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Are the liquidity and collateral roles of asset bubbles different?

Lise Clain-Chamosset-Yvrard*, Xavier Raurich† and Thomas Seegmuller‡

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Abstract

Several recent papers introduce different mechanisms to explain why asset bubbles are observed in periods of larger growth. These papers share common assumptions, heterogeneity among traders and credit market imperfection, but differ in the role of the bubble, used to provide liquidities or as collateral in a borrowing constraint. In this paper, we introduce heterogeneous traders by considering an overlapping generations model with households living three periods. Young households cannot invest in capital, while adults have access to investment and face a borrowing constraint. Introducing bubbles in a quite general way, encompassing the different roles they have in the existing literature, we show that the bubble may enhance growth when the borrowing constraint is binding. More significantly, our results do not depend on the - liquidity or collateral - role attributed to the bubble. We finally extend our analysis to a stochastic bubble, which may burst with a positive probability. Because credit and bubble are no more perfectly substitutable assets, the liquidity and collateral roles of the bubble are not equivalent. Growth is larger when bubbles play the liquidity role, because the burst of a bubble used for liquidity is less damaging to agents who invest in capital.

JEL classification: D15; E44; G11.

Keywords: Bubble; Liquidity; Collateral; Crowding-in effect; Growth.

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1 Introduction

The financial crises of recent years have led to a renewed interest in the study of the interplay between the financial and real spheres of the economy. In particular, several contributions try to understand why episodes of speculative bubbles are associated to periods of economic expansions and why bubble crashes are sources of recession. These phenomena are illustrated in several works, for example, in Caballero *et al.* (2006), Martin and Ventura (2012), Brunnermeier *et al.* (2013) and Kindleberger and Aliber (2015). These contributions are also challenging because seminal papers show that the existence of rational bubbles in dynamic general equilibrium models is associated to lower GDP per capita (Tirole (1985)) or growth (Grossman and Yanagawa (1993)). This is the so-called crowding-out effect of the bubble.

Most of the papers that try to reconcile the existence of rational bubbles with the empirical facts introduce financial imperfections embodied in borrowing constraints (see Miao (2014) for a short survey) and heterogeneous agents to have different types of traders on the asset markets. The recent and growing literature about rational bubbles with financial frictions distinguishes between two growth-enhancing roles of the bubbles or, equivalently, crowding-in effects. One is the liquidity role of the bubble: agents hold at the beginning of the period the bubble and sell it to increase their productive investment (Caballero and Krishnamurthy (2006), Kocherlakota (2009), Farhi and Tirole (2012), Martin and Ventura (2012), Hirano and Yanagawa (2017), Kiyotaki and Moore (2018)).[1] The other one is the collateral role of the bubble: agents buy the bubble to increase their possibilities to borrow and use these loans to invest in capital (Kocherlakota (2009), Miao *et al.* (2015), Martin and Ventura (2016), Bengui and Phan (2018), Miao and Wang (2018)).

There are different ways to introduce heterogeneous traders. For instance, Kocherlakota (2009), Martin and Ventura (2012), Hirano and Yanagawa (2017), Kiyotaki and Moore (2018) consider heterogeneous investment projects associated to heterogeneous agents born at the same period, *i.e.* unproductive agents *vs.* productive ones. Another possibility is to consider overlapping generations with agents living three periods, as in several recent papers like Arce and Lopez-Salido (2011), Farhi and Tirole (2012), Basco (2014, 2016) or Raurich and Seegmuller (2019), among others. These contributions also consider heterogeneous investment projects, but among agents born at different periods. Heterogeneity among agents in terms of investment projects, together with credit market imperfections, makes room for an asset market that channels liquidities from unproductive agents (lenders-savers) to productive ones (borrowers-investors), which is essential for the existence of the crowding-in effect of the bubble.

In this paper, we study the differences between the liquidity and collateral

[1] Note that this liquidity role has also been emphasized in a different perspective by Woodford (1990). Instead of being concerned with bubbles, he focuses on non-neutrality of public debt. In their paper, Kiyotaki and Moore (2018) are interested in the liquidity role of fiat money, which can be seen as a rational bubble. Fiat money allows unproductive entrepreneurs to transfer some liquidities towards productive ones, who have an investment opportunity.

roles of the bubble. Our purpose is to contribute to the literature on bubbles by identifying the mechanisms behind the crowding-in effect of a bubble.

To address these issues, we consider a three-period lived agents model. We distinguish among three types of traders (young, adult and old), while only two of them can buy assets and invest in capital (young and adult). To introduce heterogeneous traders, we assume that young households do not hold capital, while adults invest in this asset expecting a positive return. It means that adults are the most and only productive investors. At each period of time, there is also a credit market in which young households and adults can save and borrow. The amount of credit is limited by a borrowing constraint.

We start by introducing bubbles considering two examples in which, at the second period of life, borrowing is constrained and collateralized by capital, *i.e.* a fundamental collateral. In the first one, the bubble is bought by young households and sold by adults. Therefore, adults can sell the bubble to invest more in capital. Selling the bubble corresponds to a transfer from the unproductive young agents to productive adults. This mechanism extends to a general equilibrium framework the liquidity effect of the bubble developed in Farhi and Tirole (2012) and it is also in line with many other existing papers like Hirano and Yanagawa (2017), Kocherlakota (2009) or Martin and Ventura (2012). In the second example, following Kocherlakota (2009) or Martin and Ventura (2016), the bubble is only bought by adults and used as a collateral in the borrowing constraint. By increasing the collateral, the bubble increases the amount borrowed, promoting a higher investment in capital. These two examples illustrate the two different roles of bubbles. We show that these two approaches lead to exactly the same equilibrium, despite the fact that the two mechanisms of the bubble seem to be *a priori* different.

Then, we propose a general model that encompasses the two previous examples and in which bubbles may have both the liquidity and collateral roles. To this end, we assume that both young and adult households may buy or sell short the bubble. Of course, the bubble bought by adults still plays the role of a bubbly collateral and capital plays the role of a fundamental collateral in the borrowing constraint. To fix ideas and to be able to analyze the dynamics in a simple way, firms produce the final good using an Ak technology, which implies endogenous growth.

We first analyze the model without borrowing constraint. All the assets, capital, credit and bubble, are perfect substitutes and households can smooth consumption without restriction. Despite the fact that there are heterogeneous traders, the bubble has a crowding-out effect on growth. As in the seminal contribution of Grossman and Yanagawa (1993), the bubbly BGP is always characterized by a lower growth than the bubbleless one.

When the borrowing constraint is binding, capital is no more perfectly substitutable with the two other assets and households cannot smooth consumption perfectly between adult and old ages. The bubble promotes investment and has a positive effect on growth when the constraint is binding, whereas the resulting increase in the interest rate has a negative effect when capital is used as collateral. When the degree of pledgeability of the fundamental collateral is small enough, the first effect dominates and the bubble enhances growth, *i.e.* has a crowding-in effect. On the contrary, when the degree of pledgeability is sufficiently large, the bubble has a crowding-out effect on growth. The main conclusion of our paper is that these results do not depend on the particular type of bubble considered. This means that there is no distinction between the liquidity and the collateral roles of the bubble.

We finally extend our analysis to a stochastic case where bubble burst occurs with a positive probability (Weil (1987)). In contrast to the model with a deterministic bubble, credit and bubbles are no more perfectly substitutable assets. Therefore, a general approach of bubbles is no more relevant, but the liquidity and collateral roles can still be compared when the borrowing constraint is binding. When a household buys the bubble for its liquidity role at the young age, she faces more risk in terms of consumption than when she buys it for its collateral role at her adult age. Therefore, the bubble has a lower size when it has a liquidity role. The opposite conclusion applies for capital investment and growth: the bubble bought for its collateral role generates more risk for capital investment because of the borrowing constraint, which depends on the expected return of the bubble. Therefore, when the bubble is stochastic, the liquidity and collateral roles are no more equivalent. A bubble used as its liquidity role generates a higher growth than a bubble used as a collateral, because the possible bubble crash is less damaging to agents who invest in capital.

In the following section, we present the two examples of models in which the bubble is either bought when young and used to provide liquidities or used as a collateral when adult. In Section 3, we introduce our general model of bubbles, considering first the framework without binding borrowing constraint. A binding borrowing constraint is introduced in a second step to show its role on the crowding-in effect of the bubble whatever the type of bubble considered. In Section 4, we extend our analysis to a stochastic bubble. Section 5 concludes and technical details are relegated to the Appendix.

2 Two examples with liquidity and collateral roles of bubbles

To motivate our general approach of bubbles, we start by presenting two models with heterogeneous traders and borrowing constraints. These heterogeneous traders can be introduced in models with infinitely-lived agents (Kocherlakota (2009), Hirano and Yanagawa (2017)) or two-period lived agents (Martin and Ventura (2012, 2016)). As in several recent papers (Arce and Lopez-Salido (2011), Basco (2014, 2016), Farhi and Tirole (2012), Raurich and Seegmuller (2019)), we introduce heterogeneity among agents in terms of investment projects considering an overlapping generations model with three-period lived agents. Therefore, agents may invest both when young or adult. In accordance with Farhi and Tirole (2012), young households do not invest in capital, while adults expect a positive return for their capital investment.[2] It means that adults are the most and only productive investors. This framework allows us to consider the two roles of the bubble mentioned above. In the first model we present, the bubble has a liquidity role when an adult household sells the bubble bought when young to invest in capital. In the second model, the bubble is bought by adult agents, and it plays the role of a collateral in the borrowing constraint.[3]

2.1 Model with bubble bought by young savers, YS

In this first model, that we denote as YS because young agents are savers, we illustrate the liquidity role of the bubble. Agents buy the bubble when they are savers, and when they become investors, they sell it to increase investment in capital. While this liquidity role was introduced in Farhi and Tirole (2012), we consider it in a general equilibrium framework. The mechanism behind this example is also in line with many other existing papers like Hirano and Yanagawa (2017), Kocherlakota (2009), or Martin and Ventura (2012).

We consider an overlapping generations economy populated by agents living for three periods. An agent is young in the first period of life, adult in the second period and old in the third period. There is no population growth. The population size of a generation is constant and normalized to one.

Each household derives utility from consumption at each period of time. Preferences of an individual born in period t are represented by the following utility function:

$$\alpha u_1(c_{1t}) + \beta u_2(c_{2t+1}) + \gamma u_3(c_{3t+2}) \tag{1}$$

where $\alpha, \beta, \gamma > 0$ and the utility functions $u_i(c_i)$ are well defined on $\mathbb{R}_+$, strictly increasing ($u_i'(c_i) > 0$) and concave ($u_i''(c_i) < 0$) on $\mathbb{R}_{++}$.

The household inelastically supplies one unit of labor when young and adult. When young, labor efficiency is one, while it is equal to $\phi > 0$ when adult.

There are three assets in the economy: capital k_t used in the production, deposits d_{it} that allow to finance loans, and an asset without fundamental value supplied in one unit, with a price b_{1t}. There is a bubble as soon as $b_{1t} > 0$.

When young, the household saves through deposits d_{1t} and can buy the bubble b_{1t}. In the next period, these two assets provide returns given by R^d_{t+1} and r_{t+1}, respectively. When adult, the household can only save through deposits d_{2t+1} and invests in capital k_{t+2}. When old, these two assets are remunerated with the returns R^d_{t+2} and q_{t+2}, respectively. Of course, when $d_{it} < 0$, the household rather contracts loans. When adult, these loans are limited by the following borrowing constraint:

$$-R^d_{t+2} d_{2t+1} \leqslant \theta q_{t+2} k_{t+2} \tag{2}$$

[2] Note that in a recent paper, Raurich and Seegmuller (2019) already investigate what happens when the young invest in capital, while adults do not have access to the capital market.

[3] In this paper, agents that trade the different assets are identified as households, whereas some papers rather speak about entrepreneurs (Kocherlakota (2009), Farhi and Tirole (2012), Hirano and Yanagawa (2017)). The difference only concerns the denomination of the agents.

where $\theta \in [0,1)$ is the degree of pledgeability. This constraint means that, when adult, the household can borrow an amount $d_{t+2} < 0$, as long as the repayment does not exceed a fraction θ of the future return from her productive investment at period $t+2$. The parameter θ is related to the financial market imperfection, where a lower θ means a stronger imperfection. Young agents will not face an equivalent constraint, because they will not be short sellers of the liquid assets, *i.e.* deposits and the bubble, but rather use them to make transfers to the next periods of life.

The budget constraints when young, adult and old faced by a household born in period t are, respectively:

$$c_{1t} + b_{1t} + d_{1t} = w_t \tag{3}$$

$$c_{2t+1} + k_{t+2} + d_{2t+1} = \phi w_{t+1} + r_{t+1} b_{1t} + R^d_{t+1} d_{1t} \tag{4}$$

$$c_{3t+2} = q_{t+2} k_{t+2} + R^d_{t+2} d_{2t+1} \tag{5}$$

The household maximizes the utility (1) under the constraints (2)-(5). We focus on equilibria where the borrowing constraint is binding. Using the first order conditions, we deduce that $r_{t+1} = R^d_{t+1}$ for all $t \geq 0$, and:

$$u_1'(c_{1t}) = \frac{\beta}{\alpha} r_{t+1} u_2'(c_{2t+1}) \tag{6}$$

$$u_2'(c_{2t+1}) = \frac{\gamma}{\beta} u_3'(c_{3t+2}) \frac{q_{t+2}(1-\theta)}{1-\theta q_{t+2}/r_{t+2}} \tag{7}$$

In addition, the borrowing constraint is binding if:

$$u_2'(c_{2t+1}) > R^d_{t+2} \frac{\gamma}{\beta} u_3'(c_{3t+2}) \tag{8}$$

Using (7), this is ensured by $q_{t+2} > r_{t+2} > \theta q_{t+2}$. Note that the equilibrium cannot satisfy $r_{t+2} > q_{t+2}$ because, in this case, adults will not invest in capital since d_{2t+1} gives a higher return. Note also that $r_{t+2} < \theta q_{t+2}$ cannot occur as it would imply that an adult could borrow an infinite amount to invest in capital without being constrained.

