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Stochastic Stability of Endogenous Growth: The AK Case

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Stochastic stability of endogenous growth: The AK case*

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Abstract

This note studies the stochastic stability of the standard AK growth model under uncertain output technology. Capital accumulation follows a stochastic linear homogenous differential equation. It's shown that exponential balanced paths, which characterize optimal trajectories in the absence of uncertainty, are not robust to uncertainty. Precisely, it's demonstrated that the economy almost surely collapses at exponential speed even though productivity is initially arbitrarily high.

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1 Introduction

The neoclassical stochastic growth models has been the subject of a quite visible economic theory literature since the celebrated Brock and Mirman's 1972 paper (see also Mirman and Zilcha, 1975, and the less known contribution of Merton, 1975). This notably includes the study of the existence of stochastic steady states and their stability. In contrast, no such a literature exists for endogenous growth models. This is partly due to the fact that many of these models rely on zero aggregate uncertainty as in the early R&D based models (see for example, Barro and Sala-i-Martin, 1995, chapters 6 and 7). When uncertainty does not vanish by aggregation as in de Hek (1999), the usual treatment consists in applying Merton's portfolio choice methodology (Merton, 1969 and 1971) to track expected growth rate and its volatility. Recently, Steger (2005) and Boucekkine et al. (2014) apply the same methodology to study the stochastic AK model for a closed economy and for a small open capital constrained economy respectively. Precisely, these authors assume the existence of balanced growth paths (as in the deterministic counterparts) and compute the associated expected growth rates and growth volatilities, without addressing the issue of stochastic stability of the selected paths.

In this note we tackle the latter issue. It's important to notice at this stage that one cannot solve the problem by adapting the available proofs in Brock and Mirman (1972) or Merton (1975). For example, strict concavity of the production function is needed in the latter in order to build up the probability measure for stability in distribution. We take a much simpler approach here taking advantage of the linearity of the production function. When uncertainty is modelled as a Brownian motion, we show that the study of stochastic stability in such a case amounts to studying stability of a standard stochastic linear differential equation. Relying on the specialized mathematical literature (Mao, 2011, or Khasminskii, 2012), we are able to conclude. Strikingly enough, we ultimately show that the typical balanced growth paths are hardly stochastically stable in our simple framework.

This note is organized as follows. Section 2 presents the stochastic AK model as considered and the inherent stochastic stability problem. Section 3 gives the stability results after some preliminary definitions.

2 The problem

Consider strictly increasing and strictly concave utility

$$\max_c E_0 \int_0^\infty U(c) e^{-\rho t} dt, \tag{1}$$

subject to

$$dk(t) = (Ak(t) - c(t) - \delta k)dt + bAkdW(t), \quad \forall t \geq 0 \tag{2}$$

where initial condition $k(0) = k_0$ is given, positive constants δ and ρ measure depreciation and time preference, respectively, $W(t)$ is one-dimensional Brownian motion, and b reads volatility. Define Bellman's value-function as

$$V(k,t) = \max_c E_t \int_t^\infty u(c) e^{-\rho t} dt.$$

Then using Merton (1969), the stochastic Hamilton-Jacob-Bellman equation is

$$\rho V(k) = \max_c \left[U(c(t)) + V_k \cdot (Ak(t) - c(t) - \delta k) + \frac{1}{2} b^2 A^2 k^2 V_{kk} \right] \tag{3}$$

with V_k reads first order derivative with respect to k. First order condition on the right hand side of (3) reads

$$U'(c) = V_k(k). \tag{4}$$

Due to the strictly concave utility, the solution of (4), $c^*(t) = c^*(k(t))$, is optimal to the right hand side of (3). Substituting this optimal choice into (3), it follows

$$\rho V(k) = U(c^*(t)) + V_k \cdot (Ak(t) - c^*(t) - \delta k) + \frac{1}{2} b^2 A^2 k^2 V_{kk}. \tag{5}$$

