family, we find useful to introduce the following notion of revealed (by the decision maker's ordinal preferences) *preference strength for one outcome over another* as applicable to the various possible ordered pairs of those outcomes. We formulate successively the definitions of weak, strict and equivalent revealed preference strength.

Definition 1 *Let x, y, x', y' be outcomes. The ordering $\succsim$ on $\mathcal{P}(X)$ is said to reveal a weakly larger preference strength for x over y than for x' over y', which we write formally as $(x, y)\ \Delta^{\succsim}\ (x', y')$, if there are two sets A and B satisfying $\#A = \#B$, $\{x, y, x', y'\} \cap (A \cup B) = \varnothing$ and $r_x^{A\cup\{x\}} = r_y^{B\cup\{y\}} = r_{x'}^{A\cup\{x'\}} = r_y^{B\cup\{y'\}}$ such that*

$$A \cup \{x\} \succsim B \cup \{y\} \text{ and } A \cup \{x'\} \precsim B \cup \{y'\}. \tag{2}$$

If at least one of $\succsim$ and $\precsim$ in (2) is strict, we then say that the preference strength for x over y is strictly *larger than that for x' over y', which we write formally as $(x, y)\ \Delta_s^{\succsim}\ (x', y')$.*

If both $\succsim$ and $\precsim$ in (2) are replaced by $\sim$, we then say that both preference strengths are equivalent, which we write formally as $(x, y)\ \Delta_e^{\succsim}\ (x', y')$.

In words, the preference strength for x over y is revealed weakly larger than the preference strength for x' over y' if there are two sets A and B with the same number of outcomes to which the respective addition of x and y—under the condition that the rank of x and y in the two enlarged sets is the same—lead to a preference for the enlarged A to the enlarged B while the similar addition of x' and y' to the two sets lead, under the same condition on the ranks, to the opposite preference. Hence, it seems that x "does more" with respect to y than x' does with respect to y', at least as judged by their addition to some sets A and B that do not contain these outcomes. Observe that Definition 1 does not preclude the two sets A and B to which the outcomes are added to be the same. It does not even rule out the possibility that these two sets be both empty. In this latter case, the "addition" of two outcomes to the same empty set amount simply to comparing those outcomes as if they were available for sure.

The interpretation of the quaternary relation $\Delta^{\succsim}$—defined on all pairs of outcomes—as evaluating the preference strength for one outcome over another is particularly clear if one assumes from the start that the ordering $\succsim$ is itself an RDWAU ordering. Indeed, for a RDWAU ordering, the fact to have, for some rank $i \in [n]$, $\{a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n\} \succsim \{b_1, \dots, b_{i-1}, y, b_{i+1}, \dots, b_n\}$ can be written, thanks to (1), as:

$$\begin{aligned}
&\sum_{g=1}^{i-1} w_g^n u(a_g) + w_i^n u(x) + \sum_{h=i+1}^{n} w_h^n u(a_h) \\
&\qquad\qquad \geq \sum_{g=1}^{i-1} w_g^n u(b_g) + w_i^n u(y) + \sum_{h=i+1}^{n} w_h^n u(b_h)
\end{aligned}$$

$$\Leftrightarrow$$

$$u(x) - u(y) \quad \geq \quad \frac{\sum_{g=1}^{i-1} w_g^n \big(u(b_g) - u(a_g)\big) + \sum_{h=i+1}^{n} w_h^n \big(u(b_g) - u(a_g)\big)}{w_i^n}. \tag{3}$$

On the other hand,

$$\{a_1, \dots, a_{i-1}, x', a_{i+1}, \dots, a_n\} \precsim \{b_1, \dots, b_{i-1}, y', b_{i+1}, \dots, b_n\}$$

can be similarly written, for the same RDWAU ordering, as:

$$u(x') - u(y') \leq \frac{\sum_{g=1}^{i-1} w_g^n \big(u(b_g) - u(a_g)\big) + \sum_{h=i+1}^{n} w_h^n \big(u(b_g) - u(a_g)\big)}{w_i^n}. \tag{4}$$

Hence, the combination of Inequalities (3) and (4) reveals indeed that $u(x) - u(y) \geq u(x') - u(y')$.

A few remarks can be made about the quaternary relation $\Delta^{\succsim}$—or equivalently the binary relation on the set $X \times X$—of Definition 1. For one thing, this quaternary relation has very little structure. When viewed as a binary relation on $X \times X$, it is not reflexive since one may well have, for some distinct outcomes x and y, that $A \cup \{x\} \succ B \cup \{y\}$ for all sets A and B with the same number of outcomes containing neither x and y such that $r_x^{A \cup \{x\}} = r_y^{B \cup \{y\}}$. It is not complete since, again, nothing rules out the possibility that, for some outcomes x, y, x' and y' both $A \cup \{x\} \succ B \cup \{y\}$ and $A \cup \{x'\} \succ B \cup \{y'\}$ hold for all sets A and B with the same cardinality that do not contain any of these outcomes and that are such that $r_x^{A \cup \{x\}} = r_y^{B \cup \{y\}} = r_{x'}^{A \cup \{x'\}} = r_{y'}^{B \cup \{y'\}}$. For sure, $\Delta^{\succsim}$ is not transitive on $X \times X$.

However, $\Delta^{\succsim}$ on $X \times X$ does satisfy Property 2 of what Krantz, Luce, Suppes, and Tversky (1971) call (Definition 3, chapter 4) an *algebraic difference structure* as established in the following (obvious) remark.

Remark 1 *For any outcomes x, y, x' and y' in X, the two following statements are equivalent:*

(i) $(x, y)\, \Delta^{\succsim}\, (x', y')$ *and,*

(ii) $(y', x') \; \Delta^{\succsim} \; (y, x)$.

We observe also that $\Delta_s^{\succsim}$ (strictly larger preference strength) of Definition 1 is *not* the asymmetric factor of $\Delta^{\succsim}$ (it is possible to have both $(x, y) \; \Delta_s^{\succsim} \; (x', y')$ and $(x', y') \; \Delta_s^{\succsim} \; (x, y)$), even though it is compatible with it. Somewhat dually, the symmetric factor of $\Delta^{\succsim}$ is compatible with $\Delta_e^{\succsim}$ of Definition 1 (equivalent preference strength) but is not equivalent to it.

2.2 Axioms

We now state and comment a bit the axioms that characterize the whole family of RDWAU orderings. The first one is an adaptation to the present setting of Peter Wakker's "trade-off consistency" condition (Wakker, 1989). It imposes some minimal consistency among comparative statements of revealed preference strength performed by the weak $\Delta^{\succsim}$ and the strict $\Delta_s^{\succsim}$. Specifically, we require that if the preference strength for x over y is revealed weakly larger than the preference strength for x' over y' through the addition of these four outcomes to two decisions A and B with the same cardinality as described in Definition 1, then one should never observe the preference strength for x over y to be revealed strictly smaller than the preference strength between x' and y'. We state this axiom as follows.

Axiom 1 *Consistency in Comparisons of Preference Strength. For no x, y, x' and y' in X should we observe both $(x, y) \; \Delta^{\succsim} \; (x', y')$ and $(x', y') \; \Delta_s^{\succsim} \; (x, y)$.*

While (relatively) natural, this consistency condition has strong implications. For one thing, it implies an "independence" axiom that has been widely discussed in the literature on additive numerical representation of orderings. In the current rank-dependent context, the independence axiom requires, in substance, that the ranking of decisions with the same number of outcomes be independent from any outcome that they have in common when the outcome has the same rank in the two decisions. For future reference, we state formally as follows this notion of comonotonic independence.

Condition 1 *Comonotonic Independence. For any distinct α and $\beta \in X$, and decisions D and D' such that $\#D = \#D'$, $(D \cup D') \cap \{\alpha, \beta\} = \varnothing$ and $r_\alpha^{D \cup \{\alpha\}} = r_\beta^{D \cup \{\beta\}} = r_\alpha^{D' \cup \{\alpha\}} = r_\beta^{D' \cup \{\beta\}}$, we have*

$$D \cup \{\alpha\} \succsim D' \cup \{\alpha\} \iff D' \cup \{\beta\} \succsim D' \cup \{\beta\}.$$

One can observe that this condition is a (significant) weakening of the restricted independence condition used by Gravel, Marchant, and Sen (2012) (and also by Nehring and Puppe (1996)), which requires the independence to hold even for a common element that may not have the same rank in the two considered decisions. The fact that Consistency in Comparison of Preference Strength implies Comonotonic Independence is established in the following lemma proved, like all formal results of the paper, in the Appendix

Lemma 1 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ that satisfies Consistency in Comparisons of Preference Strength. Then $\succsim$ satisfies Comonotonic Independence.*

The second axiom used in the characterization of the family of (continuous) RDWAU orderings is a specific continuity requirement. It requires X to be a connected set with respect to the order topology induced by the restriction of $\succsim$ to singletons.

