Working Papers / Documents de travail

Macroeconomic Volatility and Trade in OLG Economies

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WP 2014 - Nr 46

Macroeconomic volatility and trade in OLG economies*

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November 2014

Abstract: *This chapter analyzes the effect of international trade on the local stability properties of economies in a Heckscher-Ohlin free-trade equilibrium. We formulate a two-factor (capital and labor), two-good (consumption and investment), two-country overlapping generations model where countries only differ with respect to their discount rate. We consider a CES non increasing returns to scale technology in the consumption good sector and a Leontief constant returns to scale technology in the investment good sector. In the autarky equilibrium and the free-trade equilibrium, we show the existence of endogenous cycles with dynamic efficiency when the consumption good is capital intensive, the value of the elasticity of intertemporal substitution in consumption is intermediate and the degree of returns to scale is sufficiently high. Finally using a numerical simulation, we show that period-two cycles can occur in the free-trade equilibrium although one country is characterized by saddle-point stability in the autarky equilibrium.* **Keywords:** *two-sector OLG model, two-country, local indeterminacy, endogenous fluctuations, dynamic efficiency*. **JEL Classification Numbers**: C62, E32, F11, F43, O41.

*I wish to thank my advisors, C. Nourry and A. Venditti, for having introduced me to the OLG model, for their invaluable advice and continued support. I am also grateful to S. Bervoets, R. Boucekkine, H. Latzer, P. Maarek, F. Magris and T. Seegmuller for useful comments and suggestions. This paper benefited from presentation at the "European Doctoral Group in Economics Jamboree", Munich, September 2012, at the "17th Theories and Methods in Macroeconomics", Lyon, March 2013, at the "Overlapping Generations Days", Clermont-Ferrand, May 2013, and at "12th Journées Louis-André Gérard-Varet", Aix-en-Provence, June 2013.

1 Introduction

The phenomenon of globalization has expanded rapidly in recent decades. One particular feature of globalization is the increasing development of international trade. For example, in the past three decades the share of international trade in the gross domestic product has increased by a factor of two for the main OECD countries.[1] Along with this globalization, the global crisis since 2008 has evidenced that business cycles in countries are increasingly interlinked. Recent empirical studies, using industry-level data to estimate the link between macroeconomic volatility and trade openness, show countries that are more exposed to trade are more volatile.[2] A growing interest has thus emerged for understanding the effect of international trade on the instability of trading economies.

How can trade affect business cycles of trading countries? In this chapter, we attempt to address this question by analyzing the co-movement of cycle between countries in a two-good (consumption and investment) two-factor (capital and labor) two-sector (2x2x2) Heckscher-Ohlin (H-O) model in which the two factors are internationally immobile. Countries differ only with respect to their discount rate. In particular, our aim is to study the dynamic behavior of two countries. Among economists a wide-spread view is that macroeconomic volatility are not driven only by shocks on technologies or preferences, but also by changes in expectations about the fundamentals. A large literature focusing on the analysis of cycles derived from agents' belief is built on the idea of sunspot equilibria defined in Shell (1977).[3] As shown by Woodford (1986), the existence of sunspot equilibria is related to the indeterminacy of the equilibrium under perfect foresight, i.e., the existence of a continuum of equilibrium paths converging towards one steady state from the same initial value of the state variable.

The literature demonstrates that opening to international trade may have different impacts on the local stability properties of trading countries. The literature is mainly based on a two-country version of Benhabib and Nishimura (1998) who study the existence of local indeterminacy in a closed two-sector (consumption and investment) infinitely lived agent model with sector-specific externalities and social constant returns.[4] Nishimura and Shimomura (2002) consider a H-O formulation of this model where countries only differ with respect to their initial factor endowment. They show that if both countries are characterized by local indeterminacy in the autarky equilibrium, then at the opening to international trade local indeterminacy also holds in the free-trade equilibrium.[5] By contrast, Sim and Ho (2007) consider that the technologies are different across countries.[6] They prove that the world economy is characterized by saddle-point stability even if before trade one country exhibits sunspot fluctuations. Finally, Hu and Mino (2013) consider

[1]See OECD Factbook 2010: Economic, Environmental and Social Statistics.