To find an explicit solution, we take CRRA–Constant Relative Risk Aversion:

$$U(c) = \frac{c^\gamma}{\gamma}, \; 0 < \gamma < 1.$$

It's worth pointing out here that such a range of values for γ implies that $U(0) = 0$, that's instantaneous utility is bounded from below. Therefore, consumption going to zero is not ruled out from the beginning. Moreover, the assumed γ-values imply that the intertemporal elasticity of substitution (equal to $\frac{1}{1-\gamma}$) is above unity, which has the typical economic implications on the relative size of the income vs substititution effects. This will reveal important for the stochastic stability results obtained in Section 3.2. The first order condition yields the optimal choice

$$c^* = V_k^{\frac{1}{\gamma-1}}.$$

Substituting into the HJB equation (5), we have

$$\rho V(k) = V_k \cdot (A-\delta)k + \frac{1-\gamma}{\gamma} V_k^{\frac{\gamma}{\gamma-1}} + \frac{1}{2} b^2 A^2 k^2 V_{kk}. \tag{6}$$

Parameterizing the solution as

$$V(k) = H^{1-\gamma} \frac{k^\gamma}{\gamma},$$

with constant H undetermined, and substituting into (6), it is easy to obtain

$$\frac{1}{H} = \frac{\rho}{1-\gamma} + \frac{b^2 A^2 \gamma}{2} - \frac{\gamma(A-\delta)}{1-\gamma}. \tag{7}$$

Thus, the optimal choice is

$$c^* = \frac{k}{H}.$$

Then the dynamics of optimal capital accumulation follow

$$dk(t) = \left(A - \delta - \frac{1}{H}\right) k(t)dt + bAkdW(t) \tag{8}$$

which is a linear stochastic differential equation and the explicit solution is

$$k(t) = k(0)\ \exp\left\{\left[\left(A - \delta - \frac{1}{H}\right) - \frac{b^2 A^2}{2}\right] t + bAW(t)\right\}. \tag{9}$$

Two observations are in order here. First of all, it is worth pointing out that in the absence of uncertainty, that's when $b = 0$, one gets the typical results: in particular, for any initial condition $k(0) > 0$, the economy jumps on the optimal path given by (9) under $b = 0$, and the growth rate is exactly $\frac{A-\delta-\rho}{1-\gamma}$. The growth rate is strictly positive if and only if $A > \delta + \rho$ given that $0 < \gamma < 1$. Since there are no transitional dynamics, the convergence speed to the balanced growth path is infinite. Second, it is easy to see from the explicit solution above that due to the extra negative term, $-\frac{b^2 A^2}{2}$, the stability conditions may differ from the deterministic case. In order to state clearly the results of the stability, we first present the simplest linear homogenous stochastic differential equation and then introduce the definition(s) of stochastic stability. We finally apply the latter mathematical apparatus to the stochastic $AK-$growth model.

3 Stochastic stability results

Consider the typical linear Ito stochastic differential equation

$$dx(t) = ax(t)dt + bx(t)dB(t), t \geq 0$$

with initial condition $x(0) = x_0$ given, $B(t)$ standard Brownian Motion, a and b constants. The general solution takes the form

$$x(t) = x_0 \ \exp\left\{\left(a - \frac{b^2}{2}\right)t + bB(t).\right\} \tag{10}$$

As argued at the end of the previous section, an extra negative term, $-\frac{b^2}{2}$ shows up in the deterministic part of the solution compared to the solution of the counterpart deterministic differential equation (that is when $b = 0$). It's therefore easy to figure out why the noise term, $bx(t)dB(t)$, is indeed stabilizing. Incidentally, introducing some specific white noises is one common way to "stabilize" dynamics systems. The pioneering work belongs to Khasminskii (2012) and some more recent results can be found in Appleby et al. (2008) and references therein. Thus, the stability conditions under stochastic environments may well differ from the case with certainty. To tackle seriously this issue, we display some useful preliminary definitions.

3.1 Stability conditions under stochastic environments

For simplicity, we only present results of one-dimensional stochastic differential equation. First let us consider a homogenous linear Ito stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), t \geq t_0 \tag{11}$$

with initial condition $x(t_0) = x_0$ given and $B(t)$ standard Brownian Motion. Functions $f(x(t), t)$ and $g(x(t), t)$ check

$$f(0, t) = 0 \ \text{ and } \ g(0, t) = 0, \ \ \forall t \geq t_0.$$

Thus, solution $x^* = 0$ is a solution corresponding to initial condition $x_0 = 0$. This solution is also called trivial solution or equilibrium solution.