Finally, the bubble evolves according to:

$$b_{1t+1} = r_{t+1} b_{1t} \tag{9}$$

where r_{t+1} also measures the growth of the asset price bubble b_{1t+1}/b_{1t}.

Because of the binding borrowing constraint, deposits and capital have not the same return, while deposits and the bubble are perfectly substitutable assets. Using the binding borrowing constraint and the market clearing on deposits $d_{1t} + d_{2t} = 0$, we have $-d_{2t} = d_{1t} = \frac{\theta q_{t+1} k_{t+1}}{r_{t+1}}$. Then, using the budget constraints (3)-(5), the consumptions are given by:

$$c_{1t} = w_t - b_{1t} - \frac{\theta q_{t+1} k_{t+1}}{r_{t+1}} \tag{10}$$

$$c_{2t+1} = \phi w_{t+1} + b_{1t+1} + \theta q_{t+1} k_{t+1} - k_{t+2} \left(1 - \frac{\theta q_{t+2}}{r_{t+2}}\right) \tag{11}$$

$$c_{3t+2} = (1-\theta) q_{t+2} k_{t+2} \tag{12}$$

We will obtain an equilibrium of this economy substituting these three equations in the two arbitrage conditions (6) and (7).

2.2 Model with bubble bought by adult investors, AI

This model, that we call AI because adults are investors, illustrates the collateral role of the bubble introduced by Kocherlakota (2009) and Martin and Ventura (2016). Investors buy the bubble to use it as collateral for credits that finance investment in capital.

The model is the same as the YS, except for the bubble. There is still an asset without fundamental value supplied in one unit, but it is not bought by young households and sold by adults. Instead, it is bought by adults at the price b_{2t+1} and sold by old households, taking into account that its return is R_{t+2}. Therefore, the budget constraints become:

$$c_{1t} + d_{1t} = w_t \tag{13}$$

$$c_{2t+1} + k_{t+2} + d_{2t+1} + b_{2t+1} = \phi w_{t+1} + R^d_{t+1} d_{1t} \tag{14}$$

$$c_{3t+2} = q_{t+2} k_{t+2} + R^d_{t+2} d_{2t+1} + R_{t+2} b_{2t+1} \tag{15}$$

Since the household may buy the bubble when she also invests in capital, we follow Martin and Ventura (2016) assuming that the bubble is also used as collateral to borrow. We have now both fundamental and bubbly collaterals. The borrowing constraint writes now:

$$-R^d_{t+2} d_{2t+1} \leqslant \theta q_{t+2} k_{t+2} + R_{t+2} b_{2t+1} \tag{16}$$

The household can borrow an amount $d_{t+2} < 0$, as long as the repayment does not exceed θ of the future return from her productive investment, *i.e.* the fundamental collateral, and the market value of the bubble at period $t + 2$, *i.e.* the bubbly collateral. Note that the degree of pledgeability of the bubbly collateral is one. This is in accordance with Martin and Ventura (2016) and Kocherlakota (2009). In fact, Kocherlakota (2009) introduces a constraint where only the bubble serves as collateral. It corresponds to the case where $\theta = 0$. As in these papers, we will show that, with the borrowing constraint (16), debt d_{2t+1} and the bubble b_{2t+1} are perfect substitutes.

Solving the household problem, we find $R^d_{t+2} = R_{t+2}$ for all $t \geq 0$. When the borrowing constraint is binding, we also get the first order conditions (6)-(8) substituting r_{t+i} by R_{t+i}. They imply that the borrowing constraint is binding when $q_{t+2} > R_{t+2} > \theta q_{t+2}$.

As in the model YS, contrary to deposits and capital, deposits and the bubble are perfectly substitutable assets. At an equilibrium, the binding borrowing constraint and the market clearing condition on deposits, $d_{1t} + d_{2t} = 0$, imply that:

$$-d_{2t} = d_{1t} = \frac{\theta q_{t+1} k_{t+1}}{R_{t+1}} + b_{2t} \tag{17}$$

Moreover, the bubble evolves according to:

$$b_{2t+1} = R_{t+1} b_{2t} \tag{18}$$

Using (17) and (18) and the budget constraints (13)-(15), we derive the same expressions for c_{1t}, c_{2t+1} and c_{3t+2} than (10)-(12), replacing b_{1t} by b_{2t} and r_{t+1} by R_{t+1}. Therefore, the equilibrium is defined exactly by the same equations than in the previous model.

2.3 In short

The equilibrium of the models YS and AI is characterized by the same equations, which means that they share the same equilibrium. There is a perfect equivalence between the models YS and AI, despite the fact that the role of the bubble is *a priori* different. In the model YS, the bubble is introduced to provide liquidity to the investors, *i.e.* adult households. In contrast, in the model AI, the bubble is bought by adult/investors that use it as a collateral to borrow. Therefore, the bubble increases the amount borrowed by the adult and, as a result, the deposits of the young also increase.

In fact, the effect of the bubble on the savings of the young coincide in the two models. To see this, note that in the AI model, they are given by $d_{1t} = \theta q_{t+1} k_{t+1}/R_{t+1} + b_{2t}$, whereas in the YS model, they are given by $d_{1t} + b_{1t} = \theta q_{t+1} k_{t+1}/r_{t+1} + b_{1t}$. It explains that the two mechanisms play exactly the same role and the bubble has finally the same effect in both models, as follows from the fact that the reduced forms of the two models are identical.

3 A general approach with bubble bought by young savers and adult investors

To confirm this equivalence result, we develop now a general model that encompasses both formulations described above. It admits the models YS and AI as particular cases. Moreover, it allows us to study the crowding-in effect of the bubble in a more general way. In the following, we start by presenting the production. Then, we will present our general framework. We consider first that there is no binding borrowing constraint to show that bubbles cannot enhance growth even if there are heterogeneous traders. Finally, we introduce a borrowing constraint and discuss in detail the crowding-in versus crowding-out effects of the bubble.

3.1 Production sector

To simplify the dynamic analysis, we introduce a simple Ak type technology. Aggregate output is produced by a continuum of firms, using labor, l_t, and capital, k_t, as inputs. In addition, production benefits from an externality that summarizes a learning-by-doing process, and allows to have sustained growth.

Following Frankel (1962) or Ljungqvist and Sargent (2004, chapter 14), this externality depends on the average capital-labor ratio.

Letting $a_t \equiv k_t/l_t$, $\bar{a}_t$ represents the average ratio of capital over labor. Firms produce the final good using the following technology:

$$y_t = F(k_t, \bar{a}_t l_t)$$

The technology $F(k_t, \bar{a}_t l_t)$ has the usual neoclassical properties, *i.e.* a strictly increasing and concave production function satisfying the Inada conditions, and ~~is~~ homogeneous of degree one with respect to its two arguments.

Profit maximization under perfect competition implies that the wage w_t and the return of capital q_t are given by:[4]

$$w_t = F_2(k_t, \bar{a}_t l_t)\bar{a}_t \tag{19}$$

$$q_t = F_1(k_t, \bar{a}_t l_t) \tag{20}$$

All equilibria we will consider are symmetric ones, *i.e.* $a_t = \bar{a}_t$. Let us define $s \equiv F_1(1,1)/F(1,1) \in (0,1)$ the capital share in total production and $A \equiv F(1,1) > 0$. Using (19) and (20), we deduce that:

$$w_t = (1-s)Aa_t \equiv w(a_t) \tag{21}$$

$$q_t = sA \tag{22}$$

which give the wage and the return of capital at an equilibrium.

3.2 The model without binding borrowing constraint

We generalize the models developed in Section 2 by providing a framework that encompasses them. To disentangle between the roles of heterogeneous traders and borrowing constraints, we first consider a model with a perfect credit market.

The main change with respect to the YS and AI models concerns the asset without fundamental value, which is a bubble if it is positively valued. There are two possible interpretations of this asset consistent with our framework. One possibility is to assume that there are two assets without fundamental value supplied in one unit each one. They are bubbles if their prices are positive. In this case, a household can buy one of these two assets when young at price b_{1t} and the other one when adult at price b_{2t+1}. They have *a priori* different returns, denoted by r_{t+1} and R_{t+2}, respectively. Another possibility is to assume that there is only one asset without fundamental value supplied in one unit. Then, b_{1t} and b_{2t+1} represent the price times the share of this asset bought when young and adult, respectively. Of course, in such a case, the returns of b_{1t} and b_{2t} are the same, *i.e.* $r_t = R_t$ for all t. Moreover, having $b_{it} < 0$ for $i = 1$ or $i = 2$ becomes possible, and means that the household is a short-seller of this asset at time t.

[4] We denote by $F_i(.,.)$ the derivative with respect to the ith argument of the function.

Since bubbles can be purchased in the first two periods of life, the budget constraints write:

$$c_{1t} + d_{1t} + b_{1t} = w_t \tag{23}$$
$$c_{2t+1} + k_{t+2} + d_{2t+1} + b_{2t+1} = \phi w_{t+1} + R^d_{t+1} d_{1t} + r_{t+1} b_{1t} \tag{24}$$
$$c_{3t+2} = q_{t+2} k_{t+2} + R^d_{t+2} d_{2t+1} + R_{t+2} b_{2t+1} \tag{25}$$

These budget constraints encompassed the constraints (3)-(5) and (13)-(15), as particular cases.

A household maximizes the utility (1) under the budget constraints (23)-(25). Solving the household problem, we deduce that all assets are substitutable, *i.e.* $q_{t+1} = R^d_{t+1} = r_{t+1} = R_{t+1}$, and therefore:

$$u_1'(c_{1t}) = \frac{\beta}{\alpha} R_{t+1} u_2'(c_{2t+1}) \tag{26}$$
$$u_2'(c_{2t+1}) = \frac{\gamma}{\beta} R_{t+2} u_3'(c_{3t+2}) \tag{27}$$

Let us introduce $x_{1t} = b_{1t} + d_{1t}$ and $x_{2t} = b_{2t} + d_{2t}$. This means that $x_{1t} + x_{2t} = b_{1t} + b_{2t}$ because $d_{1t} + d_{2t} = 0$ at the equilibrium on the deposit market. Therefore, we distinguish between two situations:

- $x_{1t} + x_{2t} = 0$ if there is no bubble, *i.e.* $b_{1t} = b_{2t} = 0$;
- $x_{1t} + x_{2t} > 0$ if there is a bubble on at least one asset, *i.e.* $b_{1t} > 0$ and/or $b_{2t} > 0$, or if b_{1t} and b_{2t} represent the same asset, in which case a bubble exists if $b_{1t} + b_{2t} > 0$, but either b_{1t} or b_{2t} may be negative.

In any case, the assets evolve according to:

$$x_{1t+1} + x_{2t+1} = R_{t+1}(x_{1t} + x_{2t}) \tag{28}$$

Note that this equation encompasses the different form of bubbles introduced in the models YS and AI, and is even more general.

Using (23)-(25) and the equality between the different asset returns, the life-cycle budget constraint writes:

$$c_{1t} + \frac{c_{2t+1}}{R_{t+1}} + \frac{c_{3t+2}}{R_{t+1}R_{t+2}} = w_t + \frac{\phi w_{t+1}}{R_{t+1}} \tag{29}$$

When the utility is log-linear, *i.e.* $u_i(c_i) = \ln c_i$ for $i = 1, 2, 3$, we easily deduce that the consumptions are given by:

$$c_{1t} = \frac{\alpha}{\alpha + \beta + \gamma} \left(w_t + \frac{\phi w_{t+1}}{R_{t+1}} \right) \tag{30}$$
$$c_{2t+1} = \frac{\beta R_{t+1}}{\alpha + \beta + \gamma} \left(w_t + \frac{\phi w_{t+1}}{R_{t+1}} \right) \tag{31}$$
$$c_{3t+2} = \frac{\gamma R_{t+1} R_{t+2}}{\alpha + \beta + \gamma} \left(w_t + \frac{\phi w_{t+1}}{R_{t+1}} \right) \tag{32}$$

Using the budget constraints (23)-(25), $x_{1t} = b_{1t} + d_{1t}$ and $x_{2t} = b_{2t} + d_{2t}$, we obtain:

$$x_{1t} = \frac{\beta+\gamma}{\alpha+\beta+\gamma} w_t - \frac{\alpha\phi w_{t+1}/R_{t+1}}{\alpha+\beta+\gamma} \tag{33}$$

$$k_{t+2} + x_{2t+1} = \frac{\gamma}{\alpha+\beta+\gamma}(\phi w_{t+1} + R_{t+1} w_t) \tag{34}$$

Taking into account that the population size of each generation is constant and normalized to one, the equilibrium in the labor market in efficient units requires $l_t = 1+\phi$. Hence, we have $k_{t+1} = a_{t+1} l_{t+1} = a_{t+1}(1+\phi)$.

Let us define $\widetilde{x}_{it} \equiv x_{it}/[(1+\phi)a_t]$ and $g_{t+1} \equiv a_{t+1}/a_t$. Using (21) and (22), equations (28), (33) and (34) rewrite:

$$b_{t+1} g_{t+1} = sAb_t \tag{35}$$

$$\widetilde{x}_{1t} = \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)} g_{t+1} \tag{36}$$

$$\widetilde{x}_{2t} = \frac{\gamma}{\alpha+\beta+\gamma}\left(\frac{\phi(1-s)A}{1+\phi} + \frac{s(1-s)A^2}{1+\phi}\frac{1}{g_t}\right) - g_{t+1} \tag{37}$$

with:

$$\begin{aligned} b_t \equiv \widetilde{x}_{1t} + \widetilde{x}_{2t} &= \frac{\beta+\gamma(1+\phi)}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} + \frac{\gamma}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{1+\phi}\frac{1}{g_t} \\ &-g_{t+1}\left[1 + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}\right] \equiv \Omega(g_t, g_{t+1}) \end{aligned} \tag{38}$$

Therefore, substituting (38) in (35), an equilibrium is a sequence $\{g_t\}_{t\geqslant 0}$, given $g_0 \geqslant 0$, which satisfies:

$$\Omega(g_{t+1}, g_{t+2}) g_{t+1} = sA\Omega(g_t, g_{t+1}) \tag{39}$$

where g_t is predetermined. By inspection of equation (39), two BGPs may exist, a bubbly one and a bubbleless one.

