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Abstract: In a seminal article, Obstfeld (1994) showed that growth and welfare gains from international risk-sharing arise in a continuous-time stochastic AK model. More precisely, he proved that a portfolio shift from safe and low-return capital to riskier and high-return capital triggers an unambiguous increase in growth. In this note I stress necessary and sufficient conditions ensuring stochastic stability of the exponential balanced-growth path, an issue that has not been addressed by Obstfeld. Not surprisingly, stability requires the average of the wealth growth rate to be positive, which makes clear how mean growth should be defined. Differently, though, Obstfeld defines mean growth as the growth rate of average wealth, which is larger than the mean growth rate of wealth under the maintained assumption that wealth is log-normally distributed, because the latter growth concept is risk-adjusted. The two notions of mean growth have very different comparative statics properties both for economies that hold some risk-free capital and for economies that fully specialize in risky capital. Different from Obstfeld's results, international financial integration increases the stability-related mean growth rate for both complete and incomplete specialization, if risk aversion takes on moderate values and provided that the intertemporal substitution elasticity is smaller than one. Although the welfare computations presented by Obstfeld are preserved, because they ultimately depend on parameter values, this note shows that stochastic stability sheds new light on the mechanisms that trigger growth changes under financial integration and underlines the intuition behind them.

Keywords: International Financial Integration, Endogenous Growth, Stochastic Stability

Journal of Economic Literature Classification Numbers: F34, F43, O40

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1 Introduction

Two decades ago was published a seminal article by Obstfeld (1994), which shows how large welfare gains from international risk-sharing arise in a continuous-time stochastic AK model. The intuition for why such large welfare benefits materialize is developed first, in section I of that paper, in the closed-economy version of the model where individuals may invest in risk-free capital and risky capital. The central result is that if the economy holds *some* risk-free capital, a fall in exogenous risk unambiguously leads to an increase in the share of wealth invested in risky capital. In other words, a portfolio shift toward riskier capital triggers a return effect that dominates the possibly adverse effect of a higher propensity to consume out of total wealth. As a consequence, growth and welfare go up. In contrast, goes the intuition, complete specialization in risky capital results in an ambiguous effect of exogenous risk on growth, that is entirely governed by whether the intertemporal substitution elasticity is larger or smaller than one.

In this note I show that such results are based on a definition of mean growth that seems natural absent any concern about stability of the balanced-growth path but that turns out to be misleading if the issue of stability is addressed, as it should be. I make clear why well-known conditions ensuring stochastic stability of the balanced-growth path (exposed in Mao, 2011, and used in Boucekkine and Zou, 2014) relate to an alternative definition of mean growth. To understand why a confusion may arise, it is important to stress that Obstfeld (1994) considers an AK model where risk is introduced via diffusion equations that involve geometric Brownian motions. In such a setting, the dynamics of the economy that result from optimal portfolio and savings decisions are described by a linear, stochastic differential equation with fixed coefficients and one might wrongly think that two correct concepts of growth are available to freely choose from. The first defines mean growth as the growth rate of average wealth, which is the one used by Obstfeld (1994) and many others (see e.g. Jones and Manuelli, 2003, Steger, 2005). The alternative definition, which emerges naturally in the quest for stability conditions that is developed below, is to define mean growth as the average growth rate of wealth, as opposed to the growth rate of average wealth. Because by assumption wealth is log-normally distributed, it follows by Jensen's inequality that the average growth rate of wealth - second notion - is lower than the growth rate of average wealth - first notion. Not surprisingly, stochastic stability holds if and only if the average growth rate is positive, a condition that is stronger than the requirement that the growth rate of average wealth be positive.