[2]See Kose et al. (2010) and di Giovanni and Levchenko (2009).

[3]See also Azariadis (1981) and Cass and Shell (1983).

[4]They show that sunspot fluctuations arise provided that the investment good sector is more capital intensive than the consumption good sector from the social perspective and less capital intensive from the private perspective, and that the elasticity of intertemporal substitution in consumption is large enough.

[5]Iwasa and Nishimura (2014) extend Nishimura and Shimomura (2002) by introducing a consumption capital good. They show that opening to international trade can generate sunspot fluctuations in the world market.

[6]One country is characterized by sector-specific externality and the other country is not.

different trade structures with lending and borrowing, and show that international trade produces endogenous cycles in both countries, even if before trade sunspot cycles do not exist in the two countries.

This literature focuses on the infinitely-lived agent model. In this framework, local indeterminacy necessarily requires the presence of market imperfection such as sector-specific externalities. It implies that any equilibrium is Pareto inefficient. By contrast, in a two-sector overlapping generations (OLG) model Nourry and Venditti (2011) show that local indeterminacy together with dynamic efficiency can occur without any market imperfection. In OLG models, Pareto efficiency is associated with under-accumulation of capital stock with respect to the Golden Rule. Reichlin (1986) shows how the co-existence of Pareto efficiency and local indeterminacy in OLG models is an important question in terms of stabilization policies. If sunspot fluctuations occur under dynamic efficiency, a fiscal policy can simultaneously stabilize the economy and reach the Pareto optimal steady state.[7] Most models that investigate the existence of sunspot cycles in the OLG model are closed economy.[8] Few exceptions are Bajona and Kehoe (2008) and Mountford (1998) who show the occurrence of local indeterminacy in a free trade H-O equilibrium. However, two points have to be made. First, in contrast to the literature with infinitely-lived agent model, they do not discuss the effect of international trade on the local stability properties of trading economies. Second, these two contributions do not consider the efficiency property of the dynamic equilibrium, as a result they do not mention the issue raised by Reichlin (1986).

The purpose of this chapter is to investigate the impact of international trade on the local stability properties in Pareto efficient economies. We consider a two-sector two-country OLG version of Nourry and Venditti (2011) with one consumption good and one investment good. Our formulation differs from Nourry and Venditti (2011) in two dimensions. First, Nourry and Venditti (2011) show that local indeterminacy is likely to occur under dynamic efficiency if the sectoral technologies are closed enough to Leontief functions. We thus express our model with a CES technology in the consumption good sector and a Leontief technology in the investment good sector. Second, we suppose non increasing returns to scale on the consumption good sector. This permits us to guarantee a non-degenerate social production function at the world level. As in Galor and Lin (1997) we consider a H-O environment in which the two countries differ only with respect to their discount rate. We consider two levels of market integration. First, the two countries are in an autarky regime meaning that goods and factors are traded only on their respective domestic market. Second, both countries are in a trade regime implying that goods are traded on the international market with no transaction cost. In the trade regime, lending and borrowing are not permitted and the factors of production are internationally immobile.

In the autarky regime, we show the existence of endogenous cycles together with dynamic efficiency when the consumption good is capital intensive, the value of the elasticity of intertemporal substitution in consumption is intermediate and the degree of returns to scale is high enough. This result extends the main findings of Nourry and Venditti (2011) by considering a non increasing returns to scale technology in the consumption good sector. In the trade regime, we

[7] Nourry and Venditti (2011) extends this result to a two-sector OLG model.