Proposition 1 *When there is no binding borrowing constraint, a bubbly BGP* (b^*, g^*) *coexists with the bubbleless BGP* $(0, g^{**})$ *if:*

$$\frac{\alpha(s+\phi) + (\beta+\gamma)s(1+\phi)}{(1-s)[\beta+\gamma(2+\phi)]} < 1 \tag{40}$$

In addition, we always have $g^* < g^{**}$. *Finally, the bubbly BGP is a saddle, the bubbleless BGP is a locally indeterminate sink, and one converges to these two steady states with oscillations.*

Proof. See Appendix A. ■

The bubbleless BGP always exists, while the existence of the bubbly one requires (40). A direct inspection of equation (38) shows that there is a positive bubble if the savings, which are equivalent to $\Omega(g_t, g_{t+1}) + g_{t+1}$, are high enough.

Proposition 1 shows that growth is lower with a bubble than without. We deduce that the bubble has a crowding-out effect when the borrowing constraint is not binding. We can understand this result by computing total savings at time t. Using (33) and (34), the sum of the savings of young households and adults is equal to:

$$k_{t+1} + x_{1t} + x_{2t} = \frac{\gamma}{\alpha+\beta+\gamma} R_t w_{t-1} + \frac{\beta+\gamma(1+\phi)}{\alpha+\beta+\gamma} w_t - \frac{\alpha\phi w_{t+1}/R_{t+1}}{\alpha+\beta+\gamma} \quad (41)$$

Total savings does not depend on the existence of the bubble (see the right-hand side of equation (41)). This implies that the bubble has a crowding-out effect on future capital, since part of the savings are used to purchase the bubble. It is important to outline that this result occurs despite the fact that there are heterogeneous traders (young and adult) with heterogeneous opportunities to invest in capital (strictly positive return for an adult and zero return for a young).

3.3 Households face binding borrowing constraints

We introduce in the previous model a borrowing constraint that restricts the amount borrowed by agents investing in capital, *i.e.* adults. We adopt the borrowing constraint introduced in the model AI (Section 2.2): the household can borrow an amount $d_{2t+1} < 0$, as long as the repayment does not exceed a fraction θ of the future return from her productive investment and the market value of the bubble at period $t+2$. As in the model AI, two assets serve as a collateral: the capital and the bubble bought at the adult age. The first one can be identified as a fundamental collateral and the second one as a bubbly one. The degree of pledgeability of the fundamental collateral is measured by $\theta \in [0, 1)$. The borrowing constraint is then given by:[5]

$$-R^d_{t+2} d_{2t+1} \leqslant \theta q_{t+2} k_{t+2} + R_{t+2} b_{2t+1} \quad (42)$$

A household maximizes her utility function (1) facing the budget constraints (23)-(25) and the borrowing constraint (42). Solving the household problem, we deduce that $R^d_{t+1} = r_{t+1}$, $R^d_{t+2} = R_{t+2}$ and therefore:

$$u'_1(c_{1t}) = \frac{\beta}{\alpha} R_{t+1} u'_2(c_{2t+1}) \quad (43)$$

When the borrowing constraint is binding, *i.e.* $u'_2(c_{2t+1}) > R_{t+2}\frac{\gamma}{\beta}u'_3(c_{3t+2})$, we also have:

$$u'_2(c_{2t+1}) = \frac{\gamma}{\beta} u'_3(c_{3t+2}) \frac{q_{t+2}(1-\theta)}{1-\theta q_{t+2}/R_{t+2}} \quad (44)$$

[5] As it is shown in Appendix H, if we introduce a degree of pledgeability of the bubbly collateral less than one, b_{2t+1} requires a higher return than debt, while the bubble bought when young b_{1t} has the same return than debt. This means that, at an equilibrium, b_{2t+1} has a higher return than b_{1t}. Then, the bubble b_{1t} disappears in the long run, which is not an interesting configuration for our analysis.

which implies that the constraint is binding if:

$$q_{t+2} > R_{t+2} > \theta q_{t+2} \tag{45}$$

All the liquid assets, debt/deposits and bubbles traded at the different ages, are perfectly substitutable assets, whereas they are not perfect substitutes of capital because of the binding borrowing constraint. Using this perfect substitutability of liquid assets, $x_{1t} = b_{1t} + d_{1t}$ and $x_{2t} = b_{2t} + d_{2t}$, the binding constraint becomes:

$$R_{t+2}x_{2t+1} = -\theta q_{t+2}k_{t+2} \tag{46}$$

and (28) rewrites as:

$$R_{t+1}x_{1t} = x_{1t+1} + x_{2t+1} + \theta q_{t+1}k_{t+1} \tag{47}$$

Substituting (28), (46) and (47) in the budget constraints (23)-(25), we get:

$$c_{1t} = w_t - x_{1t} - x_{2t} - \theta \frac{q_{t+1}k_{t+1}}{R_{t+1}} \tag{48}$$

$$c_{2t+1} = \phi w_{t+1} + x_{1t+1} + x_{2t+1} + \theta q_{t+1}k_{t+1} - k_{t+2}\left(1 - \frac{\theta q_{t+2}}{R_{t+2}}\right) \tag{49}$$

$$c_{3t+2} = (1-\theta)q_{t+2}k_{t+2} \tag{50}$$

The first order conditions (43) and (44), and the condition to ensure a binding credit constraint (45) are identical to the conditions in the models YS and AI. This holds whatever the utility functions $u_i(c_i)$ are. If we compare the consumptions, we observe that (48)-(50) are a generalization of equations (10)-(12) whether b_{1t} is substituted or not by b_{2t} and r_{t+1} by R_{t+1}. When $x_{1t} + x_{2t} = b_{1t}$, we obtain the model YS in which the bubbly asset is bought only by young households to provide liquidity in the adult age. When $x_{1t} + x_{2t} = b_{2t}$, we obtain the model AI in which adults buy the bubble to relax the credit constraint. The approach developed in this section encompasses the two previous models, generalizes them, and allows us to consider other configurations. We can, for instance, think about the situation where both b_{1t} and b_{2t+1} are positive, *i.e.* the bubbly asset is bought both by young households and adults, or where b_{1t} is positive but b_{2t+1} negative, meaning that adults are short-sellers of the bubble to finance capital investment, while young agents buy this asset which gives it a positive value.

3.4 Discussion of the related literature

Before analyzing the equilibrium in detail and the crowding-in versus crowding-out effect of the bubble, we more specifically compare our model with the closest related literature, namely Farhi and Tirole (2012), Hirano and Yanagawa (2017), Martin and Ventura (2012, 2016) and Kocherlakota (2009).

Farhi and Tirole (2012) is a special case of the model with $b_{2t} = 0$ for all t. We generalize their approach considering a general equilibrium model, meaning that prices and incomes are endogenous. Furthermore, we do not have to introduce an adding asset representing outside liquidity, called trees, which is required for their result.

Hirano and Yanagawa (2017) is quite similar to our framework when $b_{2t} = 0$ for all t. Despite the fact that they consider infinitely-lived agents, they distinguish between high and low productive agents. Young and adult households correspond to them in our framework, taking the extreme case where the first ones are completely unproductive. Having $b_{2t} = 0$ in our framework means that the bubble is used only to transfer resources from less to more productive agents, as in their paper.

The mechanism for the crowding-in effect of the bubble highlighted by Martin and Ventura (2012) is also encompassed in our framework. They consider two-period lived overlapping generations in which agents are heterogeneous because their investments have different returns. Despite the fact that they have no credit, the bubble enhances growth because it reallocates resources from less to more productive traders, as in our framework with $b_{1t} > 0$ and $b_{2t} = 0$. In our model, it corresponds to a situation where the unproductive young agents buy the bubble from the productive adult ones. Note that in contrast to Martin and Ventura (2012), we will not need any exogenous bubble shocks to have a crowding-in effect of the bubble.

Our model also generalizes Martin and Ventura (2016) when $b_{1t} = 0$ for all t. Indeed, we also have workers that provide credits (young agents in our framework) to some investors (adults in our framework), but this heterogeneity among agents comes from the three periods lifetime. In their model, the credit constraint also has two types of collateral, one related to the value of the firm and one associated to the bubble. However, note that in contrast to Martin and Ventura (2016), bubbles have a crowding-in effect without adding bubble shocks in our paper.

The credit constraint investigated by Kocherlakota (2009) can also be seen as a particular case of our model if we set $\theta = 0$. In Kocherlakota (2009), both productive and unproductive traders hold the bubble, which corresponds to the case where $b_{it} > 0$ for $i = 1, 2$ in our framework. The young savers, which represent the unproductive traders, do not face any binding borrowing constraint in our model. However, it is important to underline that this case where $\theta = 0$ corresponds to a configuration with strong financial imperfections in our framework, since capital does no more play the role of collateral.

3.5 Constrained equilibrium

We characterize the constrained equilibrium. To obtain simple and easily interpretable expressions, we assume in the rest of the paper a log-linear utility function, meaning that $u_i(c_i) = \ln c_i$ for $i = 1, 2, 3$.

Substituting (21), (22) and (48)-(50) in (44) evaluated one period before, we obtain:

$$a_{t+1} = \frac{\gamma}{(\beta + \gamma)(1 + \phi)} \frac{\phi w(a_t) + x_{1t} + x_{2t} + \theta q_t (1 + \phi) a_t}{1 - \theta q_{t+1}/R_{t+1}} \equiv a^s_{t+1} \tag{51}$$

This equation determines the amount invested in capital given the assets hold from the previous period. We call it the asset supply, a^s_{t+1}, because despite her labor income, an adult sells the bubble $x_{1t}+x_{2t}$ and the deposits corresponding to the fundamental collateral $\theta q_t(1+\phi)a_t$ to finance her investment in capital. The term $1/(1-\theta q_{t+1}/R_{t+1})$ corresponds to an investment multiplier explained by the fact that adults borrow to invest in capital, which serves as a collateral. It corresponds to a leverage effect. Moreover, note that using (51), total savings at time t, $(1+\phi)a_{t+1}+x_{1t}+x_{2t}$, positively depend on $x_{1t}+x_{2t}$, which differs from the model without binding constraint (see equation (41)).

Substituting now (21), (22) and (48)-(50) into (43), we get:

$$a_{t+1} = R_{t+1}\frac{\frac{\beta+\gamma}{\alpha+\beta+\gamma}w(a_t)-(x_{1t}+x_{2t})}{\theta(1+\phi)sA+\frac{\alpha}{\alpha+\beta+\gamma}\phi(1-s)A} \equiv a^d_{t+1} \tag{52}$$

This equation is called the asset demand, a^d_{t+1}. It results from the trade-off between consumption when young and adult, and re-expresses the asset demand of young households, which buy the bubble $x_{1t}+x_{2t}$ and the loans collateralized by capital $\theta q_{t+1}(1+\phi)a_{t+1}/R_{t+1}$.

Taking into account that q_t is constant and a_t is predetermined, these two equations represent respectively the asset supply a^s_{t+1} and demand a^d_{t+1} as functions of the interest factor R_{t+1}, for a given level of the bubble $x_{1t}+x_{2t}$. As illustrated in Figure 1, they allow us to understand the effect of the bubble on capital under a binding borrowing constraint.[6]

Given q_t and a_t, an increase in the level of the bubble $x_{1t}+x_{2t}$ leads to an increase in the asset supply and a decrease in the asset demand, all other things equal (see equations (51) and (52)). In Figure 1, this increase in the level of the bubble translates into an upward shift in the asset supply curve a^s_{t+1} and a downward shift in the asset demand curve a^d_{t+1}. The effect on capital a_{t+1} may be positive. Accordingly, a higher level of the bubble may be in accordance with an increase in capital. In such a case, the bubble has a crowding-in effect on capital.

To understand the basic mechanism, consider first that $b_{2t}=0$ and $b_{1t}>0$, so that $x_{1t}+x_{2t}=b_{1t}$. Following Farhi and Tirole (2012), we identify the positive effect of b_{1t} on a^s_{t+1} as a liquidity effect. Adults sell the bubble to the young households, which corresponds to a liquidity transfer from the unproductive young households to the productive adults. If the bubble increases, the young households finance less loans collateralized by capital $\theta q_{t+1}(1+\phi)a_{t+1}/R_{t+1}$, which pushes up the interest R_{t+1}. This resulting increase reduces the investment multiplier $1/(1-\theta q_{t+1}/R_{t+1})$, and thus capital. The larger the degree of pledgeability θ is, the more important this last decrease. The net effect on investment of the bubble depends on the interplay between the liquidity effect and the multiplier effect.

When $b_{1t}=0$ and $b_{2t}>0$, we obtain $x_{1t}+x_{2t}=b_{2t}$. We have the same investment multiplier because bubble and credit are perfectly substitutable as-

[6] Farhi and Tirole (2012) use such a methodology to highlight the effect of the bubble in their framework.