The contribution of this note is to show that very different comparative statics results obtain when one uses the second definition of mean growth, as one should in view of stability conditions. More precisely, mean growth happens to be enhanced by financial integration under conditions that would possibly lead to the opposite conclusion if one were to use the definition of mean growth advocated in Obstfeld (1994). This property is most striking in a specialized economy, where for example a fall in exogenous risk results in larger growth even if the intertemporal substitution elasticity is smaller than one, despite the fact that a portfolio shift does not happen. Before moving on to the technical analysis, I should stress that the welfare computations provided by Obstfeld (1994) are completely unaltered by stability considerations, because they of course ultimately depend on parameter values and not on how one defines mean growth. However, stability unequivocally imposes a definition of mean growth that leads to very different comparative statics and, therefore, sheds new light on the mechanisms that trigger growth changes. For instance, it turns out that both risk aversion and the intertemporal substitution elasticity matter to determine whether or not financial integration boosts growth. Section 3, in particular, shows how conflicting return and variance effects help explain what happens in intuitive terms. A summary of the results unveiled in this note is that although growth gains from international financial integration turn out to be smaller than previously thought, they are likely to be positive both for specialized economies and for economies that hold some risk-free capital, under reasonable parameter values.

The paper is organized as follows. Section 2 presents conditions for stochastic stability of the exponential balanced-growth path and the associated definition of mean growth. Section 3 then performs comparative statics analysis and compares results with those obtained by Obstfeld (1994).

2 Stochastic Stability and the Definition of Mean Growth

Obstfeld (1994) considers an AK version of the optimal portfolio model developed in Merton (1969) and Samuelson (1969). The following equation describes optimal wealth accumulation and is identical to equation [14] derived in Obstfeld (1994)[1]:

$$dW = [\omega\alpha + (1-\omega)i - \mu]Wdt + \omega\sigma W dz \qquad (1)$$

[1] For clarity, I use square brackets to label equations that appear in Obstfeld (1994) and round brackets for equations in this paper.

where $\alpha > 0$ and $i > 0$ are the mean returns of risky capital and risk-free bonds respectively, $\mu > 0$ is the average propensity to consume out of total wealth, dz is a Wiener process and $\sigma^2 \geq 0$ is the exogenous variance of the return on risky capital . In equation (1), $1 \geq \omega \geq 0$ denotes the share of wealth invested in risky capital and its expression is given in equation [11], that is:

$$\omega \equiv \frac{\alpha - i}{R\sigma^2} > 0 \tag{2}$$

Two cases occur, depending on whether ω is smaller than or equal to one. We will refer to the first case as incomplete specialization - when the economy holds some risk-free bond - and to the second case as complete specialization - when the economy has all its wealth invested in risky capital. It is important to notice a major difference between the two configurations: when $\omega < 1$, Obstfeld (1994) shows that i equals r, the mean return on risk-free capital such that $r < \alpha$, so that a fall in exogenous risk σ^2 always results in a portfolio shift away from risk-free capital, that is, ω goes up. This first case occurs when $R\sigma^2 > \alpha - r$, that is if (utility adjusted) risk is large enough to prevent complete specialization. If, however, $R\sigma^2 < \alpha - r$, specialization is complete because risk is small enough to ensure $\omega = 1$. In that case a fall in exogenous risk triggers a rise in risk-free return $i = \alpha - R\sigma^2$ that compensates for the fall in σ^2 so that the economy keeps all its wealth in risky capital and enjoys lower risk.

Conditions for stochastic stability of linear differential equations with fixed coefficients are stated in chapter 4 of Mao (2011) and applied in Boucekkine and Zou (2014) to a simpler AK model. Stated in terms of a generic diffusion equation $dX = \phi X dt + \theta X dz$, the unique solution of which is $X(t) = X(0)\exp\{(\phi - 0.5\theta^2)t + \theta z(t)\}$, the main result is that the variable X tends exponentially to infinity with probability one, as time goes to infinity, if and only if $\phi - 0.5\theta^2 > 0$, whereas X tends to zero with probability one if and only if $\phi - 0.5\theta^2 < 0$. A straightforward application of such conditions leads to the following lemma:

Lemma 2.1 (Stochastic Stability of the Balanced-Growth Path)

Wealth tends exponentially to infinity, along a balanced growth path, with probability one when time tends to infinity if and only if $\omega\alpha + (1-\omega)i - \mu > 0.5\omega^2\sigma^2$.

