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Matching with Recall

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Abstract

We study a two-period one-to-one dynamic matching environment in which agents meet randomly and decide whether to match early or defer. Crucially, agents can match with either partner in the second period. This "recall" captures situations where, e.g., a firm and worker can conduct additional interviews before contracting. Recall has a profound impact on incentives and on aggregate outcomes. We show that the likelihood to match early is non-monotonic in type: early matches occur between the good-but-not-best agents. The option value provided by the first-period partner provides a force against unraveling, so that deferrals occur under small participation costs.

Keywords: Dynamic matching, unraveling, recall.

JEL Classification: C78, D47, D82, D83.

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1 Introduction

Many decentralized matching markets are inherently dynamic and non-stationary. In labor markets for entry-level professionals, workers and employers typically search for matches during a well-defined window, conducting interviews sequentially. Some participants exit the market early, while others spend more time searching. The pool of participants thus evolves over time. In dynamic matching frameworks, a standard simplifying assumption is that two agents who meet but do not immediately match cannot match at a later date. Yet in practice, to the contrary, a worker and an employer may meet and then each conduct additional interviews before later deciding to match. How do matching outcomes change if agents can keep track of previous meetings?

We provide the first analysis of *recall* in a non-stationary matching environment. The analysis is built on the two-period, two-sided matching model studied by Damiano, Li and Suen (2005), into which we incorporate recall and study its effects (see Section 2). Agents have heterogeneous types and prefer to be matched with a partner of higher type. Agents are randomly paired at the first period, observe their partner's type, and decide whether to match and exit, by mutual consent, or to defer. Unmatched agents are again randomly paired in the second period, after which they have a final opportunity to form matches by mutual consent. Agents in the second period each have two potential partners to match with. With common preferences, the stable pairing is generically unique (see Lemma 1).[1]

We identify key intuitive implications of the fact that recall generates two feasible pairings for agents in the second period. First, higher-type agents expect to obtain a better partner in the second period, since they are more likely to be able to match with the better of their two options. This implies that higher types are more selective in early matches. Second, some agents remain unmatched in equilibrium. Notice that both of an agent's options in the second period may mutually prefer their other option

[1]For example, consider the highest-type agent: they must match with their preferred partner in any stable matching, so this pair is matched and removed, and the process repeats with the remaining agent of highest type, and so on.

to the given agent. Third, the highest-type agents almost never match early since, for them, this risk is negligible. In fact, they are almost sure to match with the better of their two partners in the second period, so that waiting can only benefit them. Fourth, even low-type agents can get lucky in the second period, as is the case when they meet a better partner in the second period, who loses their first-period partner to a blocking pair. We show that this implies that low-type agents are sufficiently picky in the first period that they never match early.

Our first set of results characterize equilibrium strategies (see Section 3). Since stability pins down the second period outcome, we focus on matching decisions in the first period. Lemma 2 shows that best responses are essentially unique and take the form of a threshold function, meaning that if an agent is willing to match early with a type v, then they are willing to match with any other type $v' > v$. While best responses have this simple structure, equilibrium analysis is complicated by the necessity of finding a fixed point in an appropriate function space. Theorem 1 proves the existence of an equilibrium with deferrals in symmetric threshold strategies, when the pool of agents is sufficiently large. The argument is complex, and is contained in the Appendix. Theorem 2 shows that this equilibrium displays the intuitions described above.

Notice that, in equilibrium, the set of agents who match early contains neither low-type agents nor the highest-type agents. It is the good-but-not-best agents who match early and, as a result, early matches are highly assortative. The (endogenous) pool of types in the second period displays a non-monotonicity, as the good types are under-represented relative to the low types and the highest types. Section 4 is devoted to a thorough analysis of the implications of this finding. We contrast our findings with the results from Damiano, Li and Suen (2005) in order to understand the impact of recall on matching outcomes. The qualitative features of our equilibrium are quite distinct from the no-recall case: if agents can remember their past partners, the matching patterns are fundamentally altered.

Further, we explore the welfare implications of recall. We identify countervailing effects and quantify them through large-scale numerical simulations.[2] In the no-recall benchmark, two agents who meet in the second period always match and hence late matches are conditionally random and every agent obtains a partner. By contrast, under recall second-period matches are assortative and leave some agents isolated. In addition, we find that there are fewer early matches under recall than under no recall, and those that do occur are more highly assortative. Overall, welfare is substantially higher under recall, showing that the benefits of increased assortativity outweigh the costs of leaving agents unmatched.

We also look at the counterfactual impact of prohibiting early matches in our framework, forcing all agents to meet a second partner before matching. Even though this change (weakly) expands feasible partnerships for every market participant, we find that it decreases welfare. This is explained by two related reasons. First, eliminating early matches increases the number of unmatched agents. Second, some first-period meetings are very likely to end up in matches. Including these agents in the second period imposes a negative externality on others by effectively limiting their opportunities to meet viable partners. The early matches that arise endogenously under recall are thus welfare-improving.

Finally, observe that under recall, the first-period meeting confers option value in the second period matching. In contrast, without recall, the first period partner is forgotten and there is no option value. The option value under recall is non-trivial, even for the lowest type agents who know that they will not match in the first period. This fact has an important implication regarding the possibility for the market to unravel. Damiano, Li and Suen (2005) show that the introduction of a small participation cost causes the market to unravel. The key mechanism is that, without recall, an agent who knows they will not match in the first period is unwilling to participate at any positive cost, and so all low types abstain from early participation. But with recall, the option value provides sufficient incentive to all agents to continue participating. In Section 5, we show that our equilibrium survives

[2] We run these simulations under the assumptions that types are uniformly distributed over $[0, 1]$ and that the utility from a match is equal to the product of the partners' types.

the introduction of a small participation cost, preventing the large drop in welfare associated with unraveling in the no-recall case.

Our analysis contributes, first, to the literature on dynamic, non-stationary matching. In an important paper, Damiano, Li and Suen (2005) provide one of the first analyses of matching in a non-stationary environment. They study a two-period, two-sided matching model under no recall, non-transferable utilities, and when agents prefer to be matched with higher-types partners. In this framework, agents who meet in the second period always match. This leads to a type-independent threshold strategy in the first period: an agent wants to match with its first-period partner when the partner's type is larger than the expected type in the second period, producing a uniform threshold. The equilibrium is then characterized by a simple scalar fixed point equation. The likelihood to match early decreases weakly with type and the equilibrium with deferral unravels with a small participation cost. We introduce recall into this framework and show that it has a profound impact. As discussed above, matchings in the second period can, and do, happen between agents who met in the first period. In equilibrium, higher types are more selective in the first period, leading to type-dependent threshold function. This equilibrium is then characterized by a functional fixed point equation. The likelihood to match early is non-monotonic with type and the equilibrium with deferral is robust to small participation costs.

Other studies analyze early matching and unravelling in similar contexts.[3] Li and Suen (2000) and Li and Rosen (1998) assume that types are initially uncertain and consider risk averse preferences with transfers. In their setup, early contracting provides insurance and agents who appear most promising may match early. Echenique and Pereyra (2016) consider non-transferable utilities and assume that types are initially unknown. They show that an early offer by one agent induces a negative externality on the rest of the market, and that this yields strategic unraveling in equilibrium. Du and Livne (2016) assume that some agents enter only in the second

[3] Roth and Xing (1994) document how many markets suffer from unravelling and discuss possible causes. Contributions on unravelling include Fainmesser (2013), Kagel and Roth (2000), Halaburda (2010), Ostrovsky and Schwarz (2010), Niederle and Roth (2009), Roth (1991), Sönmez (1999).

period. They look at the impact of transfers on market outcomes and find that with non-transferable utilities, some agents may want to match early and sequentially stable equilibria may fail to exist.

These studies assume that in the second period, agents know the types of every other agent present in the market and can match with every market participant.[4] By contrast, we consider a fully decentralized framework. As in Damiano, Li and Suen (2005), agents must first meet to learn each other's types and to be able to match. In our setup, market participants are risk-neutral, know their own types, and are present in the first period

Search and matching is a distinct strand of the literature, building on Mortensen (1982), Mortensen (1988), Diamond (1982), and Pissarides (2000). Several papers develop a stationary search and matching analysis of models with non-transferrable utilities and preferences to be matched with higher types, see e.g. Burdett and Coles (1997), Adachi (2003), Smith (2006). These papers generally characterize stationary equilibrium strategies, including the finding of block segregation. In these frameworks, however, recall has no impact: an agent who is unacceptable today remains unacceptable in the future.

A small but interesting literature looks at recall in a stationary search and matching environment with transferable utilities. Carrillo-Tudela, Menzio and Smith (2011) consider a model of job search where unemployed workers meet firms at random and firms make take-it-or-leave-it wage offers. Workers keep track of previous encounters with potential employers and induce bidding wars between them. They show that recall allows workers to obtain higher wages, providing a possible resolution of Diamond's paradox. We show that the implications of recall are very different, however, in a non-stationary framework. In the stationary environment of Carrillo-Tudela, Menzio and Smith (2011), workers always match with their current partner on the equilibrium path. In addition, firms' behavior does not depend on

[4] In particular, this means that matching in the second period is perfectly assortative in Echenique and Pereyra (2016) and Du and Livne (2016). Matching is generally not perfectly assortative in our framework as agents can only match if they have previously met.

their own previous encounters. Recall mainly helps workers capture a larger share of the gains from trade. By contrast in our framework, agents do, in fact, match with previously met partners. Matching prospects depend on partners' partners and, more generally, on the entire network of potential partnerships. Recall in our setup thus fundamentally alters who matches with whom and the timing of the equilibrium matches.[5]

Recall is a standard assumption in the literature on consumer search, see e.g. Wolinsky (1986), Armstrong (2017), and Choi, Dai and Kim (2018). In this literature, consumers search sequentially for products and then buy the searched product which brings highest payoff. There are at least two main differences with our framework. First, conditional on prices, decisions of a consumer are not affected by the decisions of other consumers. Decisions of when to stop searching and which product to buy are single decision maker problems. By contrast, the incentives of an agent in our framework depend on what others do, leading to much more complex strategic interactions. Second, a consumer can always buy a product they searched in the past. There is no risk that the product disappears if not bought right away. By contrast, this risk plays a key role in our framework. An agent is not guaranteed she will be able to match later with a partner met today.[6]

2 Model

In this Section, we describe a dynamic model of decentralized matching with recall. Agents meet potential partners randomly over two periods. "Recall" means that agents keep track of previous meetings and can later match with those partners. At the end of the second period, an agent may feasibly match with either of the

[5]Carrillo-Tudela and Smith (2017) introduce on the job search into the stationary framework of Carrillo-Tudela, Menzio and Smith (2011). With on the job search and recall, some agents who lose their jobs can find a new job right away and this affects aggregate output.

[6]Interestingly, there are contexts where this risk may appear in consumer search. Think, for instance, about sellers having limited quantities of products to sell. Such capacity constraints would introduce features related to our analysis: a consumer who does not buy a product right away would risk losing her chance of purchasing later, in ways that depend on others' purchase decisions.

two partners she has met. Our main objective is to understand the equilibrium properties of this matching process, with a particular emphasis on identifying the impact of recall on matching outcomes.

Our framework is most closely related to the model of dynamic, decentralized matching of Damiano, Li and Suen (2005), with the primary distinction being our inclusion of recall. Matching is two-sided and one-to-one. For clarity, we adopt the terminology of marriage markets, with women and men, throughout the paper, but our framework applies more generally to buyers and sellers, workers and firms, colleges and students, etc. Let $\mathcal{W}$ denote the set of women, $\mathcal{M}$ the set of men, with $\mathcal{N} = \mathcal{W} \cup \mathcal{M}$ the set of all agents. We assume that there are as many women as men, and the total number of agents is a finite $N \geq 6$ with $\#\mathcal{W} = \#\mathcal{M} = \frac{N}{2}$. Each woman and man has a type, drawn independently and identically from a probability distribution H. We assume that the support of H is an interval $[a, b]$ and that H admits a strictly positive density h. Let $w_i \in [a, b]$ denote a realized type of a woman i and m_j a realized type of a man j. We omit the subscripts when there is no ambiguity.