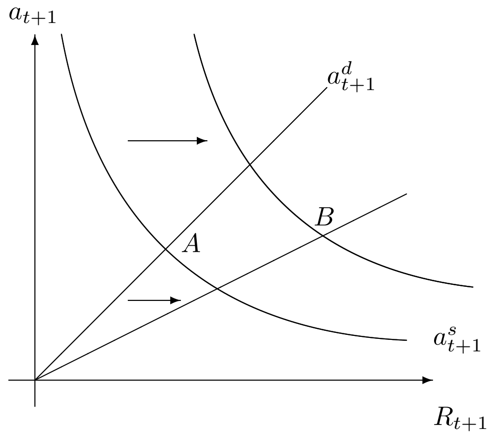

Figure 1: The effect of bubble on capital

sets and the borrowing constraint is binding. Credits used by adults to finance capital raise with the bubble and capital, because of the existence of both bubbly and fundamental collaterals. Therefore, the loans d_{1t} provided by young households to finance these credits increase with the bubble. These assets are sold when the young household becomes adult. This explains that the bubble size has a positive effect on a^s_{t+1}, which plays exactly the same role than the liquidity effect identified above.

To summarize, in the model YS, a liquidity effect exists due to the savings of young households, part of which is devoted to buy the bubble. In the model AI, the collateral role of the bubble raises the savings of young households through the loans provided. Our analysis does not only show that both mechanisms determine the same equilibrium and lead to the same result, but that they belong to a more general formulation according to which the equilibrium conditions depend neither on b_{1t}, nor on b_{2t}, but on the value of $b_{1t} + b_{2t}$.

Using the two equilibrium conditions (51) and (52), we get:

$$\begin{aligned}\frac{\theta q_{t+1}}{R_{t+1}} &= 1 - \frac{\gamma}{\beta+\gamma}\frac{\phi w(a_t) + x_{1t} + x_{2t} + \theta q_t(1+\phi)a_t}{(1+\phi)a_{t+1}} \\ &= \frac{\frac{\beta+\gamma}{\alpha+\beta+\gamma}w(a_t) - x_{1t} - x_{2t}}{(1+\phi)a_{t+1} + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi w(a_{t+1})}{\theta q_{t+1}}} \qquad (53)\end{aligned}$$

which means

$$(1+\phi)a_{t+1} = \frac{(1+\phi)a_{t+1}}{(1+\phi)a_{t+1}+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi w(a_{t+1})}{\theta q_{t+1}}}\left[\frac{\beta+\gamma}{\alpha+\beta+\gamma}w(a_t)-x_{1t}-x_{2t}\right] + \frac{\gamma}{\beta+\gamma}\left[\phi w(a_t)+x_{1t}+x_{2t}+\theta q_t(1+\phi)a_t\right] \quad (54)$$

An equilibrium satisfies (28) and (54), taking into account that R_{t+1} is defined by (53) and the borrowing constraint is binding for $q_{t+1} > R_{t+1} > \theta q_{t+1}$.

By direct inspection of equation (54), we see that if either $\alpha = 0$ or $\phi = 0$, we get a negative relationship between investment in capital (per unit of labor) a_{t+1} and the bubble size $x_{1t}+x_{2t}$. The crowding-out effect of the bubble dominates. When there is neither consumption when young nor labor income when adult, there is a high incentive when young to transfer purchasing power to the next periods of life to postpone consumption using the bubble and deposits. Any increase in the bubble implies a strong decrease in the deposits used to finance loans collateralized by capital $\theta q_{t+1}(1+\phi)a_{t+1}/R_{t+1}$. It implies, in particular, a strong increase in the interest, which has a negative effect on the investment multiplier, and therefore on capital.

When $\alpha > 0$ and $\phi > 0$, this is no more always the case. Indeed, when α is high, the savings rate of young households is low, and when ϕ is high, the savings of young households are low because the labor income when adult is high. In these last cases, the decrease due to the bubble in the deposits used to finance loans collateralized by capital is limited, making the crowding-in effect of the bubble possible.

3.6 Crowding-in versus crowding-out effect of the bubble

We analyze the existence of BGPs and the dynamics of the bubble. We study in particular whether the bubble enhances growth or not. We end this section giving an intuition for the crowding-in effect of the bubble.

3.6.1 BGPs and dynamics

Since we consider an endogenous growth framework, let us define $g_{t+1} = a_{t+1}/a_t$ and $b_t = (x_{1t}+x_{2t})/[(1+\phi)a_t]$. Using (21) and (22), equations (28) and (54) rewrite:

$$g_{t+1}b_{t+1} = R_{t+1}b_t \quad (55)$$

$$g_{t+1} = \frac{\theta s(1-s)A(\beta+\gamma)}{\alpha\phi(1-s)+\theta s(1+\phi)(\alpha+\beta+\gamma)} + \frac{\gamma A}{\beta+\gamma}\left[\frac{\phi}{1+\phi}(1-s)+\theta s\right] + b_t\left[\frac{\gamma}{\beta+\gamma} - \theta s\frac{\alpha+\beta+\gamma}{\alpha\frac{\phi}{1+\phi}(1-s)+\theta s(\alpha+\beta+\gamma)}\right] \equiv F(b_t) \quad (56)$$

Using (53), the interest factor is given by:

$$R_{t+1} = \theta s A g_{t+1} \frac{1 + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t} \equiv R(g_{t+1}, b_t) \tag{57}$$

Substituting (57) into (55), we get the dynamic equation:

$$b_{t+1} = \theta s A \frac{1 + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t} b_t \equiv G(b_t) \tag{58}$$

with $b_t < \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}$. This equation gives the dynamics of the ratio of bubble over capital, which is a non-predetermined variable. Given the sequence of $\{b_t\}_{t\geqslant 0}$, we deduce the growth factor at each period of time using (56).

There exist two BGPs, the bubbleless one $(\underline{b}, \underline{g}) = (0, F(0))$ and the bubbly one $(\bar{b}, \bar{g}) = (\bar{b}, F(\bar{b}))$, with:

$$\bar{b} \equiv \frac{\beta+\gamma-\alpha\phi}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - \theta s A \tag{59}$$

where $\bar{b} > 0$ if:

$$\phi < \frac{\beta+\gamma}{\alpha} \quad \text{and} \quad \theta < \frac{\beta+\gamma-\alpha\phi}{\alpha+\beta+\gamma}\frac{(1-s)}{s(1+\phi)} \equiv \theta_a \tag{60}$$

and

$$\bar{g} = A\left[s\theta + (1-s)\frac{\gamma}{\alpha+\beta+\gamma}\right] \tag{61}$$

Taking into account conditions (60), we can easily prove that the bubbleless steady state $\underline{b}$ is stable and the bubbly steady state $\bar{b}$ is unstable. Therefore, there are three types of equilibria depending on agents' expectations:

- there is no bubble, $b_t = \underline{b} = 0$;
- there is a persistent bubble, $b_t = \bar{b} > 0$;
- there is a bubble that decreases and converges to 0 for all $0 < b_t < \bar{b}$.

If individuals initially choose b_t such that $\bar{b} < b_t < \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}$, the bubble would increase along this equilibrium, and it would eventually crash after a finite number of periods when b_t crosses the upper bound $\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}$. Therefore, rational individuals will never buy the bubble at such a price, and hence, this is never an equilibrium. All equilibria must satisfy $0 \leqslant b_t \leqslant \bar{b}$.[7]

Of course, these equilibria should satisfy the binding borrowing constraint, *i.e.* $q_{t+1} > R_{t+1} > \theta q_{t+1}$.

[7]Note that if the bubbly BGP $\bar{b}$ does not exist, *i.e.* $\phi > \frac{\beta+\gamma}{\alpha}$ or $\theta > \theta_a$, we can easily show, using the same arguments than above, that the only equilibrium is the bubbleless BGP.

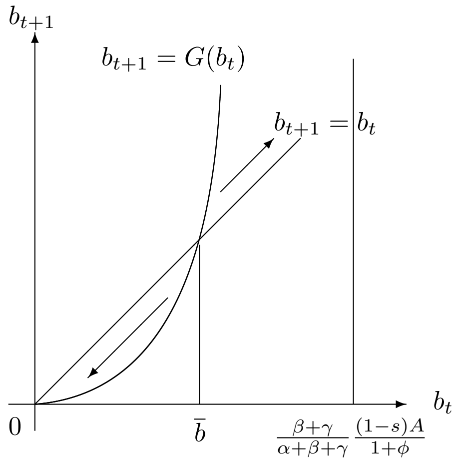

Figure 2: Dynamics of bubble

Lemma 1 *Any equilibrium $b_t \in [0, \overline{b}]$ satisfies the binding borrowing constraint if $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \theta_b$, with:*

$$\theta_b \equiv \alpha\phi \frac{1 - \frac{1-s}{s}\frac{\gamma}{\alpha+\beta+\gamma}}{\alpha\phi + \gamma(1+\phi)} \tag{62}$$

Proof. See Appendix B. ■

Using this lemma, we deduce the existence of the different types of equilibria with bubble:

Proposition 2 *Assuming $\phi < \frac{\beta+\gamma}{\alpha}$, $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \min\{\theta_a, \theta_b\}$, there exist three types of equilibria:*

1. *a bubbleless BGP $b = \underline{b} = 0$;*
2. *a bubbly BGP $b = \overline{b}$;*
3. *any sequence $b_t \in (0, \overline{b})$, which decreases and converges to 0.*

The bubble has a crowding-in effect on growth if and only if there is a positive relationship between b_t and g_{t+1}. The crowding-out effect dominates when the bubble decreases growth. The following proposition summarizes the main results:

Proposition 3 *Let*

$$\widehat{\theta} \equiv \frac{\gamma}{\beta}\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)} \tag{63}$$

1. *If we have* $\phi < \frac{\beta}{\alpha}$ *and* $\frac{s}{1-s} > \frac{\gamma}{\beta(1+\phi)} \frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma}$, *the bubble has a crowding-in effect on growth if* $\theta < \widehat{\theta}$, *has no effect on growth if* $\theta = \widehat{\theta}$ *and has a crowding-out effect on growth if* $\widehat{\theta} < \theta < \min\{\theta_a, \theta_b\}$.[8]

2. *If either* $\frac{\beta}{\alpha} \leqslant \phi < \frac{\beta+\gamma}{\alpha}$ *or* $\frac{\gamma}{\beta(1+\phi)} \frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma} \geqslant \frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$, *the bubble has a crowding-in effect on growth for all* $\theta < \min\{\theta_a, \theta_b\}$.

Proof. See Appendix C. ■

This proposition shows that when $\theta < \widehat{\theta}$, the bubble boosts growth. In this case, the crowding-in effect of the bubble dominates its crowding-out effect. As a direct implication, the bubbly BGP features a higher growth than the bubbleless one, *i.e.* $\overline{g} > \underline{g}$.[9] In contrast, when $\theta > \widehat{\theta}$, the crowding-out effect of the bubble dominates its crowding-in effect and $\overline{g} < \underline{g}$. A lower θ, *i.e.* a stronger credit market imperfection, reinforces the crowding-in effect of the bubble.

This result can be related to Hirano and Yanagawa (2017) who also analyzes an endogenous growth model, but with heterogeneous infinitely-lived agents. In their framework, there is also a level of θ such that below it, the crowding-in effect dominates, whereas above it, the bubble has a crowding-out effect. However, in contrast to us, the existence of the bubble requires a minimum value for θ. The main difference lies in the returns of investment opportunities. In their framework, all investment opportunities have a positive return, whereas in our model, young agents do not invest in capital because they expect a zero return.

More importantly, our results hold whatever the type of bubble considered, *i.e.* either young agents buy the bubble ($b_{1t} > 0$), or adults ($b_{2t} > 0$), or both. Our results even hold if some agents are short sellers of the bubble ($b_{1t} < 0$ or $b_{2t} < 0$).

3.6.2 Economic interpretation

We would like first to understand why the bubble always has a crowding-out effect when there is no binding borrowing constraint, while the bubble can promote growth when the adults / investors are constrained. When there is no binding borrowing constraint, households can perfectly smooth consumption and all assets are perfect substitutes, *i.e.* have the same return. In this case, consumptions linearly depend on the life-cycle income (see (30)-(32)), which implies that total savings depend on a weighted sum of discounted wages (see (41)). Hence, savings do not depend on the bubble. As a direct implication, any increase of the bubble implies a decrease of the new investment in capital.

When adults face a binding borrowing constraint, households can no more smooth consumption without any restrictions, and capital is no more substitutable for credit and bubble. The consumptions depend now on the asset

[8]Note that $\phi < \frac{\beta}{\alpha}$ and $\frac{s}{1-s} > \frac{\gamma}{\beta(1+\phi)} \frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma}$ ensure that $\widehat{\theta} < \min\{\theta_a, \theta_b\}$.

[9]We also observe that the interest factor at the bubbleless BGP $\underline{R}$ is lower than at the bubbly BGP, $\overline{R}$.

holdings and, therefore, the savings too. This opens the door to mechanisms for which the bubble has a crowding-in effect on capital.

Since the borrowing constraint is binding, adults use the bubble and credit to finance capital, meaning that $x_{2t} = b_{2t} + d_{2t}$ is negative. In a way, they transfer purchasing power from the old age. In contrast, young households use deposits and the bubble to postpone consumption from the first period of life, explaining that $x_{1t} = b_{1t} + d_{1t}$ is positive. This allows households to increase investment, because adult individuals have more liquidities to buy capital (liquidity effect). This can also be interpreted as a transfer from the unproductive young traders to the productive adult ones. Both these effects enhance investment in capital. Note that these effects could be achieved with either the bubble or with credit. In what follows, we explain the specific effects of the bubble.