Axiom 2 *Fixed Cardinality Continuity. For any decision D, the sets*

$$\{(b_1, \ldots, b_{\#D}) \in X^{\#D} : \{b_1, \ldots, b_{\#D}\} \succsim D\}$$

and

$$\{(b_1, \ldots, b_{\#D}) \in X^{\#D} : \{b_1, \ldots, b_{\#D}\} \precsim D\}$$

are closed in the product topology.

We observe that this continuity axiom is limited to comparisons of decisions with the same number of outcomes. It does not impose any continuity on the comparisons of decisions with differing number of outcomes.

The third—and last—axiom is a (significant) weakening of the well-known Gärdenfors (1976) principle discussed in the literature on ignorance (and notably in Barberà and Pattanaik (1984), Bossert (1989), Fishburn (1984) and Kannai and Peleg (1984)). It is formulated as follows.

Axiom 3 *Weak Gärdenfors Principle. For every decision $D = \{d_1, \ldots, d_n\} \in \mathcal{P}(X)$, one has $\{d_1\} \precsim D \precsim \{d_n\}$.*

In words, the Weak Gärdenfors Principle requires any decision to be weakly better than its worst outcome received certainly and, symmetrically, to be weakly worse than its best outcome received certainly. It is important to notice that the Weak Gärdenfors Principle is the only axiom that restricts the ranking of decisions with different numbers of possible outcomes. The fact that this restriction is, in fact, limited to the ranking of any uncertain (ambiguous) decision *vis-à-vis* certain (non-ambiguous) ones is also noteworthy.

In order to prove our main result, we introduce some additional terminology. Let $a_1, \ldots, a_k$ be some finite list of outcomes for some integer $k \geq 3$. We say that $a_1, \ldots, a_k$ form a *standard sequence* if $\{a_i, a_{i+1}\} \; \Delta_e^{\succsim} \; \{a_{i+1}, a_{i+2}\}$ for all $i \in [k-2]$. In plain English, $a_1, \ldots, a_k$ form a standard sequence if any two pairs of adjacent outcomes in the sequence exhibit the same preference strength for their first outcome over their second. Hence, a standard sequence is made of outcomes who are either increasingly favorable or decreasingly favorable (when received for sure) at a "constant rate".

3 Results

3.1 The general family of continuous RDWAU orderings

The characterization of the RDWAU family will be achieved by imposing the following rather mild richness condition on both the universe X and the set $\mathcal{P}(X)$ of all its non-empty finite subsets. This condition is not necessary for the RDWAU family of ordering in the sense that we may have environments where an RDWAU ordering would violate this condition.

Condition 2 *Essentialness. For any number $l \in \mathbb{N}$ of outcomes and rank $k \in [l]$, one can find decisions $D = \{d_1, \ldots, d_k, \ldots, d_l\} \in \mathcal{P}(X)$ and $D' = \{d_1, \ldots, d_{k-1}, d'_k, d_{k+1}, \ldots, d_l\}$ such that $D' \not\sim D$.*

This condition just says that any outcome of a decision is essential in the sense that one could always find a way to modify only this outcome while leaving unaffected both the other outcomes and their ranks that "would make the difference" in the evaluation of the decision.

The theorem proved in this paper is the following.

Theorem 1 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that Essentialness holds. Then $\succsim$ satisfies Consistency in Comparisons of Preference Strength, Fixed Cardinality Continuity and the Weak Gärdenfors Principle iff $\succsim$ is a RDWAU ordering as in (1). Moreover, the mapping u is unique up to a positive affine transformation and the weights w_i^n are unique.*

The proof of this theorem, provided in the appendix, proceeds in several steps. We first prove that the above axioms, without the Weak Gärdenfors Principle, characterize the RDWAU family of rankings of sets containing a specifically given number of outcomes. However, the result does not say anything about comparisons of sets containing different numbers of outcome. It does not even connect the numerical representation obtained for the ranking of sets with, say, m outcomes with that which enables the ranking of sets with, say, n outcomes. This first step of the proof is summarized in the following proposition.

Proposition 1 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that the Essentialness condition holds. Then $\succsim$ satisfies Consistency in Comparisons of Preference Strength and Fixed Cardinality Continuity iff there exist w_i^n as in (1) and a continuous function $u^n : X \to \mathbb{R}$ such that, for all decisions $A, B \in \mathcal{P}(X)$ with the same cardinality n, one has:*

$$A \succsim B \iff \sum_{i \in [n]} w_i^n u^n(a_i) \geq \sum_{i \in [n]} w_i^n u^n(b_i). \tag{5}$$

The mapping u^n is unique up to a positive affine transformation for every $n > 1$. The weights w_i^n are unique.

The second step of the proof consists in showing that any of the functions u^n that enters in the numerical representation (5) of the ordering $\succsim$ restricted to decisions with n outcomes provides a numerical representation of the ordering $\succsim$ restricted to singletons. The required result for this step is the following Lemma.

Lemma 2 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that the Essentialness condition holds. If $\succsim$ satisfies Consistency in Comparisons of Preference Strength and Fixed Cardinality Continuity, then, for any $n \in \mathbb{N}$, the function $u^n : X \to \mathbb{R}$ that enters in the numerical representation of the ordering $\succsim$ restricted to decisions with n outcomes as per (5) also numerically represents the restriction of the ordering $\succsim$ to singletons.*

Proposition 1 establishes the validity of the numerical representation (5) for the ranking of decisions with a given number of outcomes. However it does not connect together the functions u^n that enter in the definition of the numerical representation for decisions involving different numbers of outcome. With the help of Lemma 2, the next Lemma establishes that all the functions u^n that enter in the numerical representations of the orderings of decisions containing n outcomes for variable n can actually all be taken to be the same (up to a positive affine transformation). The formal statement of this lemma, proved in the Appendix, is as follows.

Lemma 3 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that the Essentialness condition holds. If $\succsim$ satisfies Consistency in Comparisons of Preference Strength and Fixed Cardinality Continuity, then, for any $n \in \mathbb{N}$, and any decisions D and D' with n possible outcomes, one has*

$$D \succsim D' \Longleftrightarrow \sum_{i \in [n]} w_i^n u(d_i) \geq \sum_{i \in [n]} w_i^n u(d_i')$$

for some continuous function $u : X \to \mathbb{R}$ uniquely defined up to a positive affine transformation. Moreover, the weights w_i^n are unique.

While Proposition 1 and Lemmas 2 and 3, which roughly establish the validity of the numerical representation RDWAU as per (1) for decisions with the same number of outcomes, make no use of the Weak Gärdenfors Principle, the rest of the proof, which establishes the validity of that same representation for comparing sets with different number of outcomes, will use this principle extensively. The last intermediate result that is required to prove Theorem 1 is the following lemma that establishes, under all the axioms, the existence of a "certain" (or non-ambiguous) equivalent to any decision.

Lemma 4 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that the Essentialness condition holds. If $\succsim$ satisfies Consistency in Comparisons of Preference Strength, Fixed Cardinality Continuity and the Weak Gärdenfors Principle, then, for any decision $D \in \mathcal{P}(X)$, there exists an outcome $\mathrm{CE}(D) \in X$ such that $\{\mathrm{CE}(D)\} \sim D$.*

The proof of Theorem 1 is then completed by showing that the numerical representation (1) shown so far to represent the ordering $\succsim$ on any two decisions containing the same number of outcomes is also valid for comparing decisions with different numbers of outcomes.