Agents care only about the type of their matched partner. Let $u_i(m|w)$ denote the utility that woman i of type w earns from matching with a man of type m and $u_j(w|m)$ the utility that man j of type m earns from matching with a woman of type w. Denote by $u_i(\emptyset|w)$ the utility derived by woman i when she remains unmatched, and similarly for $u_j(\emptyset|m)$. We consider preferences that satisfy the following assumptions:

Assumption 1. *For any woman i of type $w \in (a, b]$,*
(i): $u_i(m|w)$ is linear and strictly increasing in m, and
(ii): $u_i(a|w) = u_i(\emptyset|w)$.
The parallel assumptions hold for $u_j(w|m)$ for any man j of type m.

Assumption 1(*i*) is common in the literature. It implies that agents have aligned preferences on both sides of the market and strictly prefer to be matched with a partner of higher type. Linearity further implies that agents are risk neutral, which simplifies the intertemporal considerations since an agent's match if she defers in the first period is stochastic. Assumption 1(ii) implies that agents are indifferent between matching with the lowest type and being unmatched. We make this assumption for tractability; it allows us to normalize a, the lower bound of the type space, to zero.

Setting $a = 0$, an example of symmetric utilities satisfying Assumption 1 is $u_i(m|w) = u_j(w|m) = mw$. Utilities need not be symmetric, however, and could have individual-specific slopes.

Agents observe their own type and can match either in period $t = 1$ or $t = 2$. In period 1, a woman w randomly meets with a man m - all meetings are equally probable. Upon meeting, each agent learns the other agent's type, and decides to match now or to defer. If both agents decide to match early, they match and exit the market with the corresponding payoffs. If at least one agent decides to defer, they both remain in the market for the next period. We refer to a match in period 1 as an early match.

In period 2, every remaining woman w randomly meets with a remaining man m - all meetings where every agent meets someone new are equally probable.[7] Under recall, an agent in period 2 has two possible partners, and can notably match with the agent met in period 1. By contrast, there is no recall in Damiano, Li and Suen (2005) and an agent in period 2 simply matches with the agent met in period 2.

A key feature of recall is that it gives rise to a network of possible partnerships among remaining agents. Two agents are connected in this network if they met either in period 1 or period 2. In our framework, this network has a simple structure: each of its connected components is a circle of even size. Considering more than two periods or more meetings per period would lead to a more complex structure. By contrast, this network reduces to disconnected pairs in the usual frameworks without recall. The emergence of a nontrivial network of possible partnerships is thus a defining feature of recall.

[7]The only case where remaining agents cannot meet someone new is when only one woman and man have not matched early. Since they are then the only two agents remaining, they must match together in any case.

We assume that the matching formed in period 2 is *stable*, conditional on the feasible partnerships. Formally, denote by μ a matching in period 2, where $\mu(i) = j$ and $\mu(j) = i$ mean that woman i is matched with man j. A matching is *feasible* if $\mu(i) = j$ only if i and j met in period 1 or 2. With a slight abuse of notation, let $\mu(w_i)$ denote the type of woman w_i's partner under matching μ. Following Assumption 1(ii), set $\mu(w_i) = a$ when woman w_i is unmatched under μ. We adapt the usual notion of stable matchings under non-transferable utilities to our setup as follows, see, e.g., Roth and Sotomayor (1992). A pair (w_i, m_j) is a blocking pair if: (i) w_i and m_j met either in period $t = 1$ or $t = 2$, and (ii) $u_i(\mu(w_i)|w_i) < u_i(m_j|w_i)$ and $u_j(\mu(m_j)|m_j) < u_j(w_i|m_j)$. A feasible matching μ is stable if there is no blocking pair.

Since preferences are aligned, feasible stable matchings can be simply described and are generically unique. Consider a connected component of the network of partnerships. Pick the woman with highest type in the component. Let this woman match with her preferred partner among her two possible choices. Then, remove these two agents and repeat, picking the woman with the highest type among the remaining women, and let her match with her preferred remaining partner, etc. With probability 1, agents are never indifferent and the feasible stable matching is generically unique. We collect these observations in the following Lemma, whose proof is immediate and omitted.

Lemma 1. *For every set of meetings and early matchings, the network of possible partnerships among remaining period 2 agents is composed of disjoint circles of even size. Under Assumption (1), with probability 1 there is a unique feasible stable match in period 2. In every component of the network, the woman and the man with highest types match with their preferred feasible partner.*

In equilibrium, an agent prefers to match early when the utility of matching right away is greater than the expected utility from waiting, which, under Assumption 1, is equal to the utility of the expected type of the stable partner. This expectation, however, depends on others' decisions, and in particular on the distribution of types that match early or defer. In other words, strategic interactions in period 1 generate an endogenous type distribution in period 2, which has to be accounted for in agents' first period matching decisions. This linkage is a key feature of dynamic matching and, as we will show below, is profoundly affected by recall.

Formally, a pure strategy σ_i of woman $i \in \mathcal{W}$ is a measurable function from the men's type space into the binary set $\{1, 2\}$, where $\sigma_i(m) = 1$ when i announces she wants to match early with man m and $\sigma_i(m) = 2$ when i defers. Similarly, the pure strategy of man j is a function from the women's type space into $\{1, 2\}$. Early matches are executed by *mutual consent*: an early match takes place if and only if both agents announce they want to match early. By contrast, both agents reach the second period if at least one agent defers.

Let $\boldsymbol{\sigma}_{-i} = (\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$ denote a strategy profile of all agents other than i. From i's point of view, $\boldsymbol{\sigma}_{-i}$ induces a type distribution in period 2. If woman i of type w_i meets a man m in period 1, denote by $\mathbb{E}_{\boldsymbol{\sigma}_{-i}}[\mu(w_i) \mid w_i, m]$ the expected type of i's partner in the stable matching formed in period 2 if i defers.[8] Note the critical dependence of this expectation on m.

In our analysis, we focus on strategies where an agent announces she wants to match early if and only if she prefers to match early. This is a standard way of ruling out miscoordination on mutual consent, where agents may defer even though both agents strictly prefer to match early. We further assume that, when an agent is indifferent between matching early and deferring, she agrees to match with probability one. Formally, then, a profile $\boldsymbol{\sigma}$ is an equilibrium if for all i, j, w, m, $\sigma_i(m) = 1$ if and only if $m \geq \mathbb{E}_{\boldsymbol{\sigma}_{-i}}[\mu(w_i) \mid w_i, m]$, and $\sigma_j(w) = 1$ if and only if $w \geq \mathbb{E}_{\boldsymbol{\sigma}_{-j}}[\mu(m_j) \mid m_j, w]$.

In particular, note that everyone playing "always early" is an equilibrium in pure strategies. Indeed, assume that $\boldsymbol{\sigma}_{-i} = \mathbf{1}$. If woman i deviates and decides to defer, then she will not meet a new partner in the second period, since she and her first period partner are the only two agents present in the market. Therefore,

[8]Since the stable match is generically unique, the expectation is well-defined except when all other agents match early with probability one. In this case, i matches with her first period partner independent of her strategy, and we set $\mathbb{E}_{\boldsymbol{\sigma}_{-i}}[\mu(w_i) \mid w_i, m] = m$.

$\mathbb{E}_{\boldsymbol{\sigma}_{-i}=\mathbf{1}}[\mu(w_i) \mid w_i, m] = m$ and i is indifferent between matching early or waiting.

As a first step to characterize equilibria, we show that best responses display a natural threshold property. Say that a strategy σ_i is a *threshold strategy* if, for every w_i, there exists m^* such that $\sigma_i(m) = 1$ if $m \geq m^*$ and $\sigma_i(m) = 2$ if $m < m^*$. In a threshold strategy, an agent wants to match early if and only if the type of her period 1 partner is high enough. We derive the following result in the Online Appendix.

Lemma 2. *For any agent i and strategy profile $\boldsymbol{\sigma}_{-i}$, the best-response of agent i is unique and is a threshold strategy.*

Lemma 2 relies on showing a single-crossing property of $\mathbb{E}_{\boldsymbol{\sigma}_{-i}}[\mu(w_i) \mid w_i, m]$ with respect to m. While single-crossing is natural, its proof must deal with the following complication. As the type of one's first-period partner improves, he becomes more attractive to his second period partner as well, and so the chance of losing him in the second period is higher.[9] This result allows us to focus on threshold strategies without loss of generality. We further focus on *symmetric* threshold strategies, where agents share a common type-dependent threshold function $f(\cdot)$. In that case, $\sigma_i(m) = 1$ if and only if $m \geq f(w_i)$. Denote by $\mathbb{E}_f[\mu(w_i) \mid w_i, m]$ the expected type of i's stable partner in period 2 when all other agents follow the threshold strategy f. In equilibrium, at the threshold, woman i is indifferent between matching early, with a man of type $f(w_i)$, or deferring, and hence being matched with a man of expected type $\mathbb{E}_f[\mu(w_i) \mid w_i, f(w_i)]$. An equilibrium in symmetric threshold strategies is thus characterized by the following key fixed-point equation:

$$\forall i, w_i, f(w_i) = \mathbb{E}_f[\mu(w_i) \mid w_i, f(w_i)]. \tag{1}$$

Note that this characterization relies on a functional fixed-point equation. In the absence of recall, the analogous characterization relies on a much simpler scalar fixed-point condition. We discuss at length the complexities introduced by recall in Section 4.

[9]For this reason, single-crossing would generally fail if the type distribution had mass points.

3 Equilibrium Analysis

We now state our first main result. We say that an equilibrium is an equilibrium with deferral if matchings happen with positive probability in the second period.

Theorem 1. *Consider any distribution of types over $[a, b]$ with strictly positive density. Under Assumption 1, there exists $\bar{N} \in \mathbb{N}$ such that for every market size $N \geq \bar{N}$ there exists an equilibrium with deferral in symmetric threshold strategies.*

The proof of Theorem 1 is involved, and presented in Appendix A. We describe here its main steps and intuitions. Suppose that agents follow a symmetric weakly increasing threshold strategy $f(\cdot)$. Introduce the best-response operator $\tilde{\beta}$ as follows:

$$\forall i, w_i, \tilde{\beta}(f)(w_i) \text{ is the lowest type } m \text{ that solves } m = \mathbb{E}_f[\mu(w_i) \mid w_i, m] \tag{2}$$

Thus, $\tilde{\beta}(f)(w_i)$ is the type m with the following property. When w_i meets m in the first period, her expected stable partner in the second period, given f, is precisely m. We show in the Appendix that $\tilde{\beta}(f)$ is well-defined and is itself an increasing function.

It is important to recall that, fixing f, i's second-period expectation depends non-trivially on the type of her first period partner, since, e.g., she will generally match with that partner with positive probability. By Equation 1, an equilibrium in symmetric threshold strategies is a fixed point of the functional operator $\tilde{\beta}(\cdot)$. In particular, the constant function $f(w) = a$ corresponds to the strategy "always early" and is a fixed point of $\tilde{\beta}(\cdot)$. To show existence of an equilibrium with deferral, we focus on the subspace of weakly increasing functions f such that $f(a) \geq a + \varepsilon$ with $\varepsilon > 0$. We establish the following facts. (1) If ε is small enough, the best-response operator is a self-map over this subspace: if $f(\cdot)$ belongs to it, then $\tilde{\beta}(f)(\cdot)$ belongs to it as well. (2) This subspace is a convex compact set, and (3) The functional operator $\tilde{\beta}(\cdot)$ is continuous over the subspace. By Schauder's fixed point Theorem, we then conclude that $\tilde{\beta}$ has a fixed point in the subspace, which identifies an equilibrium with deferral in symmetric threshold strategies by construction.

A key step in establishing (1) is to show that when $f(\cdot)$ is bounded away from a, even an agent of lowest type has a strictly positive probability of matching with her second period partner. To see why, note that any agent has some chance to meet someone she does not want to match with in the first period. The endogenous type distribution of agents reaching the second period thus has full support. Together with the stability of period 2's matching, we show that this implies that the probability of matching with the preferred partner in period 2 is strictly less than 1 for all agents except for those of highest type. Thus, and even though agents of lowest type are never preferred, they may end up matched with their second period partner.