By direct inspection of equation (50), we first observe that the consumption when old does not directly depend on the level of the bubble because of the binding borrowing constraint. The redistribution from the old to the adult age only depends on capital income, because the loan demand net of the purchase or sale of the bubble is constrained by the fundamental collateral. Focusing now on consumptions when young and adult, given by (48) and (49), we easily see that the bubble induces a redistribution from the young households to the adults. If the bubble is bought when young ($b_{1t} > 0$), we have the standard liquidity effect, which may cause a crowding-in effect of the bubble. If the bubble is bought when adult ($b_{2t} > 0$), the explanation goes through the credit market. Due to the binding borrowing constraint, a higher bubble leads to more loans. To satisfy the equilibrium on the credit market, these loans are financed by deposits of young households. Therefore, a higher bubble requires more deposits by young households. Since these deposits are remunerated at the next period, this increases the liquidity transferred at the adult age. Since deposits, or credit, and bubbles are perfectly substitutable assets, the liquidity role of both types of bubbles (b_{1t} or b_{2t}) are identical. Our approach of bubbles allows us to deduce that what is important is not to know whether b_{1t} or b_{2t} is positive, negative or zero, but to know the level of $b_{1t}+b_{2t}$. Any combinations of b_{1t} and b_{2t} that keep $b_{1t} + b_{2t}$ constant give the same result. Of course, credits cannot have such a role, because deposits are entirely used to finance loans, meaning that $d_{1t} + d_{2t} = 0$, while the bubble has a positive value, $b_{1t} + b_{2t} > 0$.

Finally, the existence of a bubble, which means a higher supply of liquid assets than at the bubbleless equilibrium, also increases the cost of credit used to finance capital (R_{t+1} increases), which reduces capital investment. Indeed, the young savers should buy the bubble and reduce deposits used to finance loans collateralized by capital. Therefore, the bubble enhances growth when the degree of pledgeability θ is sufficiently small. As it is clear from the asset market described by equations (51) and (52), the higher θ, the more important the negative effect of a raise in R_{t+1} on capital. This last effect corresponds to the crowding-out effect of the bubble when the borrowing constraint is binding.

To further discuss the role of the degree of pledgeability θ, we easily see that both growth factors $\overline{g}$ and $\underline{g}$ evaluated respectively at the bubbly and bubbleless BGP increase with θ (see equations (56) and (61)). A higher θ means a higher role for the fundamental collateral and, therefore, a higher borrowing to finance capital investment. Since the crowding-in effect of the bubble dominates for $\theta < \widehat{\theta}$ and the crowding-out effect dominates for $\theta > \widehat{\theta}$, there is a positive gap between $\overline{g}$ and $\underline{g}$ when $\theta = 0$, which decreases and becomes negative when θ becomes higher than $\widehat{\theta}$. When $\theta = 0$, the gap between $\overline{g}$ and $\underline{g}$ is the highest one, but the growth rates are lower than the growth rates for higher values of θ.

4 Stochastic bubble

When the model is deterministic, all possible bubbles are perfect substitutes of credit. The liquidity and collateral roles of the bubble are equivalent despite the fact that the first role is obtained when the bubble is bought when young and the second one when the bubble is bought at the middle age. When we enrich the model with a positive probability of bubble crash, bubbles and credit are no more perfect substitutes and the temporal difference between the two roles of the bubble may make them dissimilar. Therefore, we provide a clear comparison of the bubble size and growth rate generated by these two roles attributed to the bubble, when this latter is stochastic. This comparison is relevant for several reasons, as for instance to know whether a household invests more when the bubble is bought at young or adult age. It is also relevant for governments that might like to know in which case the crash of the bubble will be more damaging, at least to facilitate the implementation of the most appropriate policy.

Following the seminal paper by Weil (1987), we assume that households may coordinate their expectations in an equilibrium where the bubble crashes, *i.e.* its value is zero in the next period. Because of the volatility of agents' expectations, there is a positive probability of bubble crash in each period of time. We examine now this type of stochastic equilibrium, and compare the liquidity and collateral roles of the bubble.

We consider a Markov process of a bubble crash. If there is no bubble at period t, there is no bubble at period $t+1$ with a probability equal to one. If there is a bubble at period t, there is a probability $\pi \in (0,1]$ such that the bubble persists in the next period and a probability $1-\pi$ such that the bubble crashes at period $t+1$. Note that a market crash in period $t+1$ means that the return of this asset without fundamental value is zero, i.e. $r_{t+1} = 0$ and/or $R_{t+1} = 0$ using the notations of Section 3. In contrast, the return on credit R^d_{t+1} remains positive, even if it is affected by the bubble crash at equilibrium.

Consider the budget constraints (23)-(25). In the stochastic case, the budget constraint when young (23) holds if there is a bubble at period t. With probability π the bubble persists in the next period and the budget constraint when adult is given by (24), whereas it is given by:

$$c_{2t+1} + k_{t+2} + d_{2t+1} = \phi w_{t+1} + R^d_{t+1} d_{1t}$$

when there is a crash of the bubble, which occurs with probability $1-\pi$. We can do the same reasoning if the bubble persists at the middle age and has therefore a probability to crash between the middle and old ages. As a direct implication, this implies that b_{1t} is no more substitutable for d_{1t} and b_{2t+1} is no more substitutable for d_{2t+1}. Therefore, we cannot use the general model introduced in Section 3 to compare the liquidity and collateral roles when the bubble is stochastic. We will rather study the YS and AI models with stochastic bubbles successively, and will compare the features of their equilibria.

4.1 Model YS with a stochastic bubble

We start by considering the model YS with a stochastic bubble. Since we focus on the liquidity role, the household buys the bubble only when young. Let us assume that a bubble exists at period t when the household is young.[10] At this period, her consumption c_{1t} is given by:

$$c_{1t} = w_t - d_{1t} - b_{1t} \tag{64}$$

At the middle-age, consumption is given by c^+_{2t+1} when the bubble persists, and by c^0_{2t+1} when the bubble crashes. Whatever the state of the nature, the borrowing constraint is assumed to be binding, *i.e.* $R^d_{t+2}d_{2t+1} = -\theta q_{t+2}k_{t+2}$ and $R^{d0}_{t+2}d^0_{2t+1} = -\theta q_{t+2}k^0_{t+2}$, where R^{d0}_{t+2}, k^0_{t+2} and d^0_{2t+1} are the interest factor on credit, the investment in capital and the loans when the bubble crashes.[11] We focus on such type of equilibria to be in accordance with the literature, as for instance Farhi and Tirole (2012) and Hirano and Yanagawa (2017), and to facilitate the comparison with the deterministic model presented in Sections 3.3-3.6. Using (4) and these constraints, we deduce that:

$$c^+_{2t+1} = \phi w_{t+1} + R^d_{t+1}d_{1t} + r_{t+1}b_{1t} - k_{t+2}\left(1 - \theta\frac{q_{t+2}}{R^d_{t+2}}\right) \tag{65}$$

$$c^0_{2t+1} = \phi w_{t+1} + R^d_{t+1}d_{1t} - k^0_{t+2}\left(1 - \theta\frac{q_{t+2}}{R^{d0}_{t+2}}\right) \tag{66}$$

In accordance with Farhi and Tirole (2012) and Hirano and Yanagawa (2017), R^d_{t+1} is the same in the two states of nature. When there is a bubble at period t, using the equilibrium on the debt market and the binding borrowing constraint, we get $R^d_{t+1}d_{1t} = \theta q_{t+1}k_{t+1}$, where $q_{t+1} = sA$, and d_{1t} and k_{t+1} are determined at period t. Therefore, R^d_{t+1} is uniquely defined in both equations (65) and (66).[12]

Since the consumption when old does not depend on the value of the bubble, the consumption will be c^+_{3t+2} when the consumption was c^+_{2t+1} at middle age

[10] When there is no bubble at period t, the economy is at the bubbleless equilibrium for all the following periods.

[11] From the first order of the household program, we could check that these borrowing constraints are binding under similar conditions than in the deterministic model, *i.e.* $q_{t+2} > R^d_{t+2} > \theta q_{t+2}$ and $q_{t+2} > R^{d0}_{t+2} > \theta q_{t+2}$.

[12] This would also arise if we would rather consider d_{1t} as debt contracts without risk.

and c^0_{3t+2} when the consumption was c^0_{2t+1} at middle age, with:

$$c^+_{3t+2} = (1-\theta)q_{t+2}k_{t+2} \tag{67}$$
$$c^0_{3t+2} = (1-\theta)q_{t+2}k^0_{t+2} \tag{68}$$

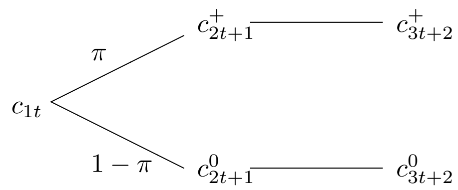

Figure 3: Consumption profile in the model YS when the bubble has a probability to crash $1-\pi$

Considering these events (see also Figure 3), the household's expected utility writes:

$$\mathbb{E}_t[\alpha u_1(c_{1t}) + \beta u_2(c_{2t+1}) + \gamma u_3(c_{3t+2})] = \alpha u_1(c_{1t}) \tag{69}$$
$$+\beta\left[\pi u_2(c^+_{2t+1}) + (1-\pi)u_2(c^0_{2t+1})\right] + \gamma\left[\pi u(c^+_{3t+2}) + (1-\pi)u_3(c^0_{3t+2})\right]$$

A household maximizes this utility function with respect to d_{1t}, b_{1t}, k_{t+2} and k^0_{t+2} under (64)-(68). Using $u_i(c_{it}) = \ln c_{it}$, we obtain:

$$\frac{\alpha}{c_{1t}} = R^d_{t+1}\beta\left(\frac{\pi}{c^+_{2t+1}} + \frac{1-\pi}{c^0_{2t+1}}\right) \tag{70}$$
$$\frac{\alpha}{c_{1t}} = r_{t+1}\beta\frac{\pi}{c^+_{2t+1}} \tag{71}$$
$$\frac{\beta}{c^+_{2t+1}}\left(1-\theta\frac{q_{t+2}}{R^d_{t+2}}\right) = (1-\theta)q_{t+2}\frac{\gamma}{c^+_{3t+2}} \tag{72}$$
$$\frac{\beta}{c^0_{2t+1}}\left(1-\theta\frac{q_{t+2}}{R^{d0}_{t+2}}\right) = (1-\theta)q_{t+2}\frac{\gamma}{c^0_{3t+2}} \tag{73}$$

As it is shown in Appendix D, using the equilibrium prices (21) and (22), the equilibrium on the debt market $d_{1t} = -d_{2t} = \theta q_{t+1}k_{t+1}/R^d_{t+1}$, the evolution of the bubble $b_{1t+1} = r_{t+1}b_{1t}$, the variables $k_t = a_t(1+\phi)$, $g_{t+1} = a_{t+1}/a_t$ and $\widetilde{b}_{1t} = b_{1t}/[(1+\phi)a_t]$, we can define an equilibrium as a sequence $\{\widetilde{b}_{1t}, g_t\}_{t\geqslant 0}$

satisfying the following two equations:

$$g_{t+1} = \frac{\theta s(1-s)A(\beta+\gamma)}{\alpha\phi(1-s)+\theta s(1+\phi)(\alpha+\beta+\gamma)} + \frac{\gamma A}{\beta+\gamma}\left[\frac{\phi}{1+\phi}(1-s)+\theta s\right] + \widetilde{b}_{1t}\left[\frac{\gamma}{\beta+\gamma} - \theta s \frac{\alpha+\beta+\gamma}{\alpha\frac{\phi}{1+\phi}(1-s)+\theta s(\alpha+\beta+\gamma)}\right] \tag{74}$$

$$\widetilde{b}_{1t+1} = \frac{\theta s A\left(1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right)}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{\pi(1-s)A}{1+\phi} - \widetilde{b}_{1t}\left[\frac{\alpha+\pi(\beta+\gamma)}{\alpha+\beta+\gamma} + \frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right]}\widetilde{b}_{1t} \equiv I(\widetilde{b}_{1t}) \tag{75}$$

The dynamics are qualitatively similar than in the deterministic model for all $\widetilde{b}_{1t} < \widehat{b}_1$ (see Section 3.6.1), with $\widehat{b}_1 \equiv \frac{\pi(1-s)A}{1+\phi} / \left[\frac{\alpha+\pi(\beta+\gamma)}{\beta+\gamma} + \frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\right]$.

Comparing with the deterministic case (see equation (58)), we note that $I(b_t) = G(b_t)$ when $\pi = 1$. For a given $\widetilde{b}_{1t}$, $I(\widetilde{b}_{1t})$ decreases with respect to π. Since $I(\widetilde{b}_{1t})$ is an increasing and convex function, it means that the higher the probability of bubble crash $1-\pi$, the smaller is the level of the bubble per unit of capital $\widetilde{b}_1$ at the stochastic bubbly BGP, with:

$$\widetilde{b}_1 = \frac{\frac{(1-s)A}{1+\phi}\frac{\pi(\beta+\gamma)-\alpha\phi}{\alpha+\beta+\gamma} - \theta s A}{\frac{\alpha+\pi(\beta+\gamma)}{\alpha+\beta+\gamma} + \frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\frac{\beta+\gamma}{\alpha+\beta+\gamma}} \tag{76}$$

which is strictly positive for $\pi\frac{\beta+\gamma}{\alpha} > \phi$ and $\theta < \frac{(1-s)[\pi(\beta+\gamma)-\alpha\phi]}{s(\alpha+\beta+\gamma)(1+\phi)}$. In addition, the higher the probability $1-\pi$, the smaller the range of $\widetilde{b}_{1t} \in (0, \widetilde{b}_1)$ converging to the bubbleless BGP.

Once the level of the bubble is given, equation (74) gives the level of growth. It is equivalent to (56), which gives the growth rate as a function of the bubble in the deterministic model. Based on this fact, if θ is sufficiently low, there is a positive link between the growth rate and the bubble size. Then, the growth rate at the stochastic bubbly BGP $\widetilde{b}_1$ is higher than at the bubbleless BGP. This means that the bubble has a crowding-in effect.

Because of the positive probability of market crash, it is risky for a young household to buy the bubble. Therefore, the bubble size is lower than at a deterministic bubbly BGP. As a direct implication, the growth rate is also lower than in the deterministic case when the bubble is productive. In this case, when she is adult and the bubble has not burst yet, the household does not face any risk, but invests less in capital because she owns less liquidities than under a deterministic bubble.