Obstfeld (1994) defines mean growth - g in his notation - as the *growth rate of average wealth*, which is given in view of equation (1) by $\omega\alpha + (1-\omega)i - \mu$ or, equivalently after plugging the expression of the share invested in risky capital, by $g(\mathbf{E}[W]) = \varepsilon(i-\delta)+(1+\varepsilon)(\alpha-i)^2/(2R\sigma^2)$, where $\varepsilon > 0$ is the elasticity of intertemporal substitution in consumption and $R > 0$ is relative risk aversion (see equation [16] in Obstfeld, 1994). Lemma 2.1 shows that $g(\mathbf{E}[W]) > 0$ is necessary but not sufficient for stochastic stability of exponential growth at a positive rate. In other words, assuming $g(\mathbf{E}[W]) > 0$ would result in convergence to *zero* wealth with probability one provided that $g(\mathbf{E}[W]) < 0.5\omega^2\sigma^2$. This is an example of the well-known fact that noise, if big enough, can significantly alter and sometimes overturn convergence: although in that case the deterministic model experiences exponential growth at a positive rate, the stochastic version converges to the trivial solution $W = 0$ with probability one (see Mao, 2011, and Boucekkine and Zou, 2014, for examples of that sort).

Lemma 2.1 therefore suggests that mean growth should be defined as the *average of the wealth growth rate*[2], that is, $\mathbf{E}[g(W)] \equiv \omega\alpha + (1-\omega)i - \mu - 0.5\omega^2\sigma^2$ which can be simplified, using the expression of ω in (2), to:

$$\mathbf{E}[g(W)] = \varepsilon(i-\delta) + \frac{(\alpha-i)^2}{2R\sigma^2}\left(1+\varepsilon-\frac{1}{R}\right) \tag{3}$$

where $\delta > 0$ is the subjective rate of time preference. A few comments are in order. Because wealth is assumed to be log-normally distributed, the property that $\mathbf{E}[g(W)] < g(\mathbf{E}[W])$ follows, of course, from Jensen's inequality: the expected value of the log of wealth is smaller than the log of expected wealth and a similar inequality applies to their derivatives with respect to time. More importantly, one goes from the first definition of mean growth, used by Obstfeld (1994), to the second, more appropriate, one by subtracting half of the (endogenous) variance of wealth, that is, $(\alpha-i)^2/(2R^2\sigma^2)$, hence the additional term $-1/R$ in equation (3). Therefore, comparative statics results are expected to be very different, as I show next.

3 Comparative Statics of Mean Growth

Straightforward computations lead to the following main result of this note.

[2] In fact, Obstfeld (1994) uses later in his paper this notion for measurement purpose, e.g. in page 1321, although not for the comparative statics analysis developed at the outset.

Proposition 3.1 (Comparative Statics of Mean Growth)

The dynamics of wealth accumulation defined in equation (1) has two regimes:

(i) if $R\sigma^2 > \alpha - r$ *(incomplete specialization): the average growth rate* $\mathbf{E}[g(W)]$ *is a decreasing function of exogenous risk* σ^2 *if and only if* $R(1+\varepsilon) > 1$*, that is, for large values of either risk aversion or of the intertemporal substitution elasticity.*

(ii) if $\alpha - r > R\sigma^2$ *(complete specialization): the average growth rate* $\mathbf{E}[g(W)]$ *is a decreasing function of exogenous risk* σ^2 *if and only if* $R(1-\varepsilon) < 1$*, that is, for small values of risk aversion and large values for the intertemporal substitution elasticity.*