We use this observation to show that agents' option value from waiting is bounded away from their default option, even for agents of the lowest type, and we are able to characterize this bound. At this stage in the proof, a difficulty is that if a woman of lowest type meets a man of lowest type in period 1 and if both end up in the same component of size 4 in period 2, they must match together.[10] It is only for this reason that we require a minimum market size. To address this issue, we show that when N is high enough, the probability that two such agents end up in a component of size greater than or equal to 6 is large enough, even if many agents are matching with high probability in the first period. We then show that when ε is small enough and N is large enough, the best-response operator is a self-map over the subspace of threshold functions, establishing (1).

We next describe key features of equilibria with deferral. We investigate their properties in more detail in the following Sections with the help of numerical simulations.

Theorem 2. *Every equilibrium with deferral in symmetric threshold strategies $f(\cdot)$ satisfies the following conditions:*

(i) $f(a) > a$ *and* $f(b) = b$

[10] In that case, the lowest type woman meets man j in period 2, the lowest type man meets woman i in period 2 and i and j had met - without matching - in period 1. Because their period 2 prospective partners are of lowest type, however, i and j match in period 2.

(ii) *$f(\cdot)$ is strictly increasing and continuous.*

(iii) *There exists an $x^* \in (a,b)$ with $f(x^*) = x^*$ such that $f(x) < x$ for every $x \in (x^*, b)$.*

Theorem 2 shows that in a symmetric equilibrium, the common threshold function starts strictly above the 45 degree line, crosses the 45 degree line at least once over the interval's interior and ends up reaching it again at the highest type from below. $f(a) > a$ since, as described above, the expected type of the period 2 partner of a lowest type agent is strictly positive. $f(b) = b$ since an agent of highest type is sure to match with his preferred partner, hence never wants to match early. $f(x) < x$ for high types since when a high-type agent meets a first period partner of the same type, she has a small chance of meeting a better partner in the second period and, even if she does, the improvement will be small. On the other hand, the agent faces some risk in deferring, as she may lose her good first-period partner in the stable match and receive a worse outcome.

While Theorem 1 does not guarantee a unique equilibrium with deferral, we conjecture that it is typically unique. To compute an equilibrium numerically, one can iterate the best response operator $\tilde{\beta}$, starting from an initial symmetric strategy. In extensive simulations, we have found that this process converges rapidly to the same functional fixed point for many initial strategies, suggesting uniqueness and stability of the equilibrium with deferral.

4 The effects of recall

4.1 The equilibrium threshold and outcome

We now turn to understanding how recall affects matching outcomes. An important benchmark is the case of two-period random matching without recall, which is studied by Damiano, Li and Suen (2005). This model corresponds to the scenario in which an early meeting that does not result in a match is permanently dissolved, as if the two agents had never met. In that case, notice that every agent who does not match early simply obtains a draw from the (endogenous) distribution of types active in the second period and matches with that partner. In particular, the quality of this matching does not depend on type and every agent has identical expectations for late matches. Therefore, the first-period threshold, x^0, also does not depend on type.

In the first period, all meetings in which both types are above x^0 match early (obtaining matches better than they expect in the second period), and all remaining pairs pass to the second period, where they are randomly re-paired to obtain their final match. Therefore, under the threshold strategy x^0, the probability of a first-period pair not matching and passing into the second period is

$$1 - (1 - H(x^0))^2 \tag{3}$$

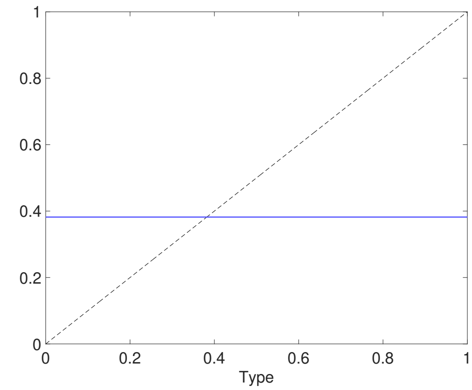
where H is the ex-ante type distribution. The equilibrium threshold x^0 is characterized by a *scalar* fixed point condition, equating this threshold to the induced expected type in the second period.

In this Section and the next we consider a simple parameterization of the model in which types are distributed uniformly on $[0, 1]$ and the productivity of any match is given by the product of types. In this case, Equation 3 simplifies to $(2x^0 - (x^0)^2)$. Since two agents match and leave the market only when each agent's type is greater than x^0, the equilibrium threshold strategy without recall, x^0, solves the following equation:

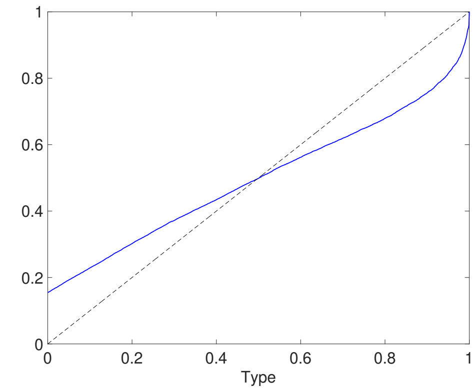
$$\begin{aligned} x^0 &= \int_0^1 \int_0^{x^0} \frac{1}{2x^0 - (x^0)^2} t dt ds + \int_0^{x^0} \int_{x^0}^1 \frac{1}{2x^0 - (x^0)^2} t dt ds \qquad (4) \\ &= \frac{3 - \sqrt{5}}{2} \approx 0.38, \qquad (5) \end{aligned}$$

where the right hand side of Equation (4) is the expected type in the second period with respect to the type distribution induced by the (constant) threshold strategy x^0.[11]

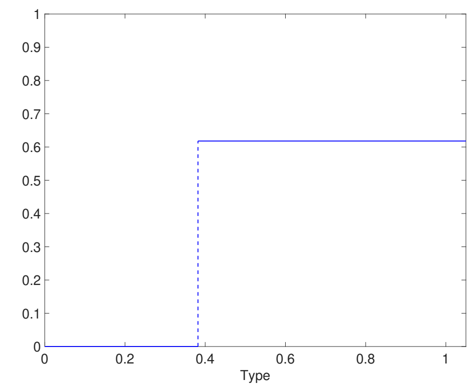
[11] To see why, note that a pair of types in the second period is drawn uniformly from an L-



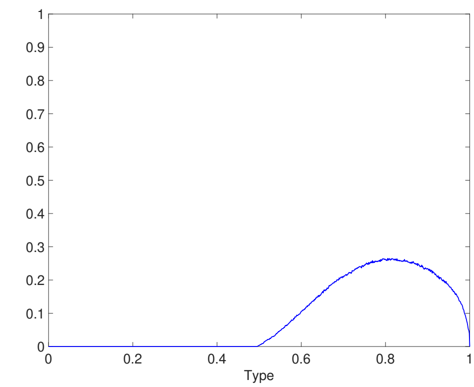
(a) Equilibrium threshold strategy, no recall



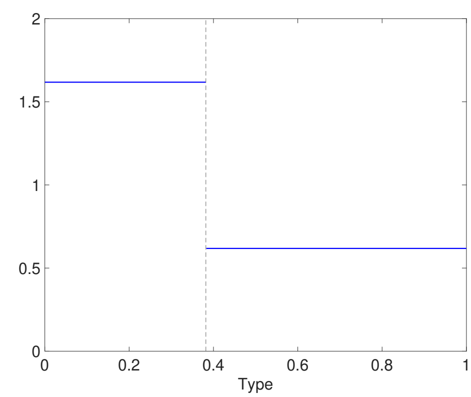
(b) Equilibrium threshold strategy, recall



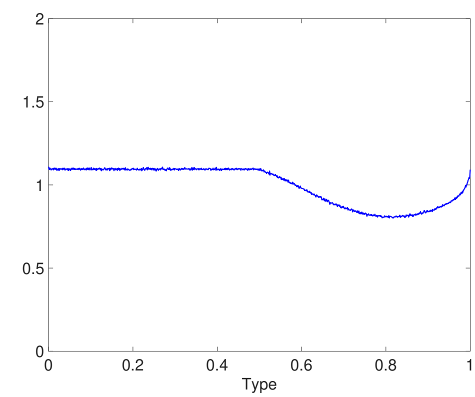
(c) Probability of matching early, no recall



(d) Probability of matching early, recall



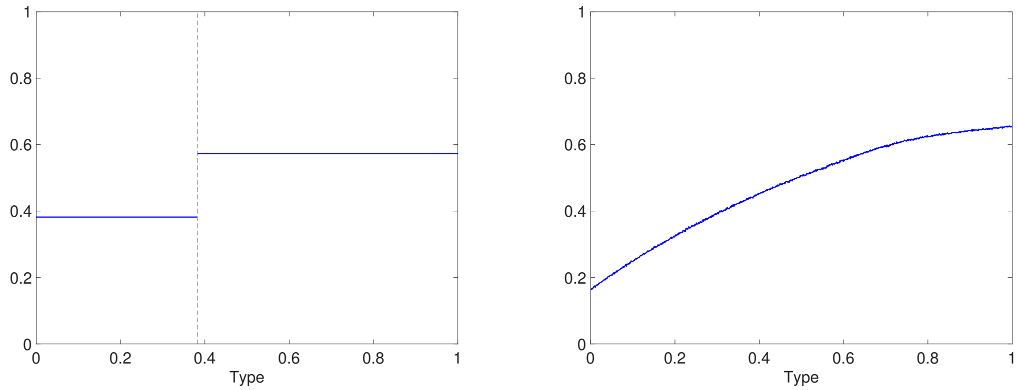
(e) Density of second period types, no recall



(f) Density of second period types, recall

Figure 1: Equilibrium outcomes, without and with recall

shaped subset of the unit square, as in Figure 5 below. Taking an expectation of the corresponding marginal distribution, the left part of this L-shaped set yields the first integral of the right hand side of Equation Equation (4), while the right part yields the second integral of the right hand side.



(a) Expected partner type, no recall **(b)** Expected partner type, recall

Figure 2: Expected type of a matched partner, without and with recall

We now compare matching outcomes with and without recall. With recall, numerical results were obtained by simulating a market with $N = 750$ participants 20,000 times. The equilibrium threshold function, and its induced properties, were computed numerically via iterated best responses starting from a threshold strategy of $\tilde{\beta}^0(x) = 1$ for all x in a discrete grid of 1000 types on the unit interval. Figure 1 first depicts the equilibrium threshold strategy, the probability of matching early, and the resulting endogenous distribution of types in the second period as a function of an agent's type. Outcomes in the benchmark case of no recall (Damiano, Li and Suen (2005)) are depicted in the Left panels, while outcomes of our model with recall are depicted in the Right panels.

The equilibrium threshold strategy is type-independent without recall (Figure 1(a)), but increasing in type and crossing the 45 degree line once from above under recall (Figure 1(b)), consistently with Theorem 2. The probability of matching early is piecewise constant - and weakly increasing - without recall: agents with a type below 0.38 have zero probability to match early while agents with a type above 0.38 have probability 0.62 to match early (Figure 2(a)). By contrast, the likelihood of matching early is continuous and non-monotonic in type under recall (Figure 2(b)). Low types do not match early because they are unacceptable to their partners, whereas the highest types rarely match early since, with high probability, they can obtain the better of their two partners after the second-stage match. This non-monotonicity is a direct consequence of recall, and has a natural interpretation in applications. Agents who are most likely to match early are good-but-not best: the mode of the distribution is equal to 0.805. As a consequence, these agents are underrepresented in the second-period type distribution under recall (Figure 1(f)). By contrast, without recall, the distribution of types present in the second period is weakly decreasing and piecewise constant (Figure 1(e)).

Next, we depict in Figure 2 the *ex-ante* expected type of a matched partner as a function of type, both under no-recall on the Left and recall on the Right.[12] We see that expected outcomes without recall are essentially binary (Figure 2(a)): low-type agents (those below x^0) expect a partner of type x^0, obtained in the second period, while high-type agents (those above x^0) expect a partner of type $\frac{1-2(x^0)^2}{2-2(x^0)^2} = \frac{3}{4}(\sqrt{5}-3) \approx 0.57$. By contrast, under recall, the expected type of a matched partner is continuous and increasing in own type (Figure 2(b)). Every agent in the second period has two feasible partners, and stability implies that agents with higher types are more likely to obtain their preferred partner in the final match. The first-period partner thus has option value in the second period and this value is higher for high-type agents since they are more likely to be able to match with their first-period partner when they so desire. All else equal, higher types hold higher expectations for their second-period outcomes, implying that they will be choosier in the first period. This yields an increasing threshold strategy and an increasing expected type of a matched partner.