3.2 Some subclasses of RDWAU orderings

The family of RDWAU orderings of decisions characterized in Theorem 1 is quite general. The price to pay for this generality is the possible inability of RDWAU criteria to restrict significantly the possible decision patterns that they allow. The rank dependent weights used by RDWAU to calculate average utility are not restricted at all, if we except the fact that they are all strictly positive and sum to 1, and are the same for all sets with the same number of outcomes. However, the weights are allowed to vary in a completely arbitrary way when possible outcomes are added—or deleted—from a decision. For example, one could imagine a RDWAU ordering that puts a weight close to 1 on the worst possible outcome of two-outcome decisions (leaving the remaining almost zero weight for best outcome) but yet reverses perspective when evaluating three-outcomes decisions by putting (almost) all the weight on the best outcome in those cases. In the following subsection, we briefly explore some possibilities of restricting somewhat the rank dependent weights of RDWAU orderings without of course going as far as making them identical as they are in the UEU ordering.

3.2.1 Extreme optimism and pessimism

A possible way of restricting the weights is through the specification of the decision maker's optimism with respect to uncertain or ambiguous decision. We are using here the term "optimism" in the common sense of "the quality of being full of hope and emphasizing the good parts of a situation, or a belief that something good will happen" (Cambridge Dictionary). There are various ways by which we can introduce this notion and its opposite—pessimism—in the current setting. We first discuss two (in our view) extreme forms of optimistic and pessimistic attitudes which, presumably, could be excluded from the range of possible behavior from the part of "reasonable" decision makers.

Definition 2 *We say that the ordering $\succsim$ on $\mathcal{P}(X)$ exhibits* ***extreme optimism*** *with respect to decision $A \in \mathcal{P}(X)$ if there is an outcome $x \in X \backslash A$ such that $\{a\} \succ \{x\}$ for every outcome $a \in A$ for which $A \cup \{x\} \succsim A$.*

*We say conversely that $\succsim$ exhibits **extreme pessimism** with respect to A if there is an outcome $x \in X \backslash A$ such that $\{x\} \succ \{a\}$ for every outcome $a \in A$ for which $A \succsim A \cup \{x\}$.*

In words, the decision maker is extremely optimistic with respect to a decision if he/she considers that receiving information about the possibility of occurrence of an outcome that is strictly worse than all the already known outcomes of the decision makes the decision weakly better. Similarly, he/she is extremely pessimistic with respect to the decision if he/she would not at all value an information about the possibility of occurrence of an outcome that is better than all the currently known outcomes of the decision.

To illustrate even further how extreme—and unreasonable—the optimism of Definition 2 is, reconsider the cancer example above, and imagine that the patient, after being offered a treatment associated to a non-ambiguous 0.5 probability of survival over five year, is informed that the treatment is, actually, a bit ambiguous and has been observed, on some sample, to provide only a 0.2 probability of survival over five years. It seems unlikely that such a news would increase (even weakly) the attractiveness of the treatment from the patient's point of view.

Vridags and Marchant (2015) have shown that the very fact of ruling out both forms of extremism from the part of a RDWAU decision maker—a requirement referred to as *dominance* in (Barberà, Bossert, and Pattanaik, 2004)— leads to a significant restrictions of the rank dependent weights used by him or her when evaluating decisions with different numbers of outcomes. Specifically, the following Proposition is proved in Vridags and Marchant (2015) (Proposition 1) (with mild changes in the notation and definition).

Proposition 2 *Let $\succsim$ be an RDWAU ordering on $\mathcal{P}(X)$. Then there are no decisions for which $\succsim$ exhibits extreme optimism or pessimism as per Definition 2 if and only the weights w_h^n (for $n \in \mathbb{N}$ and $h \in [n]$) that enter in the numerical representation (1) satisfy both*

$$\sum_{h=1}^{i} w_h^n \leq \sum_{h=1}^{i+1} w_h^{n+1} \tag{6}$$

and:

$$\sum_{h=1}^{i} w_h^n \geq \sum_{h=1}^{i} w_h^{n+1} \tag{7}$$

for all $n \in \mathbb{N}$ and $i \in [n]$.

The restrictions of the rank-dependent weights provided by Inequalities (6) and (7)—somewhat evocative of first-order dominance notions—are significant, even though they allow for quite a variety of preferences over decisions. Inequalities (7) require the cumulated weight assigned to the i worst outcomes (for every i) to decrease when the number of possible outcomes of a decision increases from n to $n-1$. This makes sense because one needs to leave room, when cumulating over the i worst outcomes of the two decision, to accommodate for the additional outcome that was not present in the n-outcome decision. Somewhat symmetrically, Inequalities (6) require the cumulated weights assigned to the i worst outcomes in the n-outcome decision—larger than those of the $n+1$-outcome decision by Inequalities (7)—to remain nonetheless smaller than the cumulated weights assigned to the $i+1$ worst outcomes in the $n+1$-outcome decision.

3.2.2 Weak optimism and pessimism

Proposition 2 provides the list of all restrictions on the rank-dependent weights of a RDWAU decision maker that result from the assumption that the decision maker is not extremely optimistic nor pessimistic as per Definition 2. What about less extreme pessimism or optimism? A plausible definition of (moderate) optimism (pessimism) for a RDWAU decision maker is the requirement that the rank-dependent weights be increasing (decreasing) with the ranking of outcomes if they were certain. Such a notion is at least somewhat compatible with the definition of optimism/pessimism given in the literature on rank-dependent expected utility models in terms of the super (sub) additivity of the Choquet capacity (see e.g. Dillenberger, Postlewaite, and Rozen (2017) or Wakker (1990)), even though nothing in the current radical uncertainty or objective ambiguity context enables the definition of such a capacity as the source of the rank-dependent weights.

There is an easy ordinal test—and definition—of optimism (pessimism) in our finite set ranking context that leads precisely to this monotonicity of the weights as definition of optimism. Consider indeed any outcomes w, x, y and z such that $(w,x)\ \Delta_e^{\succsim}\ (y,z)$ and $\{w\} \succ \{x\} \succsim \{y\} \succ \{z\}$. Hence, when certain, these four outcomes are ranked in decreasing order from w to z (with strict preference between the first two outcomes and the last two) and the preference strength for the best w over the second best z has been revealed the same—as per Definition 1—as the the preference strength for y over z. Consider then a decision D with at least two possible outcomes among which are x and y (but not w nor z) and such that the simultaneous replacement of x by w and of y by z would not affect any rank of the outcomes. Observe that the replacement of x by a more favorable w is appealing to the decision maker while the simultaneous replacement of y by the z is detrimental to him/her. However, since $(w,x)\ \Delta_e^{\succsim}$ (y,z), the preference benefit of replacing x by w is exactly the same as the preference cost of replacing y by z. Since x is ranked above y and the rank of the two options is not affected by their respective replacement by w and z, an optimistic agent—who tend to believe that something good will happen—should favour such a simultaneous replacement, while a pessimistic agent should find this very same simultaneous replacement detrimental overall. Hence we find plausible to define formally optimism (pessimism) as follows.

Definition 3 *An ordering $\succsim$ on $\mathcal{P}(X)$ is said to be weakly optimistic if for every four distinct outcomes w, x, y and $z \in X$ such that $(w,x)\ \Delta_e^{\succsim}\ (y,z)$ and*

$\{w\} \succ \{x\} \succsim \{y\} \succ \{z\}$ and every decision $A \in \mathcal{P}(X)$ such that $\{x,y\} \subset A$, $\{w,z\} \cap A = \varnothing, r_x^A = r_w^{(A\backslash\{x,y\})\cup\{w,z\}}$ and $r_y^A = r_z^{(A\backslash\{x,y\})\cup\{w,z\}}$, we have $(A\backslash\{x,y\})\cup\{w,z\} \succsim A$. The ordering is strictly optimistic if the last comparison is strict.

Weak pessimism and strict pessimism are defined similarly, with the last comparison replaced by $\precsim$ (or $\prec$).

We leave to the reader the task of verifying the following implication of this definition of weak optimism/pessimism—that is compatible with the extreme form of these notions provided in Definition 2—for a RDWAU decision maker.