Not surprisingly, comparing Proposition 3.1 and results in Obstfeld (p. 1315, 1994) shows important differences. As shown in case (*i*) of Proposition 3.1, incomplete specialization results in larger growth when exogenous risk falls down only for large enough values of either risk aversion or of the intertemporal substitution elasticity. In contrast, Obstfeld (1994) claims that a portfolio shift unambiguously improves growth, independent of R and ε. When the correct definition of mean growth is used, this is no longer true. In addition, case (*ii*) of Proposition 3.1 shows that the results obtained by Obstfeld (1994) for complete specialization can be overturned under reasonable assumptions on parameters. A striking example is the case of unitary intertemporal substitution elasticity, that is, $\varepsilon = 1$. Whereas this case implies that the growth rate of average wealth is independent of exogenous risk in Obstfeld (1994) (see his equation [17]), case (*ii*) in Proposition 3.1 shows that the average growth rate is in fact a decreasing function of exogenous risk for all values of risk aversion. This property suggests that international financial integration is likely to boost growth in economies that invest all their wealth in risky capital.

More generally, conditions ensuring that a fall in exogenous risk boosts growth for both complete and incomplete specialization become clearer under the assumption that the intertemporal substitution elasticity is smaller than one, which seems to accord better with empirical measures. Remember that in Obstfeld (1994), in this case growth unambiguously goes up under incomplete specialization whereas growth *slows down* in specialized economies, following financial integration. In contrast, Proposition 3.1 shows that using the correct definition of mean growth delivers a more contrasted picture: when $\varepsilon < 1$, a fall in exogenous risk leads to an increase in mean growth provided that relative risk aversion takes on moderate values, that is, if and only if $1/(1-\varepsilon) > R > 1/(1+\varepsilon)$. For example, the latter inequalities are met when $R = 1$. The bottom line is that because it leads to smaller exogenous risk, financial integration is expected to improve mean growth for both complete and incomplete specialization under reasonable parameter values.

So as to clarify the intuition behind the striking differences with results reported in Obstfeld (1994), I now focus on the case such that $\varepsilon = 1$, which leads to the well-known result that the average propensity to consume out of total wealth is then given by the impatience rate, that is, $\mu = \delta$. This assumption neutralizes the effect of exogenous risk on the consumption-wealth ratio, which has been described in earlier papers and in Obstfeld (1994) in particular. I now explain how a fall in exogenous risk affects mean growth. Again, two cases arise depending on the level of exogenous risk.

(*i*) if $R\sigma^2 > \alpha - r$, specialization is incomplete because exogenous risk is so large that the economy holds some risk-free capital (that is, $\omega < 1$). It follows that the risk-free interest rate $i = r$ and that the expression for mean growth simplifies to:

$$\mathbf{E}[g(W)] = r - \delta + \underbrace{\frac{(\alpha - i)^2}{R\sigma^2}}_{\text{mean return effect}} - \underbrace{0.5\frac{(\alpha - i)^2}{R^2\sigma^2}}_{\text{variance effect}} \tag{4}$$

The expression for mean growth in equation (4) reveals that two conflicting effects are at work. The return effect is such that a fall in exogenous risk σ^2 boosts welfare growth because, as shown by Obstfeld (1994), the portfolio shift away from risk-free capital increases growth under the assumption that risky capital has a larger mean return than risk-free capital. However, although ignored by Obstfeld (1994), a variance effect also materializes, essentially because a larger share in the risky asset implies that the endogenous variance of wealth goes *up* when exogenous risk goes down. Stochastic stability of the balanced-growth path requires the variance effect to be not too large but such a condition does not exclude that mean growth be a decreasing function of exogenous risk, then overturning Obstfeld's result, if risk aversion is less than one half.

(*ii*) if $\alpha - r > R\sigma^2$, specialization is complete ($\omega = 1$). It follows that the risk-free interest rate adjusts to ensure $i = \alpha - R\sigma^2 > r$ and that the expression for mean growth simplifies to:

$$\mathbf{E}[g(W)] = \alpha - \delta - \underbrace{0.5\sigma^2}_{\text{variance effect}} \tag{5}$$

Equation (5) makes clear what happens when specialization is complete. In contrast to case (*i*), there is no return effect because the economy already benefits from full specialization so that a fall in σ^2 has no effect on the mean return - there is no portfolio shift. However, a variance effect still occurs but it now has an opposite effect on mean growth compared to case (i). This is because the endogenous variance of wealth now goes *down* when exogenous risk goes down, as the risk-free return goes up to ensure that specialization remains complete in the face of a fall in risk. Quite interestingly, an analysis based on the alternative but misleading notion of mean growth, as in Obstfeld (1994), predicts that growth is not affected by such a fall in risk.