Recall has another distinct implication: the possibility of remaining unmatched. Without recall, every agent matches with a partner. Under recall, however, the stable match in the second period generally leaves some agents isolated. Figure 3 depicts the simplest such example, in which two agents out of six are isolated in the stable match.

[12] Under recall, remember that being unmatched is equivalent to being matched with a lowest type.

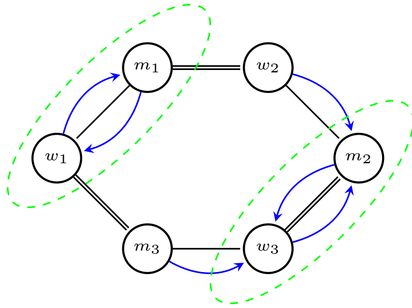


Figure 3: An example of a stable matching μ with types satisfying $m_2 > m_1 > m_3$ and $w_3 > w_1 > w_2$ where w_i and m_i meet in the first period, and where m_1 meets w_2, m_2 meets w_3 and m_3 meets w_1 in the second period. A directed arc points to the preferred partner of each agent. The resulting stable match is $\mu = \{(m_1, w_1), (m_2, w_3), (m_3, \emptyset), (w_2, \emptyset)\}$.

The decision to defer a first period match must account for this possibility. Naturally, the probability of remaining isolated in the stable match is decreasing in own type, and deferring becomes vanishingly risky for the highest type, who obtains her preferred partner with probability one. Figure 4 depicts the *ex-ante* probability of remaining isolated in equilibrium as a function of type. The good-but-not-best types match early with the highest probability since, for them, the risk of deferring is substantial, whereas the potential benefit from waiting, when the first-period partner is already good, is limited. Notice also that even the lowest type obtains a partner with probability near (but not exactly) one half.

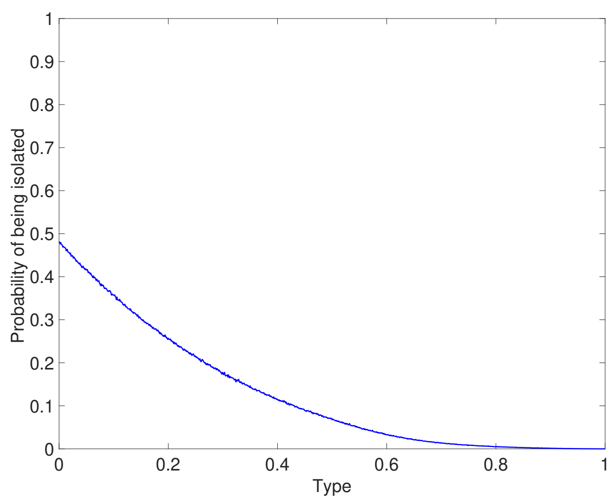


Figure 4: Probability of being isolated in equilibrium as a function of type

Finally, Figure 5 depicts the space of type-pairs, along with each agent's equilibrium threshold function, both with and without recall. The pairs of types that match early under recall are depicted by the shaded "lens", where each agent's first-period partner lies above her threshold. Without recall, and the corresponding option values in the second period, the incentives to match early are stronger. Here, early matches happen between all first-period type pairs in the upper-right quadrant, where both types exceed x^0. The set of type pairs that match early under recall is a subset of those that match early without recall; recall reduces the frequency of early matches from about 38% to about 10%. Under recall, higher types become more optimistic about the second period, as they are relatively likely to attract a better match, whereas it causes lower types to be less optimistic, since their risk of being isolated is relatively high, and their chance of attracting a better match is lower.

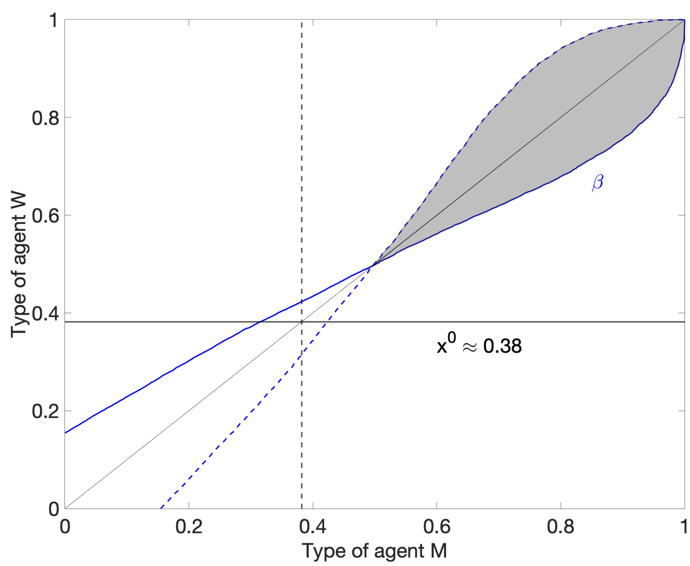


Figure 5: Equilibrium threshold strategy of a pair of agents (M, W). The solid curve and solid vertical line represent agent M's equilibrium threshold function with recall ($\tilde{\beta}$) and without recall (x^0), respectively. The dashed curve and dashed vertical line represent agent W's equilibrium threshold function with recall and without recall, respectively.

Consequently, the equilibrium thresholds with and without recall intersect: low-type agents are more inclined to match early under recall, whereas high-type agents are more inclined to match early without recall.

Finally, note that without recall, every agent prefers to meet a higher-type partner in both the first and second period, all else equal. With recall, this is no longer true: the ideal partner type for an agent can be interior. This is because higher-type partners are riskier, as their outside options are stronger.

4.2 Welfare

We turn now to analyzing the welfare properties of this matching environment, focusing on the effects of recall. To keep the analysis simple, define welfare here as the sum of the utilities of all agents. Since the match value is given by the product of types, the first-best efficient match is perfectly assortative. Specifically, the average value per match tends to $\mathbb{E}[x^2] = \frac{1}{3}$ as N tends to infinity. On the other hand, purely random matching produces an average value of $\mathbb{E}[x]^2 = \frac{1}{4}$. We normalize these average welfare levels to one and zero, respectively, and quote all subsequent welfare figures as a proportion of the first-best welfare gains on this scale. Numerical results were obtained by simulating a market with $N = 750$ participants 100,000 times.

In the benchmark scenario without recall, where all agents apply the equilibrium threshold of $x^0 \approx 0.38$, the average match value is 26.88%. Introducing recall has several distinct effects on welfare. First, fewer matches happen early, but early matches are more highly assortative than early matches without recall. Next, second-period matches are also more assortative under recall, as having multiple partners allows for endogenous assortativity that cannot arise under the purely random matching that dictates late matches without recall. Finally, some agents will remain isolated under recall, an inefficiency that negatively impacts welfare relative to the benchmark case. In sum, the average value under recall is 33.24%, exhibiting a substantial net positive welfare effect of recall. Thus, the benefits of more efficient assortativity outweigh the cost of isolated agents.[13] On average, 12.6% of agents remain unmatched under recall in equilibrium, indicating that along with the higher average value, there is more variation in value across agents.

We now quantify the effects of early matching in the presence of recall. To do

[13]Note that this conclusion is sensitive to the specification of values as the product of types.

so, we consider a model that is identical to our principal specification, but in which early matching is prohibited. In this case, every agent randomly meets two different partners, after which the stable match is implemented. The main finding is that welfare decreases to 29.28% in this specification. An important factor leading to this conclusion is that early matching reduces the number of isolated agents. Prohibiting early matches increases the frequency of isolated agents to 13.5%. We conclude that in the presence of recall, the early matches that arise endogenously are beneficial from a welfare perspective, both because they are highly assortative and because they increase the number of matched agents. Thus, a social planner would prefer to allow early matches to arise endogenously as opposed to making any attempt at prohibiting them.[14]

We now analyze in further detail the distribution of value at the equilibrium of our model with recall and endogenous early matching. Figure 6 shows the equilibrium expected partner type as a function of own type, replicating Figure 2(b), along with the 75th and 25th percentiles. The three curves are predominantly increasing, showing the assortative finding that higher types tend to match with higher types. The exception is that the 75th percentile curve is decreasing above type 0.99. The reason is that the probability of matching early is quickly vanishing for types approaching the upper bound, and matching early involves matching to a very high type partner. Thus, as an agent's type approaches the upper bound, she becomes less likely to match with a very high type, and more likely to match with her preferred partner in the second period. This is better in expectation, but the distribution of stable partners has a thinning upper tail for the highest types.

[14] It would be interesting to compare this figure against the second-best outcome of our model: the most efficient match subject to the two-period matching process. However, the efficient match is computationally complex and we have been unable to compute it for reasonable market sizes. In the second-best, the planner would implement some first-period matches, as first-period partners who remain in the market but are likely to match together eventually in the second-period impose a negative externality on others by effectively limiting their options.

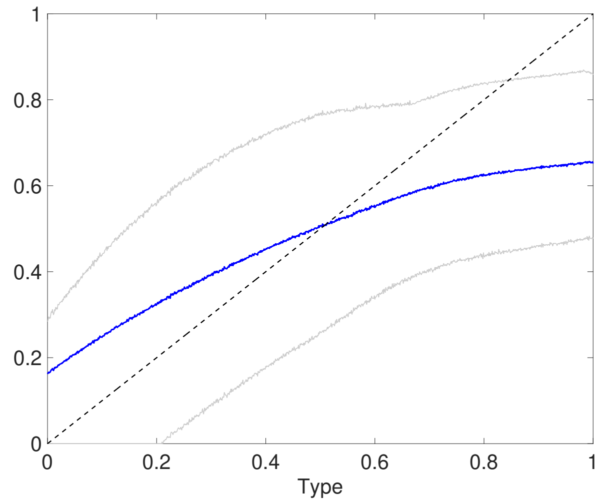
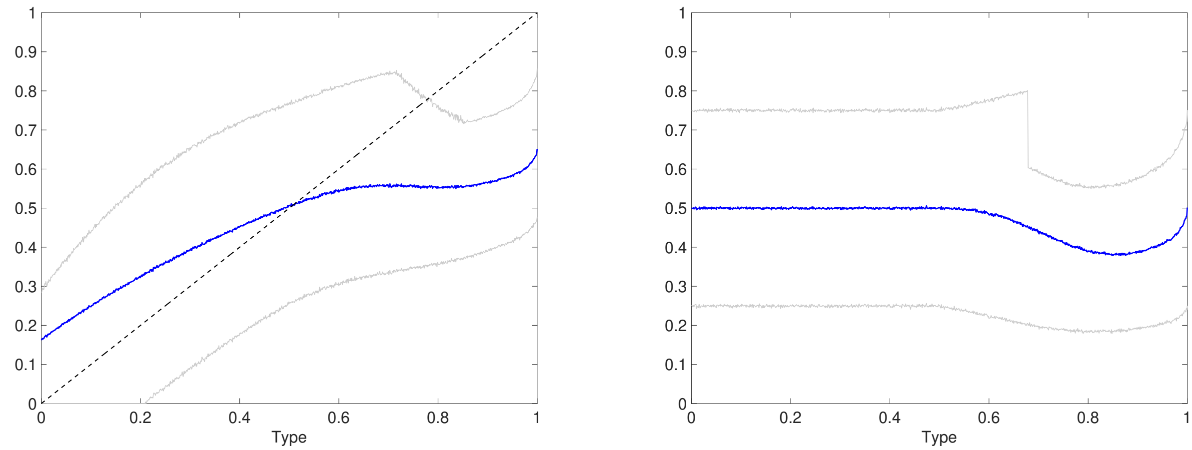


Figure 6: Expected type of matched partner, along with 75th and 25th percentiles.