Claim 1 *Let $\succsim$ be a RDWAU ordering of $\mathcal{P}(X)$ that is numerically represented as per (1) for some utility function $u : X \longrightarrow \mathbb{R}$ and some collection of strictly positive weights w_i^n ($n \in \mathbb{N}$ and $i \in [n]$) satisfying $\sum_{i\in[n]} w_i^n = 1$ for any n. Then $\succsim$ is weakly optimistic (pessimistic) if and only if $w_i^n \leq (\geq)\ w_{i+1}^n$ for every $n \in \mathbb{N}\backslash\{1\}$ and $i \in [n-1]$ and is strictly optimistic (pessimistic) if and only the inequality is strict.*

3.2.3 Constant ratio

A very simple family of RDWAU orderings that can exhibit such optimistic or pessimistic feature are those satisfying the highly specific restriction that the weights w_i^n of Expression (1) can be written as:

$$\frac{w_{i+1}^n}{w_i^n} = \rho$$

for any $n \in \mathbb{N}$ and $i \in [n]$ for some strictly positive real number ρ. Let us refer to any RDWAU ordering that satisfies this restriction as an *RDWAU ordering with constant ratio*. For such a class of RDWAU orderings, strict optimism would correspond to the requirement that $\rho > 1$. Conversely, strict pessimism would mean $\rho < 1$. Observe finally that if $\rho = 1$ (a limiting case of both weak optimism and weak pessimism), then the weights are the same for all outcomes and this brings us back to the UEU family characterized in Gravel, Marchant, and Sen (2012).

A simple observation reveals whether the real number ρ is smaller than, equal to, or larger than 1. Suppose indeed a_1, a_2 and a_3 form a standard sequence; if $\{a_1, a_2, a_3\} \precsim \{a_1, a_3\}$, then $\rho \leq 1$ and if $\{a_1, a_2, a_3\} \succsim \{a_1, a_3\}$, then $\rho \geq 1$.

The following condition is necessary and sufficient for an RDWAU ordering to exhibit a constant ratio.

Condition 3 *Rank-dependent preservation of pairwise averages equivalences. Let x_1, x_2, x_3, x_4 be outcomes such that $\{x_1\} \prec \{x_2\} \precsim \{x_3\} \prec \{x_4\}$. Let A be a set such that $A \cap \{x_1, x_2, x_3, x_4\} = \varnothing$ and $r_{x_i}^{A\cup\{x_i\}} = r_{x_j}^{A\cup\{x_j\}}$ for all $i, j \in [4]$. Then*

$$\{x_1, x_4\} \sim \{x_2, x_3\} \iff A \cup \{x_1, x_4\} \sim A \cup \{x_2, x_3\}.$$

Proposition 3 *Let $\succsim$ be a RDWAU ordering of $\mathcal{P}(X)$ as in Theorem 1. It satisfies Condition 3 iff it exhibits constant ratio.*

RDWAU orderings with constant ratio—albeit somewhat restricted—provide examples of RDWAU orderings that can be made either weakly pessimistic or optimistic, but that nonetheless satisfy both Inequalities (6) and (7). Hence, they are neither extremely optimistic nor pessimistic.

3.3 Independence of the axioms

In the following three examples, we prove the independence of the three axioms that characterize RDWAU orderings by exhibiting non-RDWAU orderings that satisfy any two of the three axioms but not the remaining one.

Example 2 *Assume that $X = \mathbb{R}$, and consider the ordering $\succsim^{\text{add}}$ on $\mathcal{P}(X)$ defined by $A \succsim^{\text{add}} B \Longleftrightarrow \sum_{i \in \#A} a_i \geq \sum_{i \in \#B} b_i$. This ordering obviously satisfies Fixed Cardinality Continuity and Consistency in Comparisons of Preference Strength. To see that it violates the Weak Gärdenfors Principle, one can simply observe that, contrary to what this principle would require, $\{1, 2\} \succ^{\text{add}} \{2\}$.*

Example 3 *Assume that $X = \mathbb{R}$, and consider the ordering $\succsim^{\min}$ on $\mathcal{P}(X)$ defined by $A \succsim^{\min} B \Longleftrightarrow a_1 \geq b_1$. This ordering obviously satisfies Fixed Cardinality Continuity and the Weak Gärdenfors Principle. To see that it violates Consistency in Comparisons of Preference Strength, one first observe that $\{2, 5\} \succ^{\min} \{1, 6\}$ and $\{3, 5\} \prec^{\min} \{4, 6\}$, which implies through Definition 1 that $(2, 1)\ \Delta_s^{\succsim^{\min}}\ (3, 4)$. The violation of Consistency in Comparisons of Preference Strength is then established by noticing that $\{0, 2\} \sim^{\min} \{0, 1\}$ and $\{0, 3\} \sim^{\min} \{0, 4\}$ and, therefore, that $(2, 1)\ \Delta_e^{\succsim^{\min}}\ (3, 4)$.*

Example 4 *Assume that $X = \mathbb{R}^2$ and, for any $x \in X$ and $i \in [2]$, let x^i denote the i^{th} component of x. Consider then the lexicographic version of the UEU ordering $\succsim^{\text{lex}}$ on $\mathcal{P}(X)$ defined, for any decisions A and B, by:*

$$A \sim^{\text{lex}} B \Longleftrightarrow \sum_{i \in [\#A]} \frac{a_i^1}{\#A} = \sum_{i \in [\#A]} \frac{b_i^1}{\#A} \text{ and } \sum_{i \in [\#A]} \frac{a_i^2}{\#A} = \sum_{i \in [\#A]} \frac{b_i^2}{\#A}$$

and by $A \succ^{\text{lex}} B$ if either

$$\sum_{i \in [\#A]} \frac{a_i^1}{\#A} > \sum_{i \in [\#A]} \frac{b_i^1}{\#A}$$

or

$$\sum_{i \in [\#A]} \frac{a_i^1}{\#A} = \sum_{i \in [\#A]} \frac{b_i^1}{\#A} \text{ and } \sum_{i \in [\#A]} \frac{a_i^2}{\#A} > \sum_{i \in [\#A]} \frac{b_i^2}{\#A}$$

Hence, the ordering $\succsim^{\text{lex}}$ compares decisions on the basis of a lexicographic combination of the symmetric average of each component of the (two-dimensional)

outcomes of those decisions. It is easy to see that this ordering violates Fixed Cardinality Continuity. Indeed, for any outcome $(a^1, a^2) \in X$, the set of outcomes $(x^1, x^2) \in X$ such that $\{(x^1, x^2)\} \succsim^{\text{lex}} \{(a^1, a^2)\}$ is not closed in X.
To see that $\succsim^{\text{lex}}$ satisfies the Weak Gärdenfors Principle, consider the decision $D = \{d_1, \ldots, d_n\}$ for some $n \in \mathbb{N}$. Since $\{d_1\} \precsim^{\text{lex}} \{d_i\}$ for all $i \in [n]$, we have, for all i, either

$$d_i^1 > d_1^1 \tag{8}$$

or

$$d_i^1 = d_1^1 \text{ and } d_i^2 \geq d_2^2 \tag{9}$$

Summing over n the inequalities or equalities (8) and (9) yields either

$$\sum_{i \in [n]} d_i^1 > n d_1^1 \Longleftrightarrow \frac{\sum_{i \in [n]} d_i^1}{n} > d_1^1 \quad \text{if (8) holds for some } i \text{ or}$$
$$\frac{\sum_{i \in [n]} d_i^1}{n} = d_1^1 \quad \text{and} \quad \frac{\sum_{i \in [n]} d_i^2}{n} \geq d_1^2 \quad \text{if (9) holds for all } i.$$

Hence, one has $\{d_1, \ldots, d_n\} \succsim^{\text{lex}} \{d_1\}$ as required by the Weak Gärdenfors Principle. The conclusion that $\{d_n\} \succsim^{\text{lex}} \{d_1, \ldots, d_n\}$ can be obtained through a similar reasoning.
We now turn to Consistency in Comparisons of Preference Strength. To show that this axiom is satisfied by the ordering $\succsim^{\text{lex}}$suppose by contradiction that it is not. This means that there exist decisions A, A', B and B' in $\mathcal{P}(X)$ satisfying $\#A = \#A' = n$ and $\#B = \#B' = m$ for some m, $n \in \mathbb{N}$ and outcomes x, y, x', $y' \in X$ such that $\{x, y, x', y'\} \cap (A \cup A' \cup B \cup B') = \varnothing$ for which one has:

$$\begin{aligned}
\{a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n\} &\succsim^{\text{lex}} \{a'_1, \ldots, a'_{i-1}, y, a'_{i+1}, \ldots, a'_n\}, & (10)\\
\{a_1, \ldots, a_{i-1}, x', a_{i+1}, \ldots, a_n\} &\precsim^{\text{lex}} \{a'_1, \ldots, a'_{i-1}, y', a'_{i+1}, \ldots, a'_n\}, & (11)\\
\{b_1, \ldots, b_{j-1}, x, b_{j+1}, \ldots, b_m\} &\prec^{\text{lex}} \{b'_1, \ldots, b'_{j-1}, y, b'_{j+1}, \ldots, b'_m\}, & (12)\\
\{b_1, \ldots, b_{j-1}, x', b_{j+1}, \ldots, b_m\} &\succsim^{\text{lex}} \{b'_1, \ldots, b'_{j-1}, y', b'_{j+1}, \ldots, b'_m\}, & (13)
\end{aligned}$$

for some $i \in [n]$ and $j \in [m]$ or, possibly, with the comparison (12) weak and the comparison (13) strict. Yet, we focus on (12) strict and (13) weak in the following sketch. From (10) we conclude:

$$\sum_{h=1}^{i-1} a_h^1 + x^1 + \sum_{h=i+1}^{n} a_h^1 > \sum_{h=1}^{i-1} a_h'^1 + y^1 + \sum_{h=i+1}^{n} a_h'^1 \tag{14}$$

or.