Relaxing the assumption that $\varepsilon = 1$ delivers similar intuitions. In case (i) the return effect dominates the variance effect so that a fall in exogenous risk fosters growth if and only if risk aversion is large enough. In case (ii) there is no return effect and the variance effect, now working in opposite direction, implies that growth improves after a fall in exogenous risk only if risk aversion is not too large when the intertemporal substitution elasticity is smaller than unity. In other words, our results about specialized economies accord with the well-documented trade-off between growth and volatility under reasonable assumptions about attitudes toward risk, for example if relative risk aversion equals one. In contrast, incomplete specialization leads to a positive relationship between the mean growth and variance of wealth under unitary risk aversion. Overall these results suggest that taking into account the variance effect on mean growth, which has been ignored by Obstfeld (1994), yields the prediction that the effects of financial integration on economies that specialize in risky capital do not qualitatively differ from those on economies that hold some risk-free capital if reasonable parameter values are assigned to risk aversion and intertemporal substitution.

To make the comparison even more transparent, I now reproduce and extend in Table 1 a numerical example given in Obstfeld (1994). More precisely, Table 1 starts with the Example 1 that is presented in pages 1318-1319 of Obstfeld (1994) and that assumes $R = 4$ and $\varepsilon = 0.5$ in particular. Table 1 compares the magnitudes of both definitions of mean growth under this parameterization and also, for robustness purpose, when $R = 1$ while all other parameter values are unchanged.

Table 1. Numerical Values of $\mathbf{E}[g(W)]$ in Left Panel and $g(\mathbf{E}[W])$ in Right Panel

	$R = 4$	$R = 1$
Autarky	1.41%	1.25%
Integration	1.75%	1.38%

	$R = 4$	$R = 1$
Autarky	1.69%	1.75%
Integration	2.00%	1.63%

In line with the analytical characterization outlined above, comparing both panels in Table 1 confirms that the mean growth rate of wealth is lower than the growth rate of mean wealth. More interestingly, comparing the rightmost columns of both panels reveals that, when $R = 1$, the conclusion regarding growth that is obtained by Obstfeld (1994) is overturned when the appropriate concept of mean growth is adopted. In fact, while the right panel predicts that growth falls (by about 12 basis points) after integration in the case of full specialization, it turns out that growth actually *goes up* (by about 13 basis points) as depicted in the left panel that uses the appropriate definition of mean growth. Let me stress that although welfare computations reported in Obstfeld (1994) are not altered at all by stability considerations, the examples in Table 1 further confirm that different comparative statics properties obtain when the stability-related concept of mean growth is used, as it should be. Aside from theoretical concerns, this is also relevant for empirical research, which typically aims at measuring the growth gains from international financial integration.

4 Conclusion

This note shows, by way of analytical results and numerical examples, that the comparative statics results derived in Obstfeld (1994) are misleading because they are based on an inappropriate notion of mean growth. Conditions ensuring that the exponential balanced-growth path is stable, in the stochastic sense, reveal that mean growth should be defined as the average growth rate of wealth, as opposed to the growth rate of average wealth. With such a definition in hand, I show that international financial integration leads to very different comparative statics results and that it is much more likely to boost growth, both for fully specialized economies that invest all their wealth in risky capital and for economies that hold some risk-free capital. This note shows that although the growth gains from financial integration are in fact smaller than previously thought, they are more likely to be positive for empirically reasonable values of relative risk aversion and the intertemporal substitution elasticity, independently of whether or not the economy invests in risk-free capital.

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