Figure 7 depicts the same curves focusing on the second-period matching market, by conditioning on agents who do not match early. While the 25th percentile curve is increasing, the 75th percentile curve is clearly nonmonotonic, and even the curve depicting the expected partner is nonmonotonic. This effect arises because, conditional on not matching early, an agent's type is negatively correlated with her first period partner's type, with whom she eventually matches with substantial probability. Thus, as an agent's type increases, her first period partner's type may decrease on average, as seen in the right panel, leading to a net decrease in her average stable match. The discontinuity in the 75th percentile of the first period partner's type arises from the fact that, at the discontinuity, the 75th percentile jumps from the type at the upper bound of the lens to the type at the lower bound of the lens.



(a) Type of stable match in equilibrium

(b) Type of first period partner in equilibrium

Figure 7: Conditional on not matching early, expected type of matched partner (Left panel) and first period partner (Right panel), along with 75th and 25th percentiles.

5 Participation costs and unravelling

We here take up the issue of how equilibrium dynamics are influenced by the introduction of small per-period participation cost c. An important result in Damiano, Li and Suen (2005) is that under no recall, even arbitrarily small costs of participation destroy the interior equilibrium and lead to a complete unravelling whereby all agents match early. As a consequence, the assortativity that arises through the sorting properties of dynamic matching are lost, and the equilibrium outcome is a purely random match. In this sense the equilibria of the no-recall environment are discontinuous at $c = 0$. By contrast, we show below that under recall an equilibrium with deferral still exists in the presence of small participation costs.

The matching process now works as follows. Prospective first-period partners are assigned through a random pairing, as before. Every agent, after learning her type, decides whether or not to pay c and participate in the first period. If both agents in a prospective pair participate, they meet and decide whether to match or defer. Otherwise, they do not meet. Notice that an agent may therefore participate, but not meet a partner, which occurs if the prospective partner does not participate.[15] Any agent who has met a partner decides whether to accept or defer, with matches executed under mutual agreement, all as before. In the second period, all remaining agents are again assigned prospective partners randomly, subject to the constraint that their second-period prospective partner is distinct from their first-period prospective partner. As before, if both agents in a prospective pairing participate in the second-period, they meet; otherwise they do not. Any unmatched agent who does not participate in the second period exits with her outside option. Finally, the (still unique) stable match is implemented from among all agents active in the second period.

The next result shows that our main equilibrium existence argument is robust to the inclusion of small participation costs.[16]

Theorem 3. *There exists $\bar{N}$ such that for every $N > \bar{N}$ there is a $c_N > 0$ such that for all $c < c_N$ the matching process with participation costs exhibits an equilibrium with deferral in symmetric threshold strategies and full participation.*

Proof. See Appendix C. ∎

There is therefore a fundamental difference introduced by the inclusion of recall in terms of unravelling. Let us provide some intuition. Without recall, and as explained by Damiano, Li and Suen (2005), agents whose types are below the threshold x^0 know they will not match early, and thus are unwilling to pay any $c > 0$ to participate in the first period. So the first period must consist exclusively of high types, whereas the second period contains low types. But this cannot be an equilibrium, as an agent near, but just below, the threshold would profit from deviating to participating in the first period, as he, being among the best possible partners in the second period, would be acceptable to his first-period partner, who will be better than his second-period partner. This deviation snowballs and leads to a situation where all agents prefer to match early: the equilibrium unravels.

[15]Our focus, however, is on exhibiting a full-participation equilibrium, in which such events are off-path.

[16]We conjecture that there is an equilibrium parametrized by c that converges to the equilibrium we identify for the $c = 0$ case above, but we have not proven this continuity.

The key insight is that under recall, the first-period meeting provides option value *even if* the agent knows she will not match early, because the first-period partner remains a viable matching opportunity in the second period. Thus even low types who know they will not match early are still willing to pay to meet a first-period partner, and the unraveling argument from the no-recall case breaks down. While any model operates under specific assumptions, our view is that the option value effect is important in applications. Any mechanism through which a current meeting provides the possibility of future benefits generates option value and will provide a force against unraveling. It is common in many settings for professionals to invest in networking, for example, where the objective is to initiate relationships that may produce downstream benefits.

6 Conclusion

We have shown that dynamic matching outcomes are impacted in fundamental ways when agents have the option of matching with others they have met previously. The effects introduced by recall seem important in the real world. Take, for instance, the annual job search that occurs in many fields. When a firm and worker interview, they can match right away. But if either side defers, in hopes of improving their outcome, so that the firm and the worker each conduct a subsequent interview with a second counterpart, it is very natural that the possibility remains of matching with each other eventually. Indeed, the choice of acceptance or deferral by the worker (and the firm) must take into account (i) the possibility of meeting a better partner, (ii) the probability that the second partner will be available to match, given that it will have other feasible options, (iii) the probability that the first partner turns out to be the best partner, but becomes unavailable due to matching with someone else, resulting in either (iii:a) matching with a worse partner, or (iii:b) remaining unmatched as the final outcome.

The result is that, in equilibrium, the higher an agent's type, the higher is her threshold for accepting an early match. The mutually agreeable early matches are highly assortative and involve relatively high-type agents – but not the very best agents, as they can afford to be patient. Since early meetings confer option value in the second period, the equilibrium is robust to the inclusion of a small participation cost, and unraveling does not occur, in contrast to the no-recall case.

We have looked at a two-period model with non-transferable utilities, aligned preferences and random meetings. It would be interesting to see how recall affects outcomes in more general non-stationary environments, e.g., if utilities are transferable, even partially, if preferences are not fully aligned, if there are more than two periods, or with a different meeting technology. Developing a full-fledged understanding of the impact of recall on non-stationary decentralized matching would be potentially fruitful and certainly challenging direction of future research. While details will undoubtedly differ, we conjecture that the key finding that recall has a first-order impact on matching outcomes will prove long-lived.

In particular, we identify three key features of recall that should apply to any decentralized, non-stationary environment. First, higher types tend to be more selective in early matches, since recall allows them to expect better future outcomes. Second, with recall, agents are linked through an evolving network of potential partnerships. The structure of this network affects realized matchings and is determined by characteristics of the meeting process. Third, meetings that do not lead to matches right away are nonetheless valuable, and such option values can help markets resist unravelling. A natural next step would then be to investigate how recall affects the functioning of real-world decentralized markets. It would be interesting to test the predictions of our model on real data, in particular regarding the non-monotonicity of matching time with respect to type. Yenerdag (2021) implements lab experiments built from our theoretical framework. He finds fairly strong support for the theoretical results.

Appendix A Proof of Theorem 1

Let $\mathcal{Y}$ be the set of all weakly increasing threshold strategies. That is,

$$\mathcal{Y} = \{f : [a,b] \to [a,b] | f(x) \geq f(y) \quad \forall x > y\}.$$

In this appendix, we define $\tilde{\beta}$, the best response operator, in a more general way than it is defined in Equation 2. We let $\tilde{\beta}$ be a mapping on $\mathcal{Y}$ defined as

$$\tilde{\beta}[f](x) \equiv \min B_f(x) \quad \forall x \in [a,b], \tag{6}$$

where $B_f(x) = \{y \in [a,b] : \mathbb{E}_f[\mu(x)|x,y] \leq y\}$. We call the mapping $\tilde{\beta}$ best response mapping. Note that in Lemma 2, we prove a key single crossing condition: for any given $x \in [a,b]$ and $f \in \mathcal{Y}$, the function $\mathbb{E}_f[\mu(x)|x,y] - y$ satisfies single crossing condition from above. In addition, in Theorem 2 we show that $\mathbb{E}_f[\mu(x)|x,y]$ is continuous in y and for any given $x \in [a,b]$ and $f \in \mathcal{Y}$, there always exist $y \in [a,b]$ that solves $\mathbb{E}_f[\mu(x)|x,y] = y$. Hence, these are sufficient to conclude that the two definitions, given in Equation 2 and Equation 6, are equivalent on $\mathcal{Y}$.

We let $L^1[a,b]$ denote the space of integrable functions with respect to the Lebesgue measure on $[a,b]$. Note that with the usual L^1 norm, $L^1[a,b]$ is a Banach space. Finally, for any fixed $\epsilon \in [a,b]$, we let $\mathcal{Y}_\epsilon$ be a subset defined as

$$\mathcal{Y}_\epsilon = \{f \in \mathcal{Y} | f(x) \geq \epsilon \quad \forall x \in [a,b]\}.$$

We prove the theorem by proving four different lemmas:

Lemma 3. *Best response mapping $\tilde{\beta}$ is well defined, and $\tilde{\beta}[f] \in \mathcal{Y}$ for all $f \in \mathcal{Y}$.*

Lemma 4. *There exists an $\epsilon \in (a,b]$ and $\bar{N} \in \mathbb{N}$ such that for all market size $N \geq \bar{N}$ the best response mapping $\tilde{\beta}$ is a self-map on $\mathcal{Y}_\epsilon$. That is,*

$$\tilde{\beta} : \mathcal{Y}_\epsilon \to \mathcal{Y}_\epsilon.$$

Lemma 5. *For all $\epsilon \in [a,b]$, $\mathcal{Y}_\epsilon$ is a convex and compact subset of Banach space $L^1[a,b]$.*

Lemma 6. *The best response mapping $\tilde{\beta} : \mathcal{Y}_\epsilon \to \mathcal{Y}_\epsilon$ is continuous in L^1.*

The proof of Lemma 4 is in the next subsection. The proofs of Lemma 3, 5 and 6 appear in the Online Appendix. We now prove Theorem 1.

Proof of Theorem 1

By Lemma 3 and Lemma 4, there exists an $\epsilon > a$ and $\bar{N} \in \mathbb{N}$ such that for all market size $N \geq \bar{N}$ the best response mapping, $\tilde{\beta}$, is well defined self map on $\mathcal{Y}_\epsilon$.

Note that $\mathcal{Y}_\epsilon$ is obviously non-empty. By Lemma 5, it is also convex and compact. Moreover, $\tilde{\beta}$ is continuous by Lemma 6. Therefore, by Schauder fixed-point theorem $\tilde{\beta}$ admits a fixed point in $\mathcal{Y}_\epsilon$. It follows from the the proof of Theorem 2 that any such fixed point satisfies Equation 1. Also, note that it is an equilibrium with deferral by construction. ■

A.1 Proof of Lemma 4

For a shorter proof, we simplify the equations with the following normalization: $a = 0$. That is, our type space is of the form $[0, b]$. Note that this is without loss of generality under Assumption 1.

Also, the following definitions and notations will be useful for the proof. We say that a strategy profile is a trivial strategy profile if, and only if, all agents match early with probability 1. We use capital letters W_j and M_j to denote random type of woman j and man j, respectively. We say that a pair (m, w) is a "period t pair" if man m and woman w meets in period $t \in \{1, 2\}$. Without loss of generality, we relabel all agents such that (m_j, w_j) is a period 1 pair for all $j = 1, 2, \ldots, \frac{N}{2}$. Now, suppose that all agents follow a threshold strategy $f \in \mathcal{Y}$. We let $A(f)$ be the set of all possible type pairs of a period 1 pair that did match early under the symmetric threshold strategy f. That is, $A(f)$ is the following set

$$A(f) = \{(w_j, m_j) \in [0, b]^2 : w_j < f(m_j) \text{ or } m_j < f(w_j)\}. \tag{7}$$

Therefore, from an agent i's point of view, a period 1 pair (w_j, m_j) $j \neq i$ that did not match early is a realization of bi-variate random vector (W_j, M_j) *conditional* on being in the set $A(f)$, denoted by $(W_j, M_j)|A(f)$. Note that the bi-variate random vector $(W_j, M_j)|A(f)$ is exchangeable. Moreover, from agent i's point of view, each $(W_j, M_j)|A(f)$ for $j = 1, 2, \ldots, i-1, i+1, \ldots, \frac{N}{2}$, is independent and identically distributed. We use $W_j|f$ and $M_j|f$ to denote the corresponding marginalizations of $(W_j, M_j)|A(f)$.