4.2 Model AI with a stochastic bubble

We now consider the model AI with a stochastic bubble to investigate the main features of the equilibrium when the bubble with a positive probability to crash plays a collateral role. We consider the behavior of a household born at period t, assuming that a bubble exists at that period. Households do not buy the bubble when young, which implies that the consumption is given by:

$$c_{1t} = w_t - d_{1t} \tag{77}$$

Depending on whether the bubble has crashed or not at the beginning of period $t+1$, the bubble is bought or not at middle age. The consumptions at middle age associated to these two events are c^{+}_{2t+1} and c^{0}_{2t+1}, respectively. They are given by:

$$c^{+}_{2t+1} = \phi w_{t+1} + R^{d}_{t+1} d_{1t} - k_{t+2} - d_{2t+1} - b_{2t+1} \tag{78}$$
$$c^{0}_{2t+1} = \phi w_{t+1} + \widetilde{R}^{d}_{t+1} d_{1t} - k^{0}_{t+2} - d^{0}_{2t+1} \tag{79}$$

In contrast to the model YS with a stochastic bubble, the return of d_{1t} is no more the same in the two states of nature. R^{d}_{t+1} is still the return of loans when the bubble persists, but $\widetilde{R}^{d}_{t+1}$ is the return when the bubble just crashes in $t+1$.

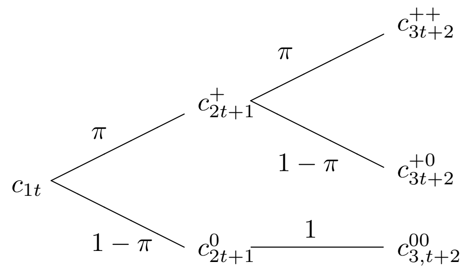

Figure 4: Consumption profile in the model AI when the bubble has a probability to crash $1-\pi$

If the bubble has crashed at middle age, there is no bubble at the following periods, which implies that the consumption when old is:

$$c^{00}_{3t+2} = q_{t+2} k^{0}_{t+2} + R^{d0}_{t+2} d^{0}_{2t+1} \tag{80}$$

If the bubble has not crashed at middle age, the consumption when old depends on whether the bubble crashes (c^{+0}_{3t+2}) or not (c^{++}_{3t+2}) at old age:

$$c^{+0}_{3t+2} = q_{t+2} k_{t+2} + \widetilde{R}^{d}_{t+2} d_{2t+1} \tag{81}$$
$$c^{++}_{3t+2} = q_{t+2} k_{t+2} + R^{d}_{t+2} d_{2t+1} + R_{t+2} b_{2t+1} \tag{82}$$

When the bubble has crashed at middle age, we assume that the borrowing constraint is binding, *i.e.* $R^{d0}_{t+2} d^{0}_{t+1} = -\theta q_{t+2} k^{0}_{t+2}$. Then, the consumptions

(79) and (80) rewrite:

$$c^0_{2t+1} = \phi w_{t+1} + \widetilde{R}^d_{t+1} d_{1t} - k^0_{t+2}\left(1-\theta\frac{q_{t+2}}{R^{d0}_{t+2}}\right) \tag{83}$$

$$c^{00}_{3t+2} = (1-\theta) q_{t+2} k^0_{t+2} \tag{84}$$

When there is a bubble at the middle age, $b_{2t+1} > 0$, the credit constraint becomes stochastic, since it depends on the expected return of bubble and on the expected cost of credit reimbursement.[13] Assuming that this (expected) constraint is binding, we have:

$$-\left[\pi R^d_{t+2} + (1-\pi)\widetilde{R}^d_{t+2}\right] d_{2t+1} = \theta q_{t+2} k_{t+2} + \pi R_{t+2} b_{2t+1} \tag{85}$$

The expected cost of borrowing (left-hand side) is equal to the expected value of the sum of the fundamental and bubbly collaterals.

Considering the consumptions at the different events (see also Figure 4), a household maximizes her expected utility:

$$\mathbb{E}_t[\alpha u_1(c_{1t}) + \beta u_2(c_{2t+1}) + \gamma u_3(c_{3t+2})] = \alpha u_1(c_{1t}) + \beta[\pi u_2(c^+_{2t+1}) + (1-\pi) u_2(c^0_{2t+1})] + \gamma\left[\pi^2 u(c^{++}_{3t+2}) + \pi(1-\pi) u_3(c^{+0}_{3t+2}) + (1-\pi) u_3(c^{00}_{3t+2})\right]$$

with respect to d_{1t}, k^0_{t+2}, k_{t+2}, b_{2t+1} and d_{2t+1} under the constraints (77), (78) and (81)-(85). The arbitrage condition between consumptions when young and adult is given by:

$$\frac{\alpha}{c_{1t}} = \frac{\beta\pi R^d_{t+1}}{c^+_{2t+1}} + \frac{\beta(1-\pi)\widetilde{R}^d_{t+1}}{c^0_{2t+1}} \tag{86}$$

If the bubble has crashed at period $t+1$, we obtain:

$$c^0_{2t+1} = \frac{\beta}{\beta+\gamma}\left(\phi w_{t+1} + \widetilde{R}^d_{t+1} d_{1t}\right) \tag{87}$$

$$k^0_{t+2}\left(1-\theta\frac{q_{t+2}}{R^{d0}_{t+2}}\right) = \frac{\gamma}{\beta+\gamma}\left(\phi w_{t+1} + \widetilde{R}^d_{t+1} d_{1t}\right) \tag{88}$$

If the bubble has not crashed at period $t+1$, we deduce the following first order conditions, considering that the borrowing constraint is binding as in equation (85):[14]

$$\left(\pi - \theta\frac{q_{t+2}}{R_{t+2}}\right)\frac{\beta}{c^+_{2t+1}} = q_{t+2}\gamma\left(\frac{\pi(\pi-\theta)}{c^{++}_{3t+2}} + \frac{\pi(1-\pi)}{c^{+0}_{3t+2}}\right) \tag{89}$$

$$\frac{\beta}{c^+_{2t+1}}\left(\pi - \frac{\pi R^d_{t+2} + (1-\pi)\widetilde{R}^d_{t+2}}{R_{t+2}}\right) = \frac{\gamma\pi(1-\pi)\widetilde{R}^d_{t+2}}{c^{+0}_{3t+2}} - \frac{\gamma\pi(1-\pi)\widetilde{R}^d_{t+2}}{c^{++}_{3t+2}} \tag{90}$$

[13] See for instance Iacovello (2015) and Quadrini (2011) who, in a different context, also consider borrowing constraints in expected terms.

[14] Using the first order conditions of the household problem, we can easily show that the borrowing constraint is binding when $q_{t+2} > \pi R_{t+2} > \theta q_{t+2}$, which is quite similar than in the deterministic case when π is close to one.

We assume that the asset d_{2t+1} is a financial instrument that gives perfect insurance in terms of consumption, *i.e.* $c^{++}_{3t+2} = c^{+0}_{3t+2}$. Using the expected borrowing constraint (85), the budget constraints (81) and (82) allow us to deduce that $c^{++}_{3t+2} = c^{+0}_{3t+2} = (1-\theta)q_{t+2}k_{t+2}$. Such an assumption, which is in accordance with Kocherlakota (2009), will facilitate the comparison with the deterministic model and the liquidity role when the bubble is stochastic (Section 4.1). In particular, we will obtain the deterministic model as a limit case when π tends to one, which is not possible otherwise.

Using (89), we immediately deduce that:

$$\pi R_{t+2} = \pi R^d_{t+2} + (1-\pi)\widetilde{R}^d_{t+2} \tag{91}$$

Therefore, the borrowing constraint (85) rewrites:

$$d_{2t+1} = -\frac{\theta q_{t+2}k_{t+2}}{\pi R_{t+2}} - b_{2t+1} \tag{92}$$

At an equilibrium on the credit market, we further have $d_{1t+1} = -d_{2t+1}$. Using these different equilibrium conditions and the budget constraint (78), the first order condition (89) implies:

$$c^{+}_{2t+1} = \frac{\beta}{\beta+\gamma}\left(\phi w_{t+1} + R^d_{t+1}d_{1t}\right) \tag{93}$$

$$k_{t+2}\left(1-\theta\frac{q_{t+2}}{\pi R_{t+2}}\right) = \frac{\gamma}{\beta+\gamma}\left(\phi w_{t+1} + R^d_{t+1}d_{1t}\right) \tag{94}$$

Using the equilibrium on the credit market, the binding credit constraint (85), the evolution of the bubble $b_{2,t+1} = R_{t+1}b_{2,t}$, and the equality $-\widetilde{R}^d_{t+2}d_{2t+1} = \theta q_{t+2}k_{t+2}$ which comes from the perfect consumption insurance, this last equation rewrites one period before as:

$$k_{t+1}\left(1-\theta\frac{q_{t+1}}{\pi R_{t+1}}\right) = \frac{\gamma}{\beta+\gamma}\left(\phi w_t + b_{2t} + \theta q_t k_t\right) \tag{95}$$

Let us introduce $a_t \equiv k_t/(1+\phi)$, $\widetilde{b}_{2t} \equiv b_{2t}/[(1+\phi)a_t]$, $g_{t+1} \equiv a_{t+1}/a_t$, and:

$$\Psi(\widetilde{b}_{2t+1}) \equiv \frac{\theta sA\left(\frac{\phi(1-s)A}{1+\phi} + \theta sA\right) + \widetilde{b}_{2t+1}\left(\frac{\pi\phi(1-s)A}{1+\phi} + \theta sA\right)}{\left(\frac{\phi(1-s)A}{1+\phi} + \theta sA\right)\left(\theta sA + \pi\widetilde{b}_{2t+1}\right)} \tag{96}$$

where $\Psi(\widetilde{b}_{2t+1}) = 1$ when $\pi = 1$ and $\Psi(\widetilde{b}_{2t+1}) > 1$ when $\pi < 1$, meaning that $\Psi(\widetilde{b}_{2t+1})$ is decreasing in π.

As it is derived in Appendix E, an equilibrium is a sequence $\{\widetilde{b}_{2t}, g_t\}_{t\geqslant 0}$

satisfying the two following equations:

$$\widetilde{b}_{2t+1} = \frac{\alpha\frac{\phi(1-s)A}{1+\phi} + \theta sA\left[\alpha + (\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\right]}{(\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\pi\frac{(1-s)A}{1+\phi} - \widetilde{b}_{2t}\left[\alpha + (\beta+\gamma)\pi\Psi(\widetilde{b}_{2t+1})\right]}\widetilde{b}_{2t} \tag{97}$$

$$\begin{aligned} g_{t+1} &= \frac{\gamma A}{\beta+\gamma}\left[\frac{\phi(1-s)}{1+\phi} + \theta s\right] + \frac{\theta s(1-s)A(\beta+\gamma)\Psi(\widetilde{b}_{2t+1})}{\alpha\phi(1-s) + \theta s(1+\phi)[\alpha + (\beta+\gamma)\Psi(\widetilde{b}_{2t+1})]} \\ &+ \widetilde{b}_{2t}\left[\frac{\gamma}{\beta+\gamma} - \frac{\theta s}{\pi}\frac{\alpha + (\beta+\gamma)\pi\Psi(\widetilde{b}_{2t+1})}{\frac{\alpha\phi(1-s)}{1+\phi} + \theta s[\alpha + (\beta+\gamma)\Psi(\widetilde{b}_{2t+1}))]}\right] \end{aligned} \tag{98}$$

To keep things as simple as possible, we focus on the existence and characterization of BGPs. By inspection of equation (97), we see that a bubbleless BGP $\widetilde{b}_2 = 0$ exists, while a stochastic bubbly BGP with a positive bubble can be defined by:

$$\widetilde{b}_2 = \frac{\frac{(1-s)A}{1+\phi}\left[(\beta+\gamma)\pi\Psi(\widetilde{b}_2) - \alpha\phi\right] - \theta sA\left[\alpha + (\beta+\gamma)\Psi(\widetilde{b}_2)\right]}{\alpha + (\beta+\gamma)\pi\Psi(\widetilde{b}_2)} \equiv B(\Psi(\widetilde{b}_2)) \tag{99}$$

In Appendix F, we show that a unique positive solution $\widetilde{b}_2 > 0$ exists if $\phi < \pi\frac{\beta+\gamma}{\alpha}$ and $\theta < \min\{\frac{\pi(1-s)}{(1-\pi)s}; \frac{1-s}{s(1+\phi)}\frac{(\beta+\gamma)\pi-\alpha\phi}{\alpha+\beta+\gamma}\}$.

When $\pi = 1$, a quick comparison of equations (56) and (58) with (97)-(98) show that we obtain of course exactly the same equilibrium than when the bubble is deterministic.

Since $\pi\Psi(\widetilde{b}_2) < 1$ and $\Psi(\widetilde{b}_2) > 1$ when $\pi < 1$, we deduce from equations (59) and (99) that the stochastic bubble $\widetilde{b}_2$ at the BGP is lower than in the deterministic case. When there is a positive probability of market crash, the bubble size is lower than when the bubble is riskless. This result is similar than for the liquidity role of the bubble, but the interpretation is different. When there is a bubble at the adult age, the household faces a form of consumption insurance at the old age. However, the loans collateralized by capital increase regarding the loans collateralized by the bubble, because the return of capital becomes relatively higher (see equation (92)). This explains the lower value of the bubble when $\pi < 1$. Finally, similarly to the model YS, equation (98) allows us to deduce that for θ low enough, the bubble has a positive effect on growth, meaning that there is still a crowding-in effect of the bubble.