Then for the stability, we take the following definitions from the Definition 2.1 and 3.1, Mao (2011). [1]

Definition 1 *(i) The equilibrium (or trivial) solution ($x^* = 0$) of equation (11) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$, such that, probability checks*

$$P\{\mid x(t; x_0, t_0) \mid < r \ \ for \ \ all \ \ t \geq t_0\} \geq 1 - \varepsilon$$

whenever $\mid x_0 \mid < \delta$. Otherwise, it is said to be stochastically unstable.

(ii) The equilibrium solution, $x^* = 0$, of equation (11) is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0, 1)$, there exists a $\delta = \delta(\varepsilon) > 0$, such that,

$$P\{\lim_{t \to \infty} \mid x(t; x_0, t_0) \mid = 0\} \geq 1 - \varepsilon$$

whenever $x_0 < \delta$.

[1]See also Khasminskii (2012), section 5.3, page 152, section 5.4, page 155, and Definition 1 in section 5.4, page 157.

(iii) The equilibrium solution, $x^* = 0$, of equation (11) is said to be stochastically asymptotically stable in the large if it is stochastically stable and, moreover, for all x_0

$$P\{\lim_{t\to\infty} \mid x(t; x_0, t_0) \mid = 0\} = 1.$$

(iv) The equilibrium solution, $x^* = 0$, of equation (11) is said to be almost surely exponential stable if

$$\lim_{t\to\infty} \sup \frac{\log \mid x(t; x_0, t_0) \mid}{t} < 0 \;\; a.s.$$

for all x_0.

For our study of AK-model and dynamic equation (8), we first consider a homogenous linear stochastic differential equation

$$dx(t) = a(t)x(t)dt + b(t)x(t)dB(t), t \geq t_0 \tag{12}$$

with initial condition $x(t_0) = x_0$ given, $B(t)$ standard Brownian Motion, $a(t)$ and $b(t)$ known functions, we have solution as

$$x(t) = x_0 \; \exp\left\{\int_{t_0}^{t} \left(a(s) - \frac{b^2(s)}{2}\right) ds + \int_{t_0}^{t} b(s)dB(s)\right\} \tag{13}$$

Then the following stability results can be demonstrated for the general linear stochastic equation (12), whose proof can be found in Mao (2011), section 4.2.7 and 4.3.8, pages 117-119 and 126-127, respectively.[2]

Proposition 1 *Consider homogenous linear stochastic equation (12) and denote $\sigma(t) = \int_{t_0}^{t} b^2(s)ds$, we have*

- *(i) $\sigma(\infty) < \infty$, then the equilibrium solution, $x^* = 0$, of equation (12) is stochastically stable if and only if*

$$\lim_{t\to+\infty} \sup \int_{t_0}^{t} a(s)ds < +\infty.$$

[2]See also Khasminskii (2012), section 5.3, page 154, and section 5.5, page 159-160.

While the equilibrium solution is stochastically asymptotically stable in the large if and only if

$$\lim_{t\to+\infty}\int_{t_0}^{t} a(s)ds = -\infty.$$

- *(ii)* $\sigma(\infty) = \infty$*, then the equilibrium solution,* $x^* = 0$*, of equation (12) is stochastically asymptotically stable in the large if*

$$\lim_{t\to+\infty}\sup\frac{\int_{t_0}^{t}\left(a(s)-\frac{b^2(s)}{2}\right)ds}{\sqrt{2\sigma(t)\log\log(\sigma(t))}} < -1,\ a.s. \tag{14}$$

- *(iii) Specially, if both* $a(t) = a$ *and* $b(t) = b$ *are constants, (14) holds if and only if*

$$a < \frac{b^2}{2}.$$

  That is, equilibrium solution, $x^* = 0$*, of (12) is stochastically asymptotically stable in the large if* $a < \frac{b^2}{2}$*.*

- *(iv) The equilibrium solution,* $x^* = 0$*, of (12) is almost surely exponentially stable if* $a < \frac{b^2}{2}$*.*

The last two results read that if $a < \frac{b^2}{2}$, then almost all sample paths of the solution will tend to the equilibrium solution $x^* = 0$ and the this convergence is exponentially fast. This is not the case in the deterministic case.

Following the above proposition, to study the stability of capital accumulation (8) in the stochastic $AK-$model, we have to study the relationship between parameters $A - \delta - \frac{1}{H}$ and bA.

3.2 Application to the stochastic AK model

It is easy to check in $AK-$model, $\sigma(t) = \int_0^t bAds = bAt$. Therefore, $\sigma(\infty) = \infty$.