We let $g_f(w_j, m_j)$ denote the probability distribution function of $(W_j, M_j)|A(f)$. Note that $g_f(w_j, m_j)$ is given by

$$g_f(w_j, m_j) = \frac{h(w_j)h(m_j)\mathbb{1}_{(w_j, m_j)}A(f)}{\mathbb{P}(A(f))}, \tag{8}$$

where $\mathbb{1}_{(w_j, m_j)}A(f)$ is the indicator function that takes value 1 if $(w_j, m_j) \in A(f)$, and 0 otherwise, and $\mathbb{P}(A(f))$ denotes the probability of $(W_j, M_j) \in A(f)$ with respect to the ex-ante type distribution. Notice that we have $\mathbb{P}(A(f)) > 0$ for all non trivial threshold strategy f. We let N_2 be the number of total agents in period $t = 2$. For each agent N_2 is a random variable in period $t = 1$. From the perspective of an agent who met with someone but did not match early in the first period, the random variable N_2 has binomial distribution with parameters $\frac{N}{2} - 1$ and $\mathbb{P}(A(f))$.

In particular,

$$\mathbb{P}_f(N_2 = 2n_2 + 2) = \binom{\frac{N}{2} - 1}{n_2} (\mathbb{P}(A(f)))^{n_2} (1 - \mathbb{P}(A(f)))^{\frac{N}{2} - 1 - n_2}, \quad \forall n_2 \in \left\{0, 1, \ldots, \frac{N}{2} - 1\right\}. \tag{9}$$

Next, given an $N_2 \geq 4$ we introduce a graph g to denote a realization of two period meetings among N_2 agents in a compact way. We assume that a graph g has the following link structure: a pair (m, w) is linked in g iff (m, w) is a period 1 or period 2 pair. Given this, we let $\mathcal{G}(\frac{N_2}{2})$ denote the set of all possible graphs.

Since every remaining agent on one side meets with a new agent on the other side, each component of a graph g must be a cycle. And, each agent can be in a component of different possible sizes. Let $\mathcal{C}(N_2)$ denote the set of all possible component sizes given that there are N_2 agents in period $t = 2$. Hence, $\mathcal{C}(N_2)$ is given by

$$\mathcal{C}(N_2) = \begin{cases} \{N_2\} & \text{if } N_2 = 4, 6 \\ \{4, 6, \ldots, N_2 - 4, N_2\} & \text{if } N_2 \geq 8. \end{cases} \tag{10}$$

Note that for all $N_2 \geq 8$, no agent can be in a component of size 2 since every agent meets with a distinct agent in each period. Similarly, no agent can be in a component of size $N_2 - 2$, since that would imply that there exists an agent who meets with the same person in both periods, which is not possible.

Next, conditional on there are N_2 agents in the second period, we let $\mathbb{P}_{N_2}(C)$ denote the probability of an agent being in a component of size $C \in \mathcal{C}(N_2)$. Note that we assume that each pair (m_j, w_j) is a period 1 pair for all $j = 1, 2, \ldots, \frac{N}{2}$. Given this, it is easy to verify the following useful inequality:

$$\mathbb{P}_{N_2}(C = 4) \leq \frac{1}{3} \quad \forall N_2 \geq 8. \tag{11}$$

For simplicity of notation, we relabel all agents as follows: For any given component size $C \in \mathcal{C}$, pairs (m_j, w_j), $j = 1, 2, \ldots, \frac{C}{2}$ are period 1 pairs. Pairs (m_j, w_{j+1}), $j = 1, 2, \ldots, \frac{C}{2} - 1$ and $(m_{\frac{C}{2}}, w_1)$ are period 2 pairs. For example, we have the following configuration for a component of size $C = 6$:

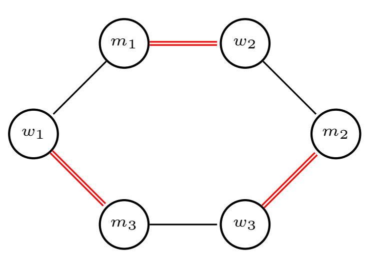


Figure 8: A realization of two period meeting process: a component of size $C = 6$. A pair linked with single line represents a period $t = 1$ pair and double line represents a period $t = 2$ pair.

Now, we are ready to state the auxiliary claims to prove Lemma 4. Without loss of generality, all the claims below are stated in terms of a generic man $i = 1$.

Claim 1. *If f is a weakly increasing threshold function with $f > a$, then*

$$\mathbb{E}_f[\mu(m_1)|m_1 = a, w_1 = a, C] \geq \mathbb{P}_f(M_2 \geq M_3, W_3 \geq W_2)\mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2] \quad \forall C \geq 6.$$

Moreover, the inequality is strict for $C \geq 8$.

Claim 2. *Fix any $m_1 \in [a, b]$. If f is a weakly increasing threshold function with $f > a$, then*

$$\mathbb{E}_f[\mu(m_1)|m_1, w_1 = a] \leq \mathbb{E}_f[\mu(m_1)|m_1, w_1 = x] \quad \forall x \in [a, b]$$

Claim 3. *Let f be a weakly increasing threshold function. If $f_c \leq f$, then*

$$\mathbb{E}_{f_c}[W_2|M_2 \geq M_3, W_3 \geq W_2] \leq \mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2].$$

The proofs of Claim 1, 2 and 3 are presented in the Online Appendix D.3, D.4 and D.5, respectively. Now, we prove Lemma 4.

Proof of Lemma 4

By Lemma 3, we know that $\tilde{\beta}[f]$ weakly increasing for all $f \in \mathcal{Y}$. Therefore, it is sufficient to show that there exists an $\epsilon > 0$ and $\bar{N}$ such that for all $N \geq \bar{N}$ $\tilde{\beta}[f](0) \geq \epsilon$ for all $f \in \mathcal{Y}_\epsilon$. To that end, fix a generic man $i = 1$. First, note that Claim 2 implies that

$$\beta[f](0) \geq \mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] \quad \forall f \in \mathcal{Y}. \tag{12}$$

Now, take any weakly increasing function f with $f(x) > 0, \forall x \in [0, b]$. And, consider the term $\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0]$:

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] = \sum_{n_2=0}^{\frac{N}{2}-1} \mathbb{P}_f(N_2 = 2n_2 + 2)\mathbb{E}_f^{N_2=2n_2+2}[\mu(m_1)|m_1 = 0, w_1 = 0], \tag{13}$$

where $\mathbb{E}_f^{N_2=2n_2+2}[\mu(m_1)|m_1 = 0, w_1 = 0]$ is the expectation conditional on $N_2 = 2n_2 + 2$ total number of agents in period $t = 2$ (including m_1 and w_1).

In addition, conditional there are N_2 agents in period 2, $\mathcal{C}(N_2)$ is the set of all possible component sizes that man m_1 can be in with w_1. Note that Equation 10 characterizes the set $\mathcal{C}(N_2)$ for any given $N_2 \geq 4$. Thus, for all $n_2 \in \{0, 1, 2, \ldots, \frac{N}{2}-1\}$ we have the following equation,

$$\mathbb{E}_f^{N_2=2n_2+2}[\mu(m_1)|m_1 = 0, w_1 = 0] = \sum_{C \in \mathcal{C}(N_2=2n_2+2)} \mathbb{P}_{N_2=2n_2+2}(C)\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0, C],$$

Note that man m_1 is matching with $w_1 = 0$ with probability 1 if and only if $C = 4$. Hence, we can write

$$\mathbb{E}_f^{N_2=2n_2+2}[\mu(m_1)|m_1 = 0, w_1 = 0] = \sum_{C \in \mathcal{C}(N_2=2n_2+2)\setminus 4} \mathbb{P}_{N_2=2n_2+2}(C)\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0, C]. \tag{14}$$

Next, note that by Claim 1 we have

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0, C] \geq \mathbb{P}_f(M_2 \geq M_3, W_3 \geq W_2)\mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2] \quad \forall C \geq 6, \tag{15}$$

with strict inequality for $C \geq 8$. Thus, combining Equation 14 and Equation 15, we obtain

$$\mathbb{E}_f^{N_2}[\mu(m_1)|m_1 = 0, w_1 = 0] \geq \mathbb{P}_{N_2}(C \geq 6)\mathbb{P}_f(M_2 \geq M_3, W_3 \geq W_2)\mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2], \quad \forall N_2 \geq 6, \tag{16}$$

with strict inequality for all $N_2 \geq 8$. Next by Equation 11 we have

$$\mathbb{P}_{N_2}(C \geq 6) \geq \frac{2}{3} \quad \forall N_2 \geq 6. \tag{17}$$

Note that the inequality given in Equation (17) is strict as long as $N_2 \neq 8$, since $\mathbb{P}_{N_2}(C \geq 6)$ is increasing in N_2 for all $N_2 \geq 8$ and it is equal to 1 for $N_2 = 6$. Therefore, by Equation (15) and Equation (17), we obtain the following strict inequality:

$$\mathbb{E}_f^{N_2}[\mu(m_1)|m_1 = 0, w_1 = 0] > \frac{2}{3}\mathbb{P}_f(M_2 \geq M_3, W_3 \geq W_2)\mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2], \quad \forall N_2 \geq 6. \tag{18}$$

Next, also note that the man m_1 matches with $w_1 = 0$ for $N_2 \leq 4$. Therefore, combining Equation 13 and Equation 18, we have the following inequality:

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] > \mathbb{P}_f(N_2 \geq 6)\frac{2}{3}\mathbb{P}_f(M_2 \geq M_3, W_3 \geq W_2)\mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2]. \tag{19}$$

Note that Equation (19) holds for every $N \geq 6$.

Now, for any weakly increasing threshold function f with $f(x) > 0$, $\forall x \in [0, b]$ we state the following observation:

$$\mathbb{P}_f(M_2 \geq M_3, W_3 \geq W_2) \geq \frac{1}{4}. \tag{20}$$

This follows from the fact that the negative (non positive) correlation between marginal random variables $W_2|f$ and $M_2|f$, of $(W_2, M_2)|f$. Hence, since $(W_2, M_2)|f$ and $(W_3, M_3)|f$ are independent and identical, Equation 20 is obvious. Therefore, from Equation 19 and Equation 20, we obtain the following inequality:

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] > \mathbb{P}_f(N_2 \geq 6)\frac{1}{6}\mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2]. \tag{21}$$

Next, consider a constant function f_c for some $c > 0$. Note that

$$\lim_{c\to 0}\mathbb{E}_{f_c}[W_2|M_2 \geq M_3, W_3 \geq W_2] > 0, \tag{22}$$

as long as the pdf of ex-ante type distribution, h, is strictly positive everywhere. We let $K^* = \lim_{c\to 0}\mathbb{E}_{f_c}[W_2|M_2 \geq M_3, W_3 \geq W_2]$. Now, take any $\epsilon \in (0, K^*]$. Now, Claim 3 implies that

$$\mathbb{E}_f[W_2|M_2 \geq M_3, W_3 \geq W_2] \geq K^* > 0 \quad \forall f \in \mathcal{Y}_\epsilon. \tag{23}$$

Thus, combining Equation 21 and Equation 23, we obtain

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] > \mathbb{P}_f(N_2 \geq 6)\frac{1}{6}K^* \quad \forall f \in \mathcal{Y}_\epsilon \text{ and } \epsilon \in (0, K^*]. \tag{24}$$

As it is stated in Equation 9, we know that N_2 has binomial distribution with parameters $\frac{N}{2} - 1$ and with success probability $\mathbb{P}(A(f))$. Also, note that if $f \in \mathcal{Y}_\epsilon$, then we have $\mathbb{P}(A(f)) \geq \mathbb{P}_{f_\epsilon}(A(f_\epsilon))$. Hence,

$$\mathbb{P}_f(N_2 \geq 6) \geq \mathbb{P}_{f_\epsilon}(N_2 \geq 6) \quad \forall f \in \mathcal{Y}_\epsilon \tag{25}$$

since $\mathbb{P}(A(f)))$ is increasing in f. Therefore, for every $N \geq 6$ we can establish the following inequality:

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] > \mathbb{P}_{f_\epsilon}(N_2 \geq 6)\frac{1}{6}K^* \quad \forall f \in \mathcal{Y}_\epsilon \text{ and } \epsilon \in (0, K^*]. \tag{26}$$

Now, fix an $\epsilon \in (0, \frac{K^*}{6})$. Then, by Equation 71, established in the proof of Claim 1, and by Equations 13-14, it is easy to verify that we can take a $\delta(\epsilon) > 0$ such that

$$\delta(\epsilon) = \inf_{f\in\mathcal{Y}_\epsilon, N\geq 6}(\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] - \mathbb{P}_{f_\epsilon}(N_2 \geq 6)\frac{1}{6}K^*) > 0. \tag{27}$$