$$\begin{aligned}
\sum_{h=1}^{i-1} a_h^1 + x^1 + \sum_{h=i+1}^{n} a_h^1 &= \sum_{h=1h}^{i-1} a_h'^1 + y^1 + \sum_{h=i+1}^{n} a_h'^1 \text{ and}\\
\sum_{h=1}^{i-1} a_h^2 + x^2 + \sum_{h=i+1}^{n} a_h^2 &\geq \sum_{h=1}^{i-1} a_h'^2 + y^2 + \sum_{h=i+1}^{n} a_h'^2 & (15)
\end{aligned}$$

Similarly, we obtain from (11):

$$\sum_{h=1}^{i-1} a_h^1 + x'^1 + \sum_{h=i+1}^{n} a_h^1 < \sum_{h=1}^{i-1} a_h'^1 + y'^1 + \sum_{h=i+1}^{n} a_h'^1 \tag{16}$$

or.

$$\begin{aligned}\sum_{h=1}^{i-1} a_h^1 + x'^1 + \sum_{h=i+1}^{n} a_h^1 &= \sum_{h=1}^{i-1} a_h'^1 + y'^1 + \sum_{h=i+1}^{n} a_h'^1 \text{ and} \\ \sum_{h=1}^{i-1} a_h^2 + x'^2 + \sum_{h=i+1}^{n} a_h^2 &\leq \sum_{h=1}^{i-1} a_h'^2 + y'^2 + \sum_{h=i+1}^{n} a_h'^2. \end{aligned} \tag{17}$$

Four cases need to be considered.

a. *(14) and (16) imply:* $x^1 - x'^1 > y^1 - y'^1$,

b. *(14) and (17) imply:* $x^1 - x'^1 > y^1 - y'^1$ *and*

$$\sum_{h=1}^{i-1} a_h^2 + x'^2 + \sum_{h=i+1}^{n} a_h^2 \leq \sum_{h=1}^{i-1} a_h'^2 + y'^2 + \sum_{h=i+1}^{n} a_h'^2,$$

c. *(15) and (16) imply:* $x^1 - x'^1 > y^1 - y'^1$ *and*

$$\sum_{h=1}^{i-1} a_h^2 + x^2 + \sum_{h=i+1}^{n} a_h^2 \geq \sum_{h=1}^{i-1} a_h'^2 + y^2 + \sum_{h=i+1}^{n} a_h'^2,$$

d. *(15) and (17) imply:* $x^1 - x'^1 = y^1 - y'^1$ *and* $x^2 - x'^2 \geq y^2 - y'^2$.

Similarly, we can derive from (12) that:

$$\sum_{h=1}^{i-1} b_h^1 + x^1 + \sum_{h=i+1}^{n} b_h^1 < \sum_{h=1}^{i-1} b_h'^1 + y^1 + \sum_{h=i+1}^{n} b_h'^1 \tag{18}$$

or

$$\begin{aligned}\sum_{h=1}^{i-1} b_h^1 + x^1 + \sum_{h=i+1}^{n} b_h^1 &= \sum_{h=1}^{i-1} b_h'^1 + y^1 + \sum_{h=i+1}^{n} b_h'^1 \text{ and} \\ \sum_{h=1}^{i-1} b_h^2 + x^2 + \sum_{h=i+1}^{n} b_h^2 &\leq \sum_{h=1}^{i-1} b_h'^2 + y^2 + \sum_{h=i+1}^{n} b_h^2, \end{aligned} \tag{19}$$

while (13) leads to:

$$\sum_{h=1}^{i-1} b_h^1 + x'^1 + \sum_{h=i+1}^{n} b_h^1 > \sum_{h=1}^{i-1} b_h'^1 + y'^1 + \sum_{h=i+1}^{n} b_h'^1 \tag{20}$$

or

$$\begin{aligned}
\sum_{h=1}^{i-1} b_h^1 + x'^1 + \sum_{h=i+1}^{n} b_h^1 &= \sum_{h=1}^{i-1} b_h'^1 + y'^1 + \sum_{h=i+1}^{n} b_h'^1 \text{ and} \\
\sum_{h=1}^{i-1} b_h^2 + x'^2 + \sum_{h=i+1}^{n} b_h^2 &\geq \sum_{h=1}^{i-1} b_h'^2 + y'^2 + \sum_{h=i+1}^{n} b_h^2. \qquad (21)
\end{aligned}$$

The four implications resulting from all the possible combinations of these expressions are:

A. *(18) and (20) yield* $x^1 - x'^1 < y^1 - y'^1$*,*

B. *(18) and (21) yield* $x^1 - x'^1 < y^1 - y'^1$ *and*

$$\sum_{h=1}^{i-1} b_h^2 + x'^2 + \sum_{h=i+1}^{n} b_h^2 \geq \sum_{h=1}^{i-1} b_h'^2 + y'^2 + \sum_{h=i+1}^{n} b_h^2,$$

C. *(19) and (20) yield* $x^1 - x'^1 < y^1 - y'^1$ *and*

$$\sum_{h=1}^{i-1} b_h^2 + x^2 + \sum_{h=i+1}^{n} b_h^2 \leq \sum_{h=1}^{i-1} b_h'^2 + y^2 + \sum_{h=i+1}^{n} b_h^2,$$

D. *(19) and (21) yield:* $x^1 - x'^1 = y^1 - y'^1$ *and* $x^2 - x'^2 > y^2 - y'^2$*.*

Since any combination of one of the cases (a)–(d) with one the cases (A)–(D) leads to an obvious contradiction, this shows that the ordering $\succsim^{\text{lex}}$ *does indeed satisfy Consistency in Comparisons of Preference Strength.*

4 Conclusion

This paper has axiomatically characterized the rather large family of criteria for decision making under ignorance or objective ambiguity that result from comparing rank-dependent weighted average utilities of the decision, for some utility function and some rank-dependent weighting scheme. It has done so by describing decisions as finite sets of outcomes - that could be either final consequences or lotteries over the same. While the rank-dependent weighted average of utility criteria look somewhat similar to the rank-dependent expected utility criteria à la Quiggin (1993) considered in decision making under risk (when decisions are described as probability distributions) or uncertainty (when decisions are described as functions from a set of states of nature to a set of consequences), they are more general than those because they can not meaningfully be described as resulting from a Choquet capacity. The rank dependent weights are, in this paper, completely arbitrary.

We have also provided some examples of possible additional restrictions that one may want to impose on the weights to make them a bit more structured. But we believe that additional work could be done in this direction. We also believe that the two main axioms—if we leave aside Fixed Cardinality Continuity—used in the characterization are quite easily amenable to experimental testing. It is our hope that future work in the area—including possibly our own—will enable progress in these directions.