4.3 Comparison of the liquidity and collateral roles of a stochastic bubble

To compare the liquidity and the collateral roles of the bubble, we first study if $\widetilde{b}_1$, given by (76), is higher or not than $\widetilde{b}_2$, defined by (99). As it is shown in Appendix F, we have $B'(\Psi) > 0$, which means that $B(\Psi(\widetilde{b}_2)) \geq B(1)$ because $\Psi(\widetilde{b}_2) \geqslant 1$. Using (76) and (99), we easily deduce that $\widetilde{b}_2 \geq B(1) > \widetilde{b}_1$.

Second, we recall that we measure the importance of the crowding-in effect as the difference between the growth rate at the bubbly and at the bubbleless BGPs, where the last one is of course the same whatever the role played by the bubble. Therefore, to determine whether the crowding-in effect is more important when the bubble has a liquidity or a collateral role, we compare the growth rates at the stochastic bubbly BGPs in the YS and AI models.

Let us call g_1 the growth factor at the stochastic bubbly BGP in the model YS and g_2 the growth factor at the stochastic bubbly BGP in the model AI. The first one identifies the liquidity effect when the bubble is stochastic, the second one the collateral effect. As it is shown in Appendix G, we have $g_1 > g_2$ if $\pi(\beta + \gamma) > \alpha\phi$ and θ low enough, which are conditions required for the existence of the stochastic BGPs.

These results are summarized in the following proposition:

Proposition 4 *Assuming that $\pi < 1$, $\pi(\beta + \gamma) > \alpha\phi$ and θ low enough, we have $\widetilde{b}_2 > \widetilde{b}_1$ and $g_1 > g_2$.*

The collateral role of the bubble generates a higher bubble than the liquidity role. However, the crowding-in effect is stronger under the liquidity role than under the collateral role when the bubble is stochastic.

When there is no risk of bubble crash, *i.e.* $\pi = 1$, the models YS and AI with stochastic bubbles are equivalent to the case of a deterministic bubble as studied in Section 3.

When $\pi < 1$, the stochastic bubbles in both models become lower than the deterministic one. At a stochastic BGP, the bubble is higher when it plays the role of collateral than the role of liquidity, because this asset is riskier in terms of consumption in the model YS than in the model AI. Indeed, when a young household buys b_1, he faces the risk of a bubble crash at the adult age. On the contrary, when an adult buy b_2 in the model AI, he faces no risk in terms of future consumption, since the loans d_2 provide a consumption insurance against the risk of a bubble crash in the last period of life.

We explain now why growth, and therefore, investment in capital are larger in the model YS than in the model AI. The bubble is not bought at the same period of life in the two models. The sunspot process affects young individuals when the bubble plays the liquidity role and affects adult individuals when the bubble is used as a collateral. Therefore, when an adult invests in capital, she knows the value of the bubble when it has a liquidity role, whereas she does not know it when it has a collateral role. As a consequence, when a household borrows d_2 and invests in the model YS, she does not expose herself to the risk of a bubble crash. In contrast, this risk persists in the model AI because the bubble crash will affect the borrowing constraint, and therefore, loans used to invest in capital. These differences in the timing explain that the liquidity and collateral roles are no more equivalent when the bubble is stochastic, unlike the deterministic case.

We underline that the results obtained when the bubble is stochastic strongly depend on the type of asset representing loans. We assume that the loans d_2

ensure perfect consumption insurance to match the deterministic model in the limit case where $\pi = 1$. Other properties could characterize loans. For instance, if we consider that loans are remunerated by the same interest rate whatever the state of nature, the borrowing constraint would no more be binding at each state of nature in the AI model. This might rule out the crowding-in effect of the bubble. Our specification has been chosen to be in accordance with existing models dealing with a bubbly collateral (Kocherlakota (2009), Martin and Ventura (2016)).

5 Concluding remarks

Recently, several papers have identified some channels through which asset bubbles promote economic activity, as it is empirically observed. Two important features in these papers are the existence of some borrowing constraints and the heterogeneity of traders' behavior. However, bubbles have different roles. The two main ones are to provide liquidities and to serve as a collateral. In this paper, we introduce heterogeneous traders by considering an overlapping generations model with three period-lived households. Only adults have access to capital investment, and face a borrowing constraint. First, we show that the roles played by a deterministic bubble, namely to provide liquidities and to be a collateral, are perfectly equivalent. Then, we introduce asset bubbles in a more general way, that encompasses the roles just mentioned but not only, and show that a bubble may enhance growth. This conclusion is true for a given value of the bubble, whatever the type and the role attributed to the bubble, and who holds this asset.

We also extend our analysis to a stochastic bubble, which may crash at each period of time with a positive probability. In this case, there is a difference between the liquidity and collateral roles of the bubble. Because the liquidity role is more damaging in terms of consumption when a bubble crashes, the bubble size is lower than when the bubble has a collateral role. On the contrary, the liquidity role generates higher growth, because it is risk-free for investment in capital.

As we have seen, the asset bubble and the credit market allow for some transfers from the young and old agents to adults who invest in capital. Of course, if the young agents invest in capital rather than the adults, the conclusions are completely different. As shown by Raurich and Seegmuller (2019), the transfers are done from the adult age to the young and old ones. In their paper, the bubble enhances production because its existence relaxes the binding credit constraint and facilitates investment. Our paper is complementary to this previous one. Using these two contributions, the conditions for the existence of a crowding-in effect of the bubble are established, regardless the investment in capital is done at the young or at the adult age.

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Appendix

A Proof of Proposition 1

Define $\widetilde{\Omega}(g) \equiv \Omega(g, g)$. Using (38), we note that $\widetilde{\Omega}'(g) < 0$. We deduce that, at a BGP, $db/dg < 0$.

Consider now equation (39) at a steady state. When there is a bubble, the growth factor is given by $g = sA \equiv g^*$, and the bubble is positive if $\widetilde{\Omega}(g^*) > 0$, which is equivalent to (40).

At a bubbleless BGP, $b = 0$ and the associated growth factor g^{**} solves $\widetilde{\Omega}(g^{**}) = 0$. Since $\widetilde{\Omega}'(g) < 0$, we easily conclude that $g^{**} > sA = g^*$.

To analyze the stability properties of these two steady states, we use (38) and (39) to obtain:

$$g_{t+2}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{(1+\phi)s}\right]+\frac{1}{g_{t+1}}\left[\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{1+\phi}+\frac{1}{g_t}\frac{\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{1+\phi}\right]$$
$$-sA\left[1+\frac{1-s}{s(1+\phi)}\frac{\alpha\phi+\beta+\gamma(1+\phi)}{\alpha+\beta+\gamma}\right]=0$$

Linearizing this equation in the neighborhood of a steady state g, we get the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$, where the trace T and the determinant D of the associated Jacobian matrix are given by:

$$T = \frac{\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{1+\phi}+\frac{\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{(1+\phi)g}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}}\frac{1}{g^2} > 0 \quad \text{(A.1)}$$

$$D = -\frac{\frac{\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{(1+\phi)g^3}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} < 0 \quad \text{(A.2)}$$

Using these two equations, we easily compute:

$$P(-1) = 1+T+D = 1+\frac{\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{(1+\phi)g^2}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} > 0, \text{ for all } g > 0 \quad \text{(A.3)}$$

We now determine:

$$P(1) = 1 - T + D = 1 - \frac{\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{(1+\phi)g^2} + \frac{2\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{(1+\phi)g^3}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} \tag{A.4}$$

At the steady state $g = g^* = sA$,

$$P(1) = 1 - \frac{(1-s)[\beta+\gamma(2+\phi)]}{\alpha(s+\phi)+(\beta+\gamma)s(1+\phi)} < 0 \tag{A.5}$$

under inequality (40). Since $P(-1) > 0$, $P(0) = D < 0$, $P(1) < 0$ and $P(+\infty) > 0$, the bubbly steady state is a saddle because the two eigenvalues satisfy $\lambda_1 > 1$ and $\lambda_2 \in (-1, 0)$.

Using $\widetilde{\Omega}(g^{**}) = 0$, equation (A.4) evaluated at the steady state $g = g^{**}$ also writes:

$$P(1) = \frac{\left(\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma} + \frac{\gamma}{\alpha+\beta+\gamma}\frac{sA}{g^{**}}\right)\frac{(1-s)A}{(1+\phi)g^{**}}\left(1-\frac{sA}{g^{**}}\right) + \frac{\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{(1+\phi)g^{**}}\left(1-\frac{s^2A^2}{g^{**2}}\right)}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} \tag{A.6}$$

Since $g^{**} > sA$, we deduce that $P(1) > 0$. Therefore, at this steady state, we have $P(-1) > 0$, $P(0) < 0$ and $P(1) > 0$, meaning that the eigenvalues are such that $\lambda_1 \in (0, 1)$ and $\lambda_2 \in (-1, 0)$.

B Proof of Lemma 1

Using (20) and (57), $R_{t+1} > \theta q_{t+1}$ is equivalent to:

$$g_{t+1}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right] > \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t \tag{B.7}$$

By inspection of (56), this inequality is always satisfied. Using again (20) and (57), $R_{t+1} < q_{t+1}$ is equivalent to:

$$\begin{aligned} RHS(b_t) \equiv &\frac{\theta\gamma}{\beta+\gamma}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right]\left[b_t + \frac{\phi(1-s)A}{1+\phi} + \theta sA\right] \\ &< (1-\theta)\left[\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t\right] \equiv LHS(b_t) \end{aligned} \tag{B.8}$$

For $b_t \leqslant \bar{b}$, we deduce that:

$$\begin{aligned} RHS(b_t) &\leqslant \left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right]\frac{(1-s)A\theta\gamma}{\alpha+\beta+\gamma} \\ LHS(b_t) &> (1-\theta)\frac{\alpha\phi}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} \end{aligned}$$

Using these last two inequalities, inequality (B.8) is satisfied if $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \theta_b$.

C Proof of Proposition 3

Assume $\phi < \frac{\beta+\gamma}{\alpha}$, $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \min\{\theta_a, \theta_b\}$. Using (56), g_{t+1} is increasing (decreasing) in b_t if and only if $\theta < \widehat{\theta}$ ($\theta > \widehat{\theta}$) and g_{t+1} does not depend on b_t if and only if $\theta = \widehat{\theta}$. Then, using (60) and (63), we can show that $\widehat{\theta} < \theta_a$ is equivalent to:

$$\phi < \frac{\beta}{\alpha} < \frac{\beta+\gamma}{\alpha}$$

Using now (62) and (63), $\widehat{\theta} < \theta_b$ if and only if:

$$\phi[\gamma(1-s) - \beta s] < \beta s - (1-s)\gamma\frac{\beta+\gamma}{\alpha+\beta+\gamma}$$

which is equivalent to $\frac{s}{1-s} > \frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma} > \frac{\gamma}{\alpha+\beta+\gamma}$. Using Proposition 2, we easily deduce the proposition.

D Derivation of an equilibrium in the model YS with a stochastic bubble

Using (66), (68) and (73), we deduce that:

$$c^0_{2t+1} = \frac{\beta}{\beta+\gamma}(\phi w_{t+1} + R^d_{t+1}d_{1t}), \tag{D.9}$$

and using (65), (67) and (72),

$$c^+_{2t+1} = \frac{\beta}{\beta+\gamma}(\phi w_{t+1} + R^d_{t+1}d_{1t} + r_{t+1}b_{1t}) \tag{D.10}$$

$$k_{t+2} = \frac{\gamma}{\beta+\gamma}\frac{\phi w_{t+1} + R^d_{t+1}d_{1t} + r_{t+1}b_{1t}}{1-\theta\frac{q_{t+2}}{R^d_{t+2}}} \tag{D.11}$$

We use now (70) and (71), and substitute the consumptions (D.9) and (D.10) to get the arbitrage condition:

$$b_{1t+1} = \frac{\pi r_{t+1}/R^d_{t+1} - 1}{1-\pi}(\phi w_{t+1} + R^d_{t+1}d_{1t}) \tag{D.12}$$

which shows that the return of the bubble r_{t+1} times the probability of bubble persistence π should be higher than the return on deposits R^d_{t+1}.

Using the equilibrium prices (21) and (22), the equilibrium on the debt market $d_{1t} = -d_{2t} = \theta q_{t+1}k_{t+1}/R^d_{t+1}$, and the variables $k_t = (1+\phi)a_t$, $g_{t+1} = a_{t+1}/a_t$ and $\widetilde{b}_{1t} = b_{1t}/[(1+\phi)a_t]$, equation (D.11) implies that:

$$g_{t+1} = \frac{\gamma}{\beta+\gamma}\frac{\frac{\phi}{1+\phi}(1-s)A + \theta sA + \widetilde{b}_{1t}}{1-\theta\frac{sA}{R^d_{t+1}}} \tag{D.13}$$

Substituting the budget constraint (64) and the consumption (D.10) in the first order condition (71), we get:

$$\left[\frac{\alpha}{\beta+\gamma}\frac{\phi}{1+\phi}(1-s)A + \theta s A\left(\frac{\alpha}{\beta+\gamma} + \pi\frac{r_{t+1}}{R^d_{t+1}}\right)\right] g_{t+1}$$
$$= r_{t+1}\left[\pi\frac{(1-s)A}{1+\phi} - \widetilde{b}_{1t}\left(\pi + \frac{\alpha}{\beta+\gamma}\right)\right] \tag{D.14}$$

while the arbitrage condition (D.12) writes:

$$\frac{(1-\pi)(1+\phi)}{\phi(1-s)A + \theta s A(1+\phi)}\widetilde{b}_{1t+1} = \pi r_{t+1}/R^d_{t+1} - 1 \tag{D.15}$$

and the evolution of the bubble $b_{1t+1} = r_{t+1}b_{1t}$ is equivalent to:

$$g_{t+1}\widetilde{b}_{1t+1} = r_{t+1}\widetilde{b}_{1t} \tag{D.16}$$

Now, we can use (D.15) to substitute R^d_{t+1} in (D.14), and deduce the return of the bubble:

$$r_{t+1} = \frac{g_{t+1}\left[\frac{\alpha}{\beta+\gamma}\frac{\phi}{1+\phi}(1-s)A + \theta s A\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right]}{\pi\frac{(1-s)A}{1+\phi} - \widetilde{b}_{1t}\left[\pi + \frac{\alpha}{\beta+\gamma} + \theta s\frac{(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\right]} \tag{D.17}$$

Using (D.15) and (D.17), equation (D.13) gives the growth factor as a function of the level of the bubble. Finally, substituting (D.17) in (D.16), we deduce the dynamic path of the bubble. The resulting equations define the dynamic system (74)-(74).