Hence, combining Equation 26 and Equation 27, for every $N \geq 6$ we obtain the following inequality

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] \geq \mathbb{P}_{f_\epsilon}(N_2 \geq 6)\frac{1}{6}K^* + \delta(\epsilon) \quad \forall f \in \mathcal{Y}_\epsilon. \tag{28}$$

Now, from Equation 9 it is obvious that $\mathbb{P}_{f_\epsilon}(N_2 \geq 6)$ is increasing in N and converges to 1 as N converges to infinity. That is, for every $\eta > 0$ there exists $\bar{N} \in \mathbb{N}$ such that $\mathbb{P}_{f_\epsilon}(N_2 \geq 6) \geq (1 - \eta)$ for all $N \geq \bar{N}$. Thus, for all $\eta \in (0, \frac{6\delta(\epsilon)}{K^*}]$, we can find an

$\bar{N} \in \mathbb{N}$ such that for all $N \geq \bar{N}$ we have

$$\mathbb{E}_f[\mu(m_1)|m_1 = 0, w_1 = 0] \geq \frac{1}{6}K^* > \epsilon \quad \forall f \in \mathcal{Y}_\epsilon. \tag{29}$$

Hence, combining Equation 12 and Equation 29 we obtain

$$\tilde{\beta}[f](0) > \epsilon \quad \forall f \in \mathcal{Y}_\epsilon, \tag{30}$$

which was to be shown. ■

Appendix B Proof of Theorem 2

Take an $\epsilon > a$ and N such that $\tilde{\beta}$ on $\mathcal{Y}_\epsilon$ admits a fixed point by Theorem 1. Assume that every agent follows a threshold strategy $f \in \mathcal{Y}_\epsilon$, not necessarily an equilibrium strategy. We are going to show that the theorem holds for $\tilde{\beta}[f]$, which in turn would imply that the theorem also holds for any equilibrium threshold strategy, since an equilibrium threshold strategy is just a fixed point of $\tilde{\beta}$. To that end, take a man i and, without loss of generality, set it to $i = 1$.

(i) Note that we have $\tilde{\beta}[f](a) > a$, which immediately follows from the fact that $\tilde{\beta}[f] \in \mathcal{Y}_\epsilon$. Now, if f is an equilibrium, then $\tilde{\beta}[f] = f > a$. For the second part, take a man $m_1 = b$ who meets with a woman w_1 in the first period. Recall that all agents strictly prefer to be matched with higher type. Thus, stability of μ implies that

$$\mathbb{E}_f[\mu(b)|b, w_1] = \mathbb{E}_f[\max\{W_2, w_1\}] \quad \forall w_1 \in [a, b]. \tag{31}$$

Moreover, since $W_2|f$ admits a strictly positive density over $[a, b]$, we have

$$\mathbb{E}_f[\mu(b)|b, w_1] > w_1 \quad \forall w_1 \in [a, b), \tag{32}$$

$$\mathbb{E}_f[\mu(b)|b, w_1] = w_1 \quad w_1 = b. \tag{33}$$

Now, note that $\tilde{\beta}[f](b) = \min\{w_1 \in [a, b] : \mathbb{E}_f[\mu(b)|b, w_1] \leq w_1\}$. Hence, by Equation (32) and Equation (33) we conclude that $\tilde{\beta}[f](b) = b$. Hence, if f is an equilibrium, then $\tilde{\beta}[f](b) = f(b) = b$. □

(ii) Fix a man $m_1 \in [a, b)$ who meets with a woman w_1 in period 1. Under any threshold strategy $f \in \mathcal{Y}_\epsilon$, there exists a strictly positive probability of being unmatched for man m_1 in the second period for every $w_1 \in (a, b]$. Too see this, for any component size of $C \geq 6$, we have

$$\mathbb{P}_f(W_2 > W_3, M_2 > m_1) > 0 \quad \mathbb{P}_f(M_{\frac{C}{2}} > m_1, W_{\frac{C}{2}} < w_1) > 0.$$

That is, there exists a strictly positive probability that (w_2, m_2) strictly prefer each other and $(w_1, m_{\frac{C}{2}})$ strictly prefer each other. Hence, for all $w_1 \in [a, b]$

we have

$$\mathbb{E}_f[\mu(m_1)|w_1, m_1] < b \quad \forall m_1 \in [a, b). \tag{34}$$

Next, given an m_1, consider a function ϕ defined as

$$\phi(w_1) \equiv \mathbb{E}_f[\mu(m_1)|m_1, w_1] - w_1. \tag{35}$$

We now claim that $\tilde{\beta}[f](m_1)$ is equal to the smallest argument that satisfies $\phi(\tilde{\beta}[f](m_1)) = 0$. To see this, recall that $\tilde{\beta}[f](m_1)$ is given by

$$\tilde{\beta}[f](m_1) = \min\{w_1 \in [a, b]|\mathbb{E}_f[\mu(m_1)|m_1, w_1] \leq w_1\} \quad \forall m_1 \in [a, b]. \tag{36}$$

Next, recall that the pdf of ex-ante type distribution, h, is assumed to be strictly positive and smooth. Moreover, f is weakly increasing. Thus, given any $m_1 \in [a, b]$, $\mathbb{E}_f[\mu(m_1)|m_1, w_1]$ is a continuous function of w_1, since w_1 only enters as a limit and integrand of corresponding well defined integrals. In fact, it is Lipschitz continuous. Therefore, by (i) and Equation (34), we can invoke intermediate value theorem and conclude that for all $m_1 \in [a, b]$ there exists a $w_1 \in [a, b]$ such that $\phi(w_1) = 0$. Moreover, we have also shown in the proof of Lemma 2 the function ϕ satisfies single crossing condition from above. Hence, all these results and Equation (36) imply that the $\tilde{\beta}[f](m_1)$ is the smallest argument that satisfies $\phi(\tilde{\beta}[f](m_1)) = 0$.[17][18]

Next, fix any $w_1 \in [a, b]$. Note that stability of μ implies that man m_1 cannot worse off by having a higher type, since all agents strictly prefer higher type. Thus, since h is assumed strictly positive and smooth, then the probability distribution function of $W_j|f$ is also strictly positive and smooth. Hence, it is sufficient to conclude that $\mathbb{E}_f[\mu(m_1)|w_1, m_1]$ is strictly increasing in m_1. This implies that $\tilde{\beta}[f]$ is strictly increasing. Hence, if f is an equilibrium, then f is strictly increasing since $\tilde{\beta}[f] = f$.

Moreover, note that given a $w_1 \in [a, b]$, $\mathbb{E}_\beta[\mu(m_1)|w_1, m_1]$ is continuous as a function of m_1, since m_1 only enters as a limit of corresponding integrals. In fact, it is Lipschitz continuous. Therefore, this implies that the smallest solution to Equation (35) changes continuously in m_1. Hence, $\tilde{\beta}[f]$ is continuous. Therefore, it implies that if f is an equilibrium threshold strategy, then f is continuous. □

(iii) Fix a man $m_1 \in [a, b)$ who meets with a woman w_1 in period 1. From man m_1's point of view, the type of his stable match in the second period, $\mu(m_1)$, is a random variable. Given, a threshold function f, m_1, w_1 and market size N, $\mu(m_1)$ can yield three different outcomes: Matching with first period partner, match-

[17] Note that this implies that if f is an equilibrium threshold strategy, then it satisfies Equation (1).

[18] This also implies that the two definitions of $\tilde{\beta}$ given by Equation (6) and Equation (2) are equivalent.

ing with second period partner, and being unmatched. Let, $\mathcal{E}_1(f, N, m_1, w_1)$, $\mathcal{E}_2(f, N, m_1, w_1)$, $\mathcal{E}_\emptyset(f, N, m_1, w_1)$ denote the events under which m_1 is matched with first period partner, second period partner and being unmatched, respectively. Therefore, we have

(a) $[\mu(m_1)|\mathcal{E}_1(f, N, m_1, w_1)] = w_1$

(b) $[\mu(m_1)|\mathcal{E}_2(f, N, m_1, w_1)] = [W_2|\mathcal{E}_2(f, N, m_1, w_1)]$

(c) $[\mu(m_1)|\mathcal{E}_\emptyset(f, N, m_1, w_1)] = a.$

Next, let $p_1(f, N, m_1, w_1)$, $p_2(f, N, m_1, w_1)$ and $p_\emptyset(f, N, m_1, w_1)$ denote the probabilities of the corresponding events. Hence, by the law of total probabilities:

$$\mathbb{E}_f[\mu(m_1)|m_1, w_1] = p_1(f, N, m_1, w_1)w_1 + p_2(f, N, m_1, w_1)\mathbb{E}[W_2|f, \mathcal{E}_2(f, N, m_1, w_1)] + p_\emptyset(f, N, m_1, w_1)a. \quad (37)$$

Now, note that $W_2|f$ admits strictly positive probability density function. Then, it is easy to verify that $\mathbb{E}[W_2|f, \mu(m_1) = W_2, w_1, m_1]$ is a continuous function of w_1 and m_1. Also, we have $\mathbb{E}[W_2|f, \mathcal{E}_2(f, N, m_1, w_1)] \in (a, b)$ for all $m_1, w_1 \in [a, b]$. Therefore, there exists a $y \in (a, b)$ such that

$$\max_{w_1, m_1 \in [a,b]} \mathbb{E}[W_2|\mathcal{E}_2(\beta, N, m_1, w_1)] = y < b. \quad (38)$$

Hence, for all $x \in [y, b)$ we can conclude:

$$\mathbb{E}_f[\mu(x)|x, x] \leq p_1(f, N, x, x)x + p_2(f, N, x, x)y + p_\emptyset(f, N, x, x)a < x, \quad (39)$$

since we have $p_1(f, N, x, x) > 0$, $p_2(f, N, x, x) > 0$ and $p_\emptyset(f, N, x, x) > 0$ for all $x \in [y, b)$. Thus, Equation (39) implies that $\tilde{\beta}[f](x) < x$ for all $x \in [y, b)$.

Finally, recall that we have previously shown $\tilde{\beta}[f](a) > a$, $\tilde{\beta}[f]$ is continuous and strictly increasing (see the proofs for (i) and (ii)). Therefore, by intermediate value theorem, there exists an $x^* < y$ such that $\tilde{\beta}[f](x^*) = x^*$ and $\tilde{\beta}[f](x) < x$ for all $x \in (x^*, b)$. The proof now follows from the fact that $\tilde{\beta}[f] = f$ if f is an equilibrium threshold strategy. □

■

Appendix C Proof of Theorem 3

We prove the theorem in two main steps. First, we assume full participation in period $t = 1$ and $t = 2$. Then, under this assumption, we prove existence of optimal symmetric threshold strategies for all participation cost $c < \bar{c}$ for some $\bar{c} \in (0, a)$. Then, under any optimal threshold strategy, we show that no remaining agent has incentive to deviate from participating in the second period market. Finally, we show that no agent has incentive to deviate from participating in the first period market.

To that end, assume full participation in both periods. Take a man i and set it to $i = 1$, without loss of generality. Fix a type $m_1 \in [a, b]$, and suppose that m_1 is meeting with a woman $w_1 \in [a, b]$ in period 1. Note that if all agents except $i = 1$ follow a symmetric threshold strategy f, then the m_1 decides to match early with w_1 if, and only if,

$$\mathbb{E}_f[\mu(m_1)|m_1, w_1] - \frac{c}{m_1} \leq w_1, \tag{40}$$

where c is a given per period cost. Now, recall that in the proof of Lemma 2, we have shown that for any fixed $m_1 \in [a, b]$ the following function

$$\mathbb{E}_f[\mu(m_1)|m_1, w_1] - w_1 \tag{41}$$

has a single crossing property from above as a function of $w_1 \in [a, b]$. Note that for any fixed $m_1 \in [a, b]$ this property still holds if we subtract any constant $\frac{c}{m_1}$ from Equation 41. That is, fixing any $m_1 \in [a, b]$, the following function

$$\mathbb{E}_f[\mu(m_1)|m_1, w_1] - w_1 - \frac{c}{m_1} \tag{42}$$

has the single crossing property (from above) as a function of w_1.