A Proofs

A.1 Lemma 1

Assume Consistency in Comparisons of Preference Strength holds and consider two distinct outcomes α and $\beta \in X$, and two decisions D, D' such that $\#D = \#D'$, $(D \cup D') \cap \{\alpha, \beta\} = \varnothing$ and $r_\alpha^{D\cup\{\alpha\}} = r_\beta^{D\cup\{\beta\}} = r_\alpha^{D'\cup\{\alpha\}} = r_\beta^{D'\cup\{\beta\}}$. Assume that $D \cup \{\alpha\} \succsim D' \cup \{\alpha\}$ and, contrary to what Comonotonic Independence requires, that $D \cup \{\beta\} \succsim D' \cup \{\beta\}$ does not hold. Since $\succsim$ is complete, we must have $D \cup \{\beta\} \prec D' \cup \{\beta\}$. By Definition 1, we thus have $(\alpha, \alpha)\ \Delta_s^{\succsim}\ (\beta, \beta)$ and $(\beta, \beta)\ \Delta_s^{\succsim}\ (\alpha, \alpha)$. This contradicts Consistency in Comparisons of Preference Strength and proves that Comonotonic Independence holds. $\square$

A.2 Proposition 1

The result being true for $n = 1$ by Debreu (1954) theorem (any continuous ordering on a topological space can be numerically represented by a utility function), consider any integer $n \geq 2$. Any decision D with $n \geq 2$ ordered elements can be represented as an ordered vector in X^n. The set of all such vectors is a subset of X^n, denoted by $\mathcal{O}^n(X)$. Let us consider n disjoint connected subsets $\{Y_1, \ldots, Y_n\}$ of X such that, for all $i \in [n-1]$ and for all $x \in Y_i, y \in Y_{i+1}$, we have $x \precsim y$. By construction, the Cartesian product $\Pi_{i=1}^n Y_i$ is a subset of $\mathcal{O}^n(X)$. The restriction of $\succsim$ to $\Pi_{i=1}^n Y_i$ satisfies Consistency in Comparisons of Preference Strength, Essentialness and Continuity. We can therefore apply Theorem 3.2 in Wakker (1993)—combined with the proof provided in Wakker (1989) that Consistency in Comparisons of Utility Differences is equivalent to a triple cancellation property—and conclude in the existence of n continuous mappings $\{u_i^n\}_{i\in[n]}$ such that, for all $A, B \in \Pi_{i\in[n]} Y_i$:

$$A \succsim B \iff \sum_{i\in[n]}^{n} u_i^n(a_i) \geq \sum_{i\in[n]}^{n} u_i^n(b_i). \tag{22}$$

The set $\mathcal{O}^n(X)$ is the union of infinitely many Cartesian products of the form $\Pi_{i\in[n]} Y_i$. The set $\mathcal{O}^n(X)$ satisfies Assumption 2.1 in Chateauneuf and Wakker (1993) and there is therefore a continuous additive representation of $\succsim$ (restricted to sets of cardinality n). That is to say, Expression (22) provides a numerical representation of the ordering $\succsim$ not only on $\Pi_{i\in[n]} Y_i$ but, also, on the whole set $\mathcal{O}^n(X)$.

For any rank $j \in [2, n]$, let A and B be any two sets of cardinality $n-1$ and x_1, x_2 and x_3 be outcomes in X not contained in either A or B such that $\{x_1\} \precsim \{x_2\} \precsim \{x_3\}$ and $r_{x_i}^{A\cup\{x_i\}} = r_{x_{i+1}}^{A\cup\{x_{i+1}\}} = r_{x_i}^{B\cup\{x_i\}} = r_{x_{i+1}}^{B\cup\{x_{i+1}\}} = j$ for $i \in [2]$. Choose also such outcomes in such a way that $A \cup \{x_i\} \sim B \cup \{x_{i+1}\}$ for $i \in [2]$. The existence of these three outcomes and two sets A and B having those features is secured by the continuity of the representation in (22) and the connectedness of X. Hence, using (22), we can write $A \cup \{x_i\} \sim B \cup \{x_{i+1}\}$ for $i \in [2]$ as

$$\sum_{h\in[j-1]} u_h^n(a_h) + u_j^n(x_1) + \sum_{h\in[j+1,n]} u_h^n(a_{h-1}) = \sum_{h\in[j-1]} u_h^n(b_h) + u_j^n(x_2) + \sum_{h\in[j+1,n]} u_h^n(b_{h-1}) \quad (23)$$

and

$$\sum_{h\in[j-1]} u_h^n(a_h) + u_j^n(x_2) + \sum_{h\in[j+1,n]} u_h^n(a_{h-1}) = \sum_{h\in[j-1]} u_h^n(b_i) + u_j^n(x_3) + \sum_{h\in[j+1,n]} u_h^n(b_{h-1}). \quad (24)$$

Subtracting (24) from (23) yields $u_j^n(x_2) - u_j^n(x_1) = u_j^n(x_3) - u_j^n(x_2)$. Let now C and D be sets of cardinality $n-1$ not containing x_1, x_2 and x_3 such that the rank of x_1, x_2 and x_3 in $\{x_i\} \cup C$ and in $\{x_i\} \cup D$ (for $i \in [3]$) is 1. Again, the existence of these two sets C and D having those features is secured by the continuity of the representation in (22) and the connectedness of the set X. We observe that, thanks to Consistency in Comparisons Preference Strength, one must have $C \cup \{x_i\} \sim D \cup \{x_{i+1}\}$ for $i \in [2]$. The same reasoning as above therefore yields $u_1^n(x_2) - u_1^n(x_1) = u_1^n(x_3) - u_1^n(x_2)$. In other words, the images of x_1, x_2, x_3 under u_j^n are equally spaced and so are they in u_1^n. Let us say that x_1, x_2 and x_3 form a grid in X with a mesh of size 1. We necessarily have that $u_j^n = \alpha_j + \beta_j u_1^n$ for some real numbers α_j, $\beta_j (\beta_j > 0)$.

By continuity and connectedness, there is $x_{1\circ 2}$ 'halfway' between x_1 and x_2. More formally, there is $x_{1\circ 2}$ such that $u_j^n(x_{1\circ 2}) - u_j^n(x_1) = u_j^n(x_2) - u_j^n(x_{1\circ 2})$. There is also $x_{2\circ 3}$ such that $u_j^n(x_{2\circ 3}) - u_j^n(x_2) = u_j^n(x_3) - u_j^n(x_{2\circ 3})$. The images of $x_1, x_{1\circ 2}, x_2, x_{2\circ 3}$ and x_3 under u_j^n are thus equally spaced and so are they under u_1^n. The outcomes $x_1, x_{1\circ 2}, x_2, x_{2\circ 3}$ and x_3 thus form a grid in X with mesh of size 1/2. We can again halve the mesh of this grid by adding the outcomes $x_{1\circ(1\circ 2)}, x_{(1\circ 2)\circ 2}, x_{2\circ(2\circ 3)}$ and $x_{(2\circ 3)\circ 3}$ to the grid. And we can make the mesh as fine as we want by repeating this process.

Let us denote the outcomes of the initial grid (mesh size = 1), by $g_1^1 = x_1, g_2^1 = x_2$ and $g_3^1 = x_3$. Similarly, we denote the elements of the second grid (mesh size = 1/2) by $g_1^2 = x_1, g_2^2 = x_{1\circ 2}, g_3^2 = x_2, g_4^2 = x_{2\circ 3}$ and $g_5^2 = x_3$, those of the third grid (mesh size = $1/2^2, g_1^3 = x_1, g_2^3 = x_{1\circ(1\circ 2)}, g_3^3 = x_{1\circ 2}, \ldots, g_8^3 = x_{(2\circ 3)\circ 3}$ and $g_9^3 = x_3$ and so on.

If x_1 is not minimal for $\succsim$ restricted to singletons, we can try to extend the grid to 'the left' of $x_1 = g_1^1$, by looking for an element g_0^1 such that $u_j^n(g_1^1) - u_j^n(g_0^1) = u_j^n(g_2^1) - u_j^n(g_1^1)$. If such a g_0^1 does not exist, then there exists a mesh size s such that there is g_0^s satisfying $u_j^n(g_1^s) - u_j^n(g_0^s) = u_j^n(g_2^s) - u_j^n(g_1^s)$.

If g_0^s is not minimal for $\succsim$, we can again extend the grid to 'the left' of g_0^s (this may require using a finer mesh). By repeating this process, we can extend the grid to the 'left' of x_1 and go as close as we wish to the minimal decisions for $\succsim$ (if any). We can also extend he grid to the 'right' of x_3 and go as close as we wish to the maximal singleton decision for $\succsim$ (if any). Since the images of all elements of a grid are equally spaced in $\mathbb{R}$ under u_j^n and u_1^n, we necessarily have that $u_j^n(x) = \alpha_j + \beta_j u_1^n(x)$ for some real numbers $\alpha_j, \beta_j (\beta_j > 0)$ and for any element x of a grid of any mesh size.