E Derivation of an equilibrium in the model AI with a stochastic bubble

Substituting (77), (87) and (93) in the arbitrage condition (86), we get:

$$\frac{\alpha}{w_t - d_{1t}} = \frac{(\beta+\gamma)\pi R^d_{t+1}}{\phi w_{t+1} + R^d_{t+1}d_{1t}} + \frac{(\beta+\gamma)(1-\pi)\widetilde{R}^d_{t+1}}{\phi w_{t+1} + \widetilde{R}^d_{t+1}d_{1t}} \tag{E.18}$$

Using (91) and (92), and the equilibrium conditions which follows from these equations, we can substitute R^d_{t+1}, d_{1t}, and $\widetilde{R}^d_{t+1}$ to obtain:

$$\frac{\alpha(\theta q_{t+1}k_{t+1} + \pi R_{t+1}b_{2t})}{\pi R_{t+1}(w_t - b_{2t}) - \theta q_{t+1}k_{t+1}} = \frac{(\beta+\gamma)\pi(\theta q_{t+1}k_{t+1} + R_{t+1}b_{2t})}{\phi w_{t+1} + \theta q_{t+1}k_{t+1} + R_{t+1}b_{2t}}$$
$$+\frac{(\beta+\gamma)(1-\pi)\theta q_{t+1}k_{t+1}}{\phi w_{t+1} + \theta q_{t+1}k_{t+1}} \tag{E.19}$$

Recall that, when the bubble exists, b_{2t} is its price at time t and R_{t+1} is the increase of this price. Accordingly, the evolution of the bubble $b_{2t+1} = R_{t+1}b_{2t}$ writes:

$$g_{t+1}\widetilde{b}_{2t+1} = R_{t+1}\widetilde{b}_{2t} \tag{E.20}$$

and, using (21) and (22), equation (95) is equivalent to:

$$g_{t+1}\left(1-\theta\frac{sA}{\pi R_{t+1}}\right)=\frac{\gamma}{\beta+\gamma}\left[\frac{\phi(1-s)A}{1+\phi}+\widetilde{b}_{2t}+\theta sA\right] \tag{E.21}$$

Using (21)-(22) again and (E.19), we obtain:

$$\frac{\alpha g_{t+1}\left(\frac{\phi(1-s)A}{1+\phi}+\theta sA\right)+\alpha R_{t+1}\widetilde{b}_{2t}}{\pi R_{t+1}\left(\frac{(1-s)A}{1+\phi}-\widetilde{b}_{2t}\right)-\theta sAg_{t+1}}=(\beta+\gamma)\Psi(\widetilde{b}_{2t+1}) \tag{E.22}$$

where $\Psi(\widetilde{b}_{2t+1})$ is defined by (96). Note that (E.22) is equivalent to:

$$\begin{aligned}&g_{t+1}\left[\alpha\frac{\phi(1-s)A}{1+\phi}+\theta sA\left(\alpha+(\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\right)\right] \\ &=R_{t+1}\left[(\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\pi\frac{(1-s)A}{1+\phi}-\widetilde{b}_{2t}\left(\alpha+(\beta+\gamma)\pi\Psi(\widetilde{b}_{2t+1})\right)\right]\end{aligned} \tag{E.23}$$

Substituting R_{t+1}/g_{t+1} in the evolution of the bubble (E.20) and in (E.21), we get (97) and (98).

F Existence and uniqueness of a stochastic bubbly BGP in the model AI

Using (99), we can show that $B'(\Psi)>0$ and $B''(\Psi)<0$ for $\theta<\frac{\pi(1-s)}{(1-\pi)s}$. Moreover, $\Psi'(\widetilde{b}_2)>0$ and $\Psi''(\widetilde{b}_2)<0$, which implies that $B(\Psi(\widetilde{b}_2))$ is increasing and strictly concave in $\widetilde{b}_2$ because $B'(\Psi)\Psi'(\widetilde{b}_2)>0$ and $B''(\Psi)\Psi'(\widetilde{b}_2)^2+B'(\Psi)\Psi''(\widetilde{b}_2)<0$.

Therefore, there is a unique positive solution $\widetilde{b}_2>0$ to equation (99) if $B(\Psi(0))>0$. This is ensured by $\phi<\pi\frac{\beta+\gamma}{\alpha}$ and $\theta<\frac{1-s}{s(1+\phi)}\frac{(\beta+\gamma)\pi-\alpha\phi}{\alpha+\beta+\gamma}$.

G Comparison of the stochastic bubbly growth rates in the YS and AI models

When $\theta=0$, the growth rates g_1 and g_2 are equal. Indeed, using (76) and (99), we have $\widetilde{b}_1=\widetilde{b}_2$ and, using (74) and (98), the expressions of the growth rates are identical in the YS and AI models.

From (74) and (76), we obtain:

$$\begin{aligned}g_1=&\frac{\theta s(1-s)A(\beta+\gamma)}{\alpha\phi(1-s)+\theta s(1+\phi)(\alpha+\beta+\gamma)}+\frac{\gamma A}{\beta+\gamma}\left[\frac{\phi}{1+\phi}(1-s)+\theta s\right]\\ &+\left[\frac{\gamma}{\beta+\gamma}-\theta s\frac{\alpha+\beta+\gamma}{\alpha\frac{\phi}{1+\phi}(1-s)+\theta s(\alpha+\beta+\gamma)}\right]\left(\frac{\frac{(1-s)A}{1+\phi}\frac{\pi(\beta+\gamma)-\alpha\phi}{\alpha+\beta+\gamma}-\theta sA}{\frac{\alpha+\pi(\beta+\gamma)}{\alpha+\beta+\gamma}+\frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\frac{\beta+\gamma}{\alpha+\beta+\gamma}}\right)\end{aligned}$$

Differentiating this equation, we deduce:

$$\left.\frac{dg_1}{d\theta}\right|_{\theta=0} = sA\left\{\begin{array}{c} \frac{(\beta+\gamma)}{\alpha\phi}+\frac{\gamma}{\beta+\gamma}-\left(\frac{\alpha+\beta+\gamma}{\alpha\phi}\right)\left(\frac{\pi(\beta+\gamma)-\alpha\phi}{\alpha+\pi(\beta+\gamma)}\right) \\ -\frac{\gamma}{\beta+\gamma}\left(\frac{[\alpha+\pi(\beta+\gamma)]+[\pi(\beta+\gamma)-\alpha\phi]\left(\frac{(1-\pi)}{\phi}\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)}{\frac{[\alpha+\pi(\beta+\gamma)]^2}{\alpha+\beta+\gamma}}\right)\end{array}\right\}$$

We focus now on g_2. We note that from (96), we obtain $\left.\frac{d\Psi(\widetilde{b}_{2t+1})}{d\widetilde{b}_2}\right|_{\theta=0}=0$. Then, using (99), we get:

$$\begin{aligned}\left.\widetilde{b}_2\right|_{\theta=0} &= \frac{(1-s)A}{1+\phi}-\frac{(1-s)A\alpha}{\alpha+(\beta+\gamma)\pi} \\ \left.\frac{d\widetilde{b}_2}{d\theta}\right|_{\theta=0} &= -\frac{sA\,(\alpha+\beta+\gamma)}{\alpha+(\beta+\gamma)\pi}\end{aligned}$$

Therefore, using (98), we deduce that:

$$\left.\frac{dg_2}{d\theta}\right|_{\theta=0} = sA\left\{\begin{array}{c} \frac{\gamma}{\beta+\gamma}+\frac{(\beta+\gamma)}{\alpha\phi}-\frac{(\alpha+\beta+\gamma)}{\alpha+(\beta+\gamma)\pi}\left(\frac{\gamma}{\beta+\gamma}\right) \\ -\left(1-\frac{(1+\phi)\alpha}{\alpha+(\beta+\gamma)\pi}\right)\left[\frac{\alpha+(\beta+\gamma)\pi}{\pi\alpha\phi}\right]\end{array}\right\}$$

We compare now the two derivatives. The inequality $\left.\frac{dg_1}{d\theta}\right|_{\theta=0} > \left.\frac{dg_2}{d\theta}\right|_{\theta=0}$ is satisfied if and only if:

$$[\pi(\beta+\gamma)-\alpha\phi] > [\pi(\beta+\gamma)-\alpha\phi]\left(\frac{\gamma\pi}{\alpha+\pi(\beta+\gamma)}\right)$$

A positive bubble at the two BGPs requires $\pi(\beta+\gamma)-\alpha\phi>0$. Thus $\left.\frac{dg_1}{d\theta}\right|_{\theta=0} > \left.\frac{dg_2}{d\theta}\right|_{\theta=0}$ holds. Since $g_1|_{\theta=0}=g_2|_{\theta=0}$, we conclude that, by continuity, $g_1>g_2$ for a sufficiently low $\theta>0$.

H Degree of pledgeability less than one for the bubbly collateral

Let us consider that the borrowing constraint is characterized by a degree of pledgeability $\xi \in (0,1)$ for the bubbly collateral. The borrowing constraint rewrites:

$$-R^d_{t+2}d_{2t+1} \leqslant \theta q_{t+2}k_{t+2}+\xi R_{t+2}b_{2t+1} \tag{H.24}$$

Except this constraint, the model is similar to the one in Section 3.3. We associate the multipliers λ_{1t}, λ_{2t+1} and λ_{3t+2} to the budget constraints (23), (24) and (25), and μ_{t+1} to the borrowing constraint (H.24). We obtain the following first order conditions:

$$\alpha u_1'(c_{1t})=\lambda_{1t}, \beta u_2'(c_{2t+1})=\lambda_{2t+1}, \gamma u_3'(c_{3t+2})=\lambda_{3t+2} \tag{H.25}$$

$$\lambda_{1t}=R^d_{t+1}\lambda_{2t+1}, \lambda_{1t}=r_{t+1}\lambda_{2t+1} \tag{H.26}$$

$$\lambda_{2t+1}=q_{t+2}(\lambda_{3t+2}+\theta\mu_{t+1}) \tag{H.27}$$

$$\lambda_{2t+1}=R^d_{t+2}(\lambda_{3t+2}+\mu_{t+1}) \tag{H.28}$$

$$\lambda_{2t+1}=R_{t+2}(\lambda_{3t+2}+\xi\mu_{t+1}) \tag{H.29}$$

Using (H.26), we easily deduce that $R^d_{t+1} = r_{t+1}$. Using (H.27)-(H.29), we get $R^d_{t+2} < q_{t+2}$ and $R^d_{t+2} < R_{t+2}$.

We deduce that at an equilibrium, $r_{t+1} = R^d_{t+1} < R_{t+1}$. This means that b_{1t} and b_{2t} cannot represent the same asset, but are different bubbly assets with different returns:

$$b_{1t+1} = r_{t+1} b_{1t} \tag{H.30}$$

$$b_{2t+1} = R_{t+1} b_{2t} \tag{H.31}$$

Let us introduce $\widehat{b}_{it} \equiv b_{it}/[(1+\phi)a_t]$. Equations (H.30) and (H.31) are equivalent to:

$$g_{t+1}\widehat{b}_{1t+1} = r_{t+1}\widehat{b}_{1t} \tag{H.32}$$

$$g_{t+1}\widehat{b}_{2t+1} = R_{t+1}\widehat{b}_{2t} \tag{H.33}$$

An equilibrium with $b_{1t} > 0$ and $b_{2t} > 0$ requires that $r_{t+1} \leqslant g_{t+1}$ and $R_{t+1} \leqslant g_{t+1}$ for an infinite number of periods.[15] Otherwise, one of the bubbly asset explodes, and can no more be bought by the households. In such a case, we have $b_{1t} = 0$ and/or $b_{2t} = 0$, which rules out the liquidity and/or the collateral role of the bubble. Assuming that the last two inequalities hold, we have:

$$\frac{\widehat{b}_{1t+1}}{\widehat{b}_{2t+1}} = \frac{r_{t+1}}{R_{t+1}}\frac{\widehat{b}_{1t}}{\widehat{b}_{2t}} \tag{H.34}$$

Since $r_{t+1} < R_{t+1}$, $\widehat{b}_{1t}/\widehat{b}_{2t}$ tends to zero in the long run. This means that with respect to the collateral role, the liquidity role of the bubble disappears in the long run.

The reason of this result is the following. The bubbly asset bought when young is a perfect substitute for debt. Therefore, both these assets have the same return. Because of the degree of pledgeability $\xi < 1$ in the borrowing constraint (H.24), the bubble bought when adult is less useful than debt. This implies that the return R_{t+2} of b_{2t+1} needs to be higher than the return of debt, otherwise this asset will never be hold by households. As a consequence, the bubbly asset bought at the adult age b_{2t} grows at a higher rate than b_{1t}, and the liquidity role of the bubble tends to disappear in the long run.

Note that, in the model, we do not impose that b_{1t} and b_{2t} represent a priori the same asset. If it was the case, they would have the same return $r_{t+1} = R_{t+1}$. Using (H.26), the first order condition (H.28) would hold as an equality, but (H.29) would become the inequality $\lambda_{2t+1} > R_{t+2}(\lambda_{3t+2} + \xi\mu_{t+1})$. This would imply that households sell short an infinite amount of the bubbly asset b_{2t}, which cannot be sustained as an equilibrium.

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