Also, note that we have proven following properties in Theorem 2:

(*i*) $\mathbb{E}_f[\mu(m_1)|m_1, w_1] > a \;\forall m_1, w_1 \in [a, b]$

(*ii*) $\mathbb{E}_f[\mu(m_1)|m_1, w_1] < b \;\forall m_1 < b, w_1 \in [a, b]$

(*iii*) $\mathbb{E}_f[\mu(m_1)|m_1, w_1] = b$ iff $m_1 = w_1 = b$

(*iv*) For all $w_1 \in [a, b]$, $\mathbb{E}_f[\mu(m_1)|m_1, w_1]$ is strictly increasing as a function of m_1

(*v*) $\mathbb{E}_f[\mu(m_1)|m_1, w_1]$ is continuous in m_1 and inw_1.

Now, for any $m_1 \in [a, b]$ and f, define a function ϕ given as

$$\phi(w_1) \equiv \mathbb{E}_f[\mu(m_1)|m_1, w_1] - w_1 - \frac{c}{m_1}. \tag{43}$$

Note that depending on $c > 0$ there may not exist a $w_1 \in [a, b]$ such that $\phi(w_1) = 0$. Given the properties $(i) - (v)$, the following condition, given eq. (44), is sufficient to guarantee the existence of a w_1 that guarantees $\phi(w_1) = 0$.

$$\mathbb{E}_f[\mu(a)|m_1 = a, w_1 = a] - a - \frac{c}{a} > 0, \tag{44}$$

Next, fix any $\epsilon > 0$ and consider the following functional map defined on $\mathcal{Y}_\epsilon$.

$$\tilde{\beta}[f](m_1) \equiv \min\{w_1 \in [a, b] : \mathbb{E}_f[\mu(m_1)|m_1, w_1] - w_1 - \frac{c}{m_1} \leq 0\} \quad \forall m_1 \in [a, b]. \tag{45}$$

Also, suppose the following condition holds.

$$\mathbb{E}_f[\mu(m_1)|m_1 = a, w_1 = a] - a - \frac{c}{a} > 0 \quad \forall f \in \mathcal{Y}_\epsilon. \tag{46}$$

Then, from the discussion above, properties $(i)-(v)$ and the single crossing condition imply that for all $m_1 \in [a, b]$ $\tilde{\beta}[f](m_1)$ is the smallest argument that makes ϕ equal to 0.

Moreover, by Claim 2 we have

$$\mathbb{E}_f[\mu(m_1)|m_1, w_1 = a] < \mathbb{E}_f[\mu(m_1)|m_1, w_1 = x] \quad \forall x \in (a, b]. \tag{47}$$

Therefore, if Equation 46 holds, then Equation 47 implies that

$$\tilde{\beta}[f](a) \geq \mathbb{E}_f[\mu(m_1)|m_1 = a, w_1 = a] - \frac{c}{a} \quad \forall f \in \mathcal{Y}_\epsilon. \tag{48}$$

Moreover, if there exists an $\epsilon > a$ such that

$$\tilde{\beta}[f](a) \geq \epsilon \quad \forall f \in \mathcal{Y}_\epsilon, \tag{49}$$

then we can conclude that the mapping $\tilde{\beta}$ given in Equation 45 is a well defined self map. Indeed, we are going to show there exists a $\epsilon > a$ and $\bar{c} > 0$ such that for all $c \leq \bar{c}$, the equation Equation 46 and Equation 49 holds, and, thus, the mapping $\tilde{\beta}$ given in Equation 45 is a well defined self map.

To that end, first note that we have $a > 0$ and by Assumption 1 matching with type a is same as being unmatched. Moreover, a man of type $m_1 = a$ who meets with a woman of type $w_1 = a$ in period 1 matches with w_1 with probability 1 whenever m_1 is in a component of size 4. Taking these into account, it is easy to verify that Equation 24 given in the proof of Lemma 4 simply becomes

$$\mathbb{E}_f[\mu(m_1)|m_1 = a, w_1 = a] > \frac{5a}{6} + \mathbb{P}_{f_\epsilon}(N_2 \geq 6)\frac{1}{6}K^* \quad \forall f \in \mathcal{Y}_\epsilon,\ \epsilon \in (a, K^*] \text{ and } N \geq 6. \tag{50}$$

Where in Equation (50), K^* is given by

$$K^* = \lim_{c \to a} \mathbb{E}_{f_c}[W_2|M_2 \geq M_3, W_3 \geq W_2] > a. \tag{51}$$

Note that the assumptions imposed on probability distribution function h are sufficient to guarantee $K^* > a$. Now, fix any $\epsilon \in (a, \frac{5a}{6} + \frac{K^*}{6})$. It is shown that we can find a $\delta(\epsilon) > 0$ such that

$$\delta(\epsilon) = \inf_{f \in \mathcal{Y}_\epsilon, N \geq 6} (\mathbb{E}_f[\mu(m_1)|m_1 = a, w_1 = a] - \frac{5a}{6} - \mathbb{P}_{f_\epsilon}(N_2 \geq 6)\frac{1}{6}K^*) > 0. \tag{52}$$

Hence, for all $N \geq 6$ we can write

$$\mathbb{E}_f[\mu(m_1)|m_1 = a, w_1 = a] \geq \frac{5a}{6} + \mathbb{P}_{f_\epsilon}(N_2 \geq 6)\frac{1}{6}K^* + \delta(\epsilon) \quad \forall f \in \mathcal{Y}_\epsilon. \tag{53}$$

Note that by Equation 9 $\mathbb{P}_{f_\epsilon}(N_2 \geq 6)$ is increasing in N and converges to 1 as N converges to infinity. That is, for every $\eta > 0$ there exists $\bar{N} \in \mathbb{N}$ such that $\mathbb{P}_{f_\epsilon}(N_2 \geq 6) \geq (1-\eta)$ for all $N \geq \bar{N}$. Thus, for all $\eta \in (0, \frac{3\delta(\epsilon)}{K^*}]$, we can find an $\bar{N} \in \mathbb{N}$ such that the following condition holds

$$\mathbb{E}_f[\mu(m_1)|m_1 = a, w_1 = a] \geq \frac{5a}{6} + \frac{1}{6}K^* + \frac{\delta(\epsilon)}{2} \quad \forall f \in \mathcal{Y}_\epsilon \text{ and } N \geq \bar{N}. \tag{54}$$

Therefore, by setting a $\bar{c} < \frac{a\delta(\epsilon)}{2}$ we can conclude that Equation 46 and Equation 49 hold for all $c \leq \bar{c}$. That is, there exists an $\bar{N} \in \mathbb{N}$ such that for all market size $N \geq N$ there exists an $\epsilon > a$ and $\bar{c} > 0$ such that for all $c < \bar{c}$ the mapping $\tilde{\beta}$ defined in Equation 45 is a well defined self map. Then, since $\tilde{\beta}$ is continuous in L^1 and $\mathcal{Y}_\epsilon$ is compact, convex and non-empty, there exists a fixed point of $\tilde{\beta}$.

Next, take any corresponding fixed point β of $\tilde{\beta}$ given that the participation cost is some $c \leq \bar{c}$. Note that β is an optimal threshold strategy under the assumption of full participation in both periods.

We now show that under the assumption of full participation in the first period, and given the threshold strategy β, no agent has incentive to deviate from participating in the second period market with participation cost c. To that end, take a man m_1 who meets with a woman w_1 in period 1 but not matched early under β. There are two cases to consider:

Case 1: $\beta(m_1) > w_1$. In this case, man m_1's utility from not participating in the second period market is at most $m_1 w_1$. On the other hand, $\mathbb{E}_\beta[\mu(m_1)|m_1, w_1]$ is the expected type of m_1's stable partner if he participates in the second period market. Note that, since β is a fixed point of $\tilde{\beta}$, we have $m_1\mathbb{E}_\beta[\mu(m_1)|m_1, w_1] - c > m_1 w_1$ (see Equation (45)). Thus, utility from participating second period market is strictly grater than not participating.

Case 2: $\beta(m_1) \leq w_1$ and $m_1 < \beta(w_1)$. If man m_1 does not participate in costly second period market, then he will be unmatched. Again, we have $m_1\mathbb{E}_\beta[\mu(m_1)|m_1, w_1] - c > am_1$ by construction. Thus, utility from participating second period market is strictly grater than not participating.

Thus, we have shown that, assuming full participation in period 1 matching market, there exists a $\bar{c} < \frac{a\delta(\epsilon)}{2}$ such that for all $c \in (0, \bar{c})$, there exists an optimal threshold strategy β such that no remaining agent has incentive to deviate from participating in the second period market.

Now, to complete the proof, we need to show that the upper bound of participation cost, $\bar{c}$, is such that no agent has incentive to deviate from participating in the first period market.

To see this, note that man m_1's expected type of his match before he participates the first period market is greater or equal to[19]

$$\int_a^b \mathbb{E}_\beta[\mu(m_1)|m_1, w_1]h(w_1)dw_1 \quad \forall m_1 \in [a, b], \tag{55}$$

where β is any optimal threshold strategy with an associated per period cost c. Expected type of m_1's match conditional on not participating in the first period market is given by

$$\mathbb{E}_\beta[\mu(m_1)|m_1, a]. \tag{56}$$

That is, not participating in the first period market is as if agent m_1 meets with $w_1 = a$ in the first period, since matching with type a is equal to being unmatched. Now, by Claim 2 we have

$$\mathbb{E}_f[\mu(m_1)|m_1, a] < \mathbb{E}_f[\mu(m_1)|m_1, w_1] \quad \forall m_1 \in [a, b], w_1 \in (a, b] \text{ and } f \in \mathcal{Y}_\epsilon. \tag{57}$$

Since h, the pdf of ex-ante type distribution, is strictly positive, then

$$\int_a^b \mathbb{E}_f[\mu(m_1)|m_1, w_1]h(w_1)dw_1 - \mathbb{E}_f[\mu(m_1)|m_1, a] > 0 \quad \forall m_1 \in [a, b], f \in \mathcal{Y}_\epsilon \tag{58}$$

Now, the left hand-side of Equation 58 is a continuous function of m_1 defined on a compact interval $[a, b]$. Then, we have

$$\min_{m_1 \in [a,b]} \left\{ \int_a^b \mathbb{E}_f[\mu(m_1)|m_1, w_1]h(w_1)dw_1 - \mathbb{E}_f[\mu(m_1)|m_1, a] \right\} > 0 \quad \forall f \in \mathcal{Y}_\epsilon \tag{59}$$

Similarly, the expectation is continuous as a function of f defined on a compact set $\mathcal{Y}_\epsilon$. Then, we have

$$C_1(\epsilon)^* = \min_{f \in \mathcal{Y}_\epsilon} \min_{m_1 \in [a,b]} \left\{ \int_a^b \mathbb{E}_f[\mu(m_1)|m_1, w_1]h(w_1)dw_1 - \mathbb{E}_f[\mu(m_1)|m_1, a] \right\} > 0. \tag{60}$$

We use $C_1(\epsilon)^*$ to denote this minimum since it is related with period 1 participation. Therefore, by setting $\bar{c} < \min\{\frac{a\delta(\epsilon)}{2}, \frac{aC_1(\epsilon)^*}{2}\}$, for all $c \in (0, \bar{c})$ there exists an optimal threshold function β such that no agent has incentive to deviate from participating in the second period matching market. Moreover, Equation (60) implies that for all

[19] Note that under a non trivial threshold strategy β, some type m_1 matches early after meeting with some w_1. In that case, the utility from matching early is higher than the second period stable match. Thus, Equation 55 is a lower bound under β.

$c \in (0, \bar{c})$ the following condition holds

$$m_1 \int_a^b \mathbb{E}_\beta[\mu(m_1)|m_1, w_1]h(w_1)dw_1 - 2c > m_1\mathbb{E}_\beta[\mu(m_1)|m_1, a] \quad \forall m_1 \in [a, b], \quad (61)$$

which in turn implies that no agent has incentive to deviate from participating in the first period matching market. Thus, this completes the proof of Theorem 3. ■

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