Consider now an element x that does not belong to any grid. We have just seen above that we can refine or extend the initial grid in order to be as close as we wish to x. Continuity of u_j^n and u_1^n then imply that $u_j^n(x) = \alpha_j + \beta_j u_1^n(x)$. This holds for all $x \in X$. The reasoning just made is valid for any rank $j \in [2, n]$. Hence we can write Equivalence (5) as:

$$A \succsim B \iff \sum_{i \in [n]} \beta_i u_1^n(a_i) \geq \sum_{i \in [n]} \beta_i u_1^n(b_i).$$

or, after defining $w_i^n = \beta_i / \sum_{i=1}^n \beta_i$ and $u^n = u_1^n$, as:

$$A \succsim B \iff \sum_{i \in [n]} w_i^n u^n(a_i) \geq \sum_{i \in [n]} w_i^n u^n(b_i).$$

and this completes the proof. $\square$

A.3 Lemma 2

Let x and y be two outcomes in X such that $\{x\} \succsim \{y\}$. For any $n \geq 2$, consider a decision D such that $\#D = n-1, \{x, y\} \cap D = \varnothing$ and $r_x^{D \cup \{x\}} = r_y^{D \cup \{y\}} = i$ for some $i \in [n]$. Again, the existence of such a decision for any given outcomes x and y such that $\{x\} \succsim \{y\}$ guaranteed by the connectedness of X and the essentialness condition. By consistency in Comparisons of Preference Strength, one must have $D \cup \{x\} \succsim D \cup \{y\}$. Thanks to Proposition 1, this latter statement can equivalently be written as

$$\sum_{h \in [i-1]} w_h^n u^n(d_h) + w_i^n u^n(x) + \sum_{h \in [i+1, n]} w_h^n u^n(d_{h-1})$$
$$\geq \sum_{h \in [i-1]} w_h^n u^n(d_h) + w_i^n u^n(y) + \sum_{h \in [i+1, n]} w_h^n u^n(d_{h-1}).$$

Since $w_i^n > 0$, this is equivalent to $u^n(x) \geq u^n(y)$.

Using a similar reasoning and the completeness of $\succsim$, one would obtain the strict inequality $u^n(y) > u^n(x)$ if one had assumed $\{x\} \prec \{y\}$ instead. Hence $\{x\} \succsim \{y\}$ if and only if $u^n(x) \geq u^n(y)$ for any $n \geq 2$. □

A.4 Lemma 3

Thanks to Proposition 1, we know that, for any $n \in \mathbb{N}$, there exist weights w_i^n as in (1) and $u^n : X \to \mathbb{R}$ such that (5) holds. Suppose $(a_i)_{i\in[p]}$ is some standard sequence for the sets $\{b\}$ and $\{c\}$ such that $\{b\} \prec \{c\} \prec \{a_i\}$ for all $i \in [p]$. By Consistency in Comparisons of Preference Strength, we must have $\{b, a_i\} \sim \{c, a_{i-1}\}$ and $\{b, a_{i+1}\} \sim \{c, a_i\}$ for all $i \in [2, p-1]$ and, applying the numerical representation of Proposition 1 with $n = 2$,

$$u^2(a_i) - u^2(a_{i-1}) = u^2(a_{i+1}) - u^2(a_i) \tag{25}$$

for all those i. Consider now any $n \in [3, \cdot]$. One can find, thanks to continuity and connectedness of X, some sets C and D of cardinality $n-1$ such that $\{a_1, \ldots, a_p\} \cap (C \cup D) = \varnothing$ and, for all $i \in [p-1]$, $r_{a_i}^{C\cup\{a_i\}} = r_{a_{i+1}}^{D\cup\{a_{i+1}\}}$. Thanks to Consistency in Comparisons of Preference Strength, one will have $C \cup \{a_i\} \sim D \cup \{a_{i-1}\}$ and $C \cup \{a_{i+1}\} \sim D \cup \{a_i\}$ for all $i \in [2, p-1]$ and, thanks to (5),

$$u^n(a_i) - u^n(a_{i-1}) = u^n(a_{i+1}) - u^n(a_i). \tag{26}$$

Thanks to the connectedness of X and continuity of $\succsim$ when restricted to singletons, we can choose the standard sequence $(a_i)_{i\in[p]}$ as fine as we want following the procedure described at the end of the proof of Proposition 1. Hence, the comparison of (25) and (26) shows that $u^2 = \gamma^n + \lambda^n u^n$ for all $n \in [3, \cdot]$. We have seen in Proposition 1 that u^n is unique up to any positive affine transformation. Since u^2 is a positive affine transformation of u^n, we can use u^2 instead of u^n in (5). We can also use u^2 instead of u^1 because, as shown in Lemma 2, any u^1 (up to a strictly increasing transformation) will do. Since we can use u^2 everywhere instead of u^n for any n (including $n = 1$), we can just define $u = u^2$ and this completes the proof. □

A.5 Lemma 4

The result being trivially true for a decision D with a single outcome, consider a decision D with $n \geq 2$ outcomes. Let us write $D = \{d_1, \ldots, d_n\}$. We define $\mathrm{CE}(D)$ to be any element of the set $u^{-1}(\sum_{i\in[n]} w_i^n u(d_i))$ where the continuous function u and the weights w_i^n are those that define the numerical representation constructed in Lemma 3, which is also a numerical representation of the restriction of the ordering $\succsim$ to singletons thanks to Lemma 2. We need to show that $u^{-1}(\sum_{i\in[n]} w_i^n u(d_i)) \neq \varnothing$ and, also, that $\{\mathrm{CE}(D)\} \sim D$.

The proof of the non-emptiness of $u^{-1}(\sum_{i\in[n]} w_i^n u(d_i))$ (and of the fact that $\{\mathrm{CE}(D)\} \sim D$) being immediate if $u(d_i) = u(d_{i+1})$ for all $i \in [n-1]$, we consider the case where $u(d_n) > u(d_1)$ and, by Lemma 2, where $\{d_n\} \succ \{d_1\}$. We observe that since $w_i^n > 0$ (by essentialness) and $u(d_n) > u(d_1)$ by assumption, one must have $u(d_n) > \sum_{i\in[n]} w_i^n u(d_i) > u(d_1)$. The continuity of u on its connected domain X implies u^{-1} is defined for any real number in the non-degenerate interval $[u(d_1), u(d_n)]$ and thus, in particular, for the real number $\sum_{i\in[n]} w_i^n u(d_i)$. Hence CE$(D)$ exists.

Suppose now that CE$(D) \in X$ is such that $\{\mathrm{CE}(D)\} \nsim D$. Let us specifically assume that $\{\mathrm{CE}(D)\} \succ D$. Let $i \in [n-1]$ be defined by $u(d_i) \leq u(\mathrm{CE}(D))$ and $u(d_{i+1}) > u(\mathrm{CE}(D))$. Consider then the sequence of decisions D^t (for $t \in \mathbb{N}$) defined by

$$D^t = \{\mathrm{CE}(D)^{-\varepsilon_1^t}, \ldots, \mathrm{CE}(D)^{-\varepsilon_i^t}, \mathrm{CE}(D)^{+\varepsilon_{i+1}^t}, \ldots, \mathrm{CE}(D)^{+\varepsilon_n^t}\}$$

for some list of n outcomes $\mathrm{CE}(D)^{+\varepsilon_h^t}$ and $\mathrm{CE}(D)^{-\varepsilon_h^t}$ such that $u(\mathrm{CE}(D)^{-\varepsilon_h^t}) = u(\mathrm{CE}(D) - \varepsilon_h^t)$, for $h \in [i]$, and $u(\mathrm{CE}(D)^{+\varepsilon_h^t}) = u(\mathrm{CE}(D) + \varepsilon_h^t)$, for $h \in [i+1, n]$ for some sequence of suitably small positive real numbers ε_h^t (for $h \in [n]$) such that $\varepsilon_1^t > \varepsilon_2^t > \ldots > \varepsilon_i^t$, $\varepsilon_n^t > \varepsilon_{n-1}^t > \ldots > \varepsilon_{i+1}^t$, $\lim_{t\to\infty} \varepsilon_j^t = 0$ for all $j \in [n]$ and

$$\sum_{h\in[i]} w_h^n \varepsilon_h^t = \sum_{h\in[i+1,n]} w_h^n \varepsilon_h^t.$$