From the above mathematical results, the capital stock tends to equilibrium $k^* = 0$ if

$$\left(A - \delta - \frac{1}{H}\right) < \frac{b^2 A^2}{2}.$$

Substituting $\frac{1}{H}$ from (7) into the above inequality, we have

$$A - \delta - \frac{\rho}{1-\gamma} - \frac{b^2 A^2 \gamma}{2} + \frac{\gamma(A-\delta)}{1-\gamma} < \frac{b^2 A^2}{2},$$

which is equivalent to

$$F(A) \equiv \frac{b^2(1-\gamma^2)A^2}{2} - A + (\rho+\delta) > 0, \quad \text{with } \ 1-\gamma^2 > 0. \tag{15}$$

Obviously, $F(A)$ is a second degree polynomial in term of A and opens upward. Denote $\Delta = 1 - 2b^2\ (1-\gamma^2)(\rho+\delta)$.

Thus, (a) if $\Delta < 0$, that is, $b^2 > \frac{1}{2(1-\gamma^2)(\rho+\delta)}$, we have $F(A) > 0$, for any A; (b) if $\Delta \geq 0$, i.e., $b^2 \leq \frac{1}{2(1-\gamma^2)(\rho+\delta)}$ then $F(A) > 0$ for $A \in (0, A_1) \cup (A_2, +\infty)$, with A_i, $i = 1, 2$, are the two positive roots of $F(A) = 0$.

The above analysis is concluded in the following:

Proposition 2 *Consider problem (1) with constraint (2). The equilibrium $k^* = 0$ is (globally) stochastically asymptotically stable in the large and almost surely exponentially stable, if one of the two following conditions hold: (a) $b^2 > \frac{1}{2(1-\gamma^2)(\rho+\delta)}$ and for any $A > 0$; or (b) $b^2 \leq \frac{1}{2(1-\gamma^2)(\rho+\delta)}$ and $A \in (0, A_1) \cup (A_2, +\infty)$, with*

$$A_1 = \frac{1 - \sqrt{1 - 2(\delta+\rho)b^2(1-\gamma^2)}}{b^2(1-\gamma^2)}, \quad A_2 = \frac{1 + \sqrt{1 - 2(\delta+\rho)b^2(1-\gamma^2)}}{b^2(1-\gamma^2)}.$$

The final proposition is striking at first glance. In contrast to the deterministic case, where the economy will optimally jump on an exponentially increasing path provided $A > \rho + \delta$, it turns out that under uncertainty, our economy almost surely collapses (at an exponential speed) for a large class of parameterizations. Two engines are driving this result. First, the size of uncertainty as captured by parameter b matters: a too large uncertainty in the sense of condition (a) of Proposition 2 will destroy the usual deterministic growth paths even if productivity is initially very high (so even if $A >> \delta + \rho$). Second, since $0 < \gamma < 1$, we are in the typical case where uncertainty boosts contemporanous consumption at the expense of savings and growth because the inherent income effects are dominated by the intertemporal substitution effects. In such a case, even if uncertainty is not large in the sense of condition (b) of Proposition 2, the usual deterministic growth paths are not robust to uncertainty. To understand more clearly the associated productivity values, it is interesting to come back to the parametric case considered by Steger (2005). Steger sets $b = 1$ and $\delta = 0$. Then, the first part of condition (b) holds for ρ small enough. Indeed, condition $1 \leq \frac{1}{2\rho(1-\gamma^2)}$ is fulfilled for ρ going to zero and given $0 < \gamma < 1$. The second part of condition (b) is more interesting. For ρ close to zero, and using elementary approximation, one can easily show that $A_1 \approx \rho$ and $A_2 \approx \frac{2-\rho(1-\gamma^2)}{1-\gamma^2}$. Condition (b) states that the economy collapses almost surely and at an exponential speed either if $A < A_1$ or $A > A_2$. Condition $A < A_1$, which amounts to $A < \rho$, is compatible with the deterministic counterpart as exponentially increasing paths require $A > \rho$ when $\delta = 0$. However, $A > A_2$ is not since $A_2 > \rho$ for ρ small enough: exponentially optimal increasing paths exist in the deterministic path but not in the stochastic counterpart where the economy optimally almost surely collapses. In such case, balanced growth is not robust to uncertainty.

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