It is clear here again that the existence of these $\mathrm{CE}(D)^{+\varepsilon_h^t}$ and $\mathrm{CE}(D)^{-\varepsilon_h^t}$ is secured by the continuity of u, the connectedness of X and the fact that $u(d_n) > \sum_{h\in[n]} w_h^n u(d_h) > u(d_1)$. We then have

$$\sum_{h\in[i]} w_h^n u(\mathrm{CE}(D)^{-\varepsilon_h^t}) + \sum_{h\in[n]:h>i} w_h^n u(\mathrm{CE}(D)^{+\varepsilon_h^t}) = u(\mathrm{CE}(D)) = \sum_{h\in[n]} w_h^n u(d_h). \quad (27)$$

By Proposition 1, (27)) implies $D \sim D^t$ for all t. By transitivity, we therefore have $\{\mathrm{CE}(D)\} \succ D^t$ for all t. We observe also that the Weak Gärdenfors Principle implies that

$$\{\mathrm{CE}(D)^{+\varepsilon_n^t}\} \succsim D^t \succsim \{\mathrm{CE}(D)^{-\varepsilon_1^t}\} \quad (28)$$

while the fact—established in Lemma 2—that u numerically represents the ordering $\succsim$ restricted to singletons implies that

$$\{\mathrm{CE}(D)^{+\varepsilon_n^t}\} \succ \{\mathrm{CE}(D)\} \succ \{\mathrm{CE}(D)^{-\varepsilon_1^t}\}. \quad (29)$$

Since $\{\mathrm{CE}(D)\} \succ D^t$ for all t, it follows from (28) and (29) and transitivity that

$$\{\mathrm{CE}(D)^{+\varepsilon_n^t}\} \succ \{\mathrm{CE}(D)\} \succ D^t \succsim \{\mathrm{CE}(D)^{-\varepsilon_1^t}\}.$$

Yet both sequences of singletons $\{\mathrm{CE}(D)^{+\varepsilon_n^t}\}$ and $\{\mathrm{CE}(D)^{-\varepsilon_1^t}\}$ converge to $\{\mathrm{CE}(D)\}$. Hence, having $\{\mathrm{CE}(D)^{+\varepsilon_n^t}\} \succ \{\mathrm{CE}(D)^{-\varepsilon_1^t}\}$ for all t holding at the limit is incompatible with the continuity of $\succsim$. The argument for the case where $D \succ \{\mathrm{CE}(D)\}$ is similar. □

A.6 Remaining of Theorem 1

We know from Proposition 1 and Lemma 3 that for any $n \in \mathbb{N}$, there is a continuous utility function $u : X \to \mathbb{R}$ and a set of weights w_i^n (for $i \in [n]$) for which (1) holds for any decisions D and D' having both n possible outcomes. For any decision D, let $U_D = \sum_{i \in [\#D]} w_i^{\#D} u(d_i)$. Consider two decisions $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ for some $n, m \in \mathbb{N}_{++}$ with $n \neq m$ such that $A \succsim B$. We need to show that $U_A \geq U_B$. By contradiction, suppose that $U_B > U_A$. Choose any suitably small strictly positive real number ε. Using Lemma 4, the continuity of u and the connectedness of X, define the sets C_n^j and C_m^j by

$$C_n^j = \{c_1^j, \ldots, c_n^j\}$$

and

$$C_m^j = \{c_1^j, \ldots, c_m^j\}$$

where, for any $i \in [m] \cup [n]$ and $j \in \mathbb{N}, c_i^j \in X$ is such that

$$c_i^j = u^{-1}\left(U_A + \frac{\varepsilon}{(\max(m,n) + 1 - i) \times j}\right).$$

By the Weak Gärdenfors Principle, we have that $\{c_n^j\} \succsim C_n^j$. By the numerical representation, $C_n^j \succ A$. We observe also that, for sufficiently large j, $U_{C_m^j}$ can be made arbitrarily close to U_A. Take therefore a fixed j, say $\bar{\jmath}$, sufficiently large for the assumption $U_B > U_A$ to imply $U_B > U_{C_m^{\bar{\jmath}}} > U_A$. Since the numerical representation holds for sets of cardinality m, we must therefore have $B \succ C_m^{\bar{\jmath}}$. By the Weak Gärdenfors Principle, $C_m^{\bar{\jmath}} \succsim \{c_1^{\bar{\jmath}}\}$ and, by the numerical representation applied to singletons, $\{c_1^{\bar{\jmath}}\} \succ \{c_1^j\}$ for all $j > \bar{\jmath}$. Thanks to Lemma 4 and transitivity, we have

$$\{c_n^j\} \succsim C_n^j \succ A \sim \{\mathrm{CE}(A)\} \succsim B \succ C_m^{\bar{\jmath}} \succsim \{c_1^{\bar{\jmath}}\} \succ \{c_1^j\},$$

for all $j > \bar{\jmath}$. But having $\{c_n^j\} \succ \{\mathrm{CE}(A)\} \succ \{c_1^{\bar{\jmath}}\} \succ \{c_1^j\}$ for all $j > \bar{\jmath}$ contradicts Fixed Cardinality Continuity (applied to singletons), since both sequences $\{c_n^j\}_{j \in \mathbb{N}}$ and $\{c_1^j\}_{j \in \mathbb{N}}$ converge to $u^{-1}(U_A) = u^{-1}(\mathrm{CE}(A))$. The converse implication that $U_A \geq U_B$ implies $A \succsim B$ is proved in the same way. $\square$

A.7 Proposition 3

Necessity. Let x_1, x_2, x_3, x_4, A be as in the premise of Condition 3. Let $\#A = n$ and $r_{x_1}^{A \cup \{x_1\}} = k$. Then $\{x_1, x_4\} \sim \{x_2, x_3\}$ imply

$$w_1^2 u(x_1) + w_2^2 u(x_4) = w_1^2 u(x_2) + w_2^2 u(x_3).$$

Obvious simplifications yield

$$\frac{w_2^2}{w_1^2} = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}.$$

Assuming a constant ratio ordering, we have

$$\frac{w^n_{k+1}}{w^n_k} = \rho = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}$$

and

$$\sum_{h\in[k-1]} w^n_h u(a_h) + w^n_k u(x_1) + w^n_{k+1} u(x_4) + \sum_{h\in[k,n]} w^n_{h+2} u(a_h)$$
$$= \sum_{h\in[k-1]} w^n_h u(a_h) + w^n_k u(x_2) + w^n_{k+1} u(x_3) + \sum_{h\in[k,n]} w^n_{h+2} u(a_h).$$

Hence $A \cup \{x_1, x_4\} \sim A \cup \{x_2, x_3\}$ and necessity is proved.

Sufficiency. Suppose $\succsim$ is a RDWAU ordering as in Theorem 1, with utility function u and weights w^n_i. Consider any four outcomes x_1, x_2, x_3, x_4 such that $\{x_1\} \prec \{x_2\} \precsim \{x_3\} \prec \{x_4\}$, $\{x_1, x_4\} \sim \{x_2, x_3\}$ and $\{x\} \prec \{x_1\}$, $\{x_4\} \prec \{y\}$ for some $x, y \in X$. The existence of such outcomes is secured by the Essentialness condition (applied to decisions made of two alternatives) and the continuity of the RDWAU ordering (combined with the connectedness of X for the order topology of $\succsim$ restricted to singletons).

Consider any $n \in [2, \cdot]$ and $k \in [n-1]$. By continuity of $\succsim$ and connectedness of X again, there are $n-2$ outcomes $a_1, \ldots, a_{n-2}$ such that $a_1 \prec \ldots \prec a_{k-1} \prec x_1 \prec x_4 \prec a_{k+2} \prec \ldots \prec a_n$. If we define $A = \{a_1, \ldots, a_{n-2}\}$, then Condition 3 implies $A \cup \{x_1, x_4\} \sim A \cup \{x_2, x_3\}$. Using the numerical representation of Theorem 1, we find

$$\frac{w^n_{k+1}}{w^n_k} = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}.$$

Since this holds for all $n \in [2, \cdot]$ and $k \in [n-1]$, the proof is complete if we define

$$\rho = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}.$$

□

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