[8] See for example Galor (1992), Venditti (2005) and Nourry and Venditti (2011).

prove the existence of sunspot cycles under dynamic efficiency with similar restriction on shares and elasticities as the autarky regime. However, the condition on these elasticities and shares to obtain sunspot cycles in the autarky and the trade regime are different. Our aim is to analyze the effect of opening to international trade on the local stability properties of trading countries while the elasticity of intertemporal substitution in consumption is varying. To answer this question we provide a numerical example and show that opening to international trade creates a contagion of period-two cycles from one country to another.

This chapter is organized as follows. Section 2 describes an economy in the autarky regime while section 3 introduces the analysis of the local dynamics of the closed economy. Section 4 provides the analysis of the two-country H-O model, the pattern of trade and the local stability analysis of the free-trade steady state. Section 5 presents a numerical simulation to determine the effect of international trade on the local stability properties of trading countries. Section 6 contains the concluding remarks and, the proofs are gathered in the appendix.

2 The autarky model

We consider a closed economy, which can be north or south, that has two goods (consumption and investment), two factors (capital and labor), and two generations (young and old) in each period. In this section, we extend the two-sector OLG model of Nourry and Venditti (2011) by considering non increasing returns to scale in the consumption good sector.[9] In the present section, to simplify the exposition we do not consider any superscripts for the north and the south. However, from section 4 when the two countries are considered at the same time we will add superscripts $\{N, S\}$ to distinguish them.

2.1 Technology

Consider a competitive economy in which there are two sectors, one representative firm for each sector and each firm producing one good. In this economy there exists two goods: one consumption good produced in quantity $Y_{0,t}$ and one investment good produced in quantity Y_t. The consumption good is taken as a numeraire. Each sector uses two factors, capital K_t and labor L_t, both factors are mobile between sectors. Depreciation of capital is complete within one period:[10]

$$K_{t+1} = Y_t \tag{1}$$

where K_{t+1} is the total amount of capital in period $t+1$. The consumption good $Y_{0,t}$ is assumed to be produced with a CES technology and the investment good Y_t is assumed to be produced with

[9] See also Drugeon et al. (2010).

[10] In a two-periods OLG model, full depreciation of capital is justified by the fact that one period is about thirty years.

a Leontief technology:[11]

$$
\begin{aligned}
Y_{0,t} = F^0\left(K_t^0, L_t^0\right) = \Theta\left\{\mu\left(K_t^0\right)^{-\rho} + (1-\mu)\left(L_t^0\right)^{-\rho}\right\}^{-\frac{\nu}{\rho}};\\
Y_t = F^1\left(K_t^1, L_t^1\right) = \min\left\{\frac{K_t^1}{\eta}, L_t^1\right\}.
\end{aligned}
\tag{2}
$$

where parameter $\mu \in (0,1)$ reflects the capital intensity in production, $\rho > -1$ is the sectoral elasticity of capital-labor substitution in the consumption good sector, $\nu > 0$ is the degree of returns to scale in the consumption good sector, $\eta \in (0,1)$ is the capital intensity in the investment good sector and $\Theta > 0$ is a normalization constant. We assume non increasing returns to scale in the consumption good sector, i.e., $\nu \leq 1$.

Labor is normalized to one and given by $L = L_t^0 + L_t^1 = 1$, and the capital stock is given by $K_t = K_t^0 + K_t^1$. The optimal allocation of factors between sectors is defined by the social production function $T(K_t, Y_t, L)$:

$$
\begin{aligned}
T(K_t, Y_t, L) = \max_{K_t^j, L_t^j, j=0,1} Y_{0,t}\\
s.t.\ Y_t \leq F^1(K_t^1, L_t^1),\ K_t^0 + K_t^1 \leq K_t,\ L_t^0 + L_t^1 \leq L
\end{aligned}
\tag{3}
$$

The social production function is the frontier of the production possibility set and gives the maximal output of the consumption good. Using equation (2) and the resource constraint, $K_t^0 + K_t^1 = K_t$ and $L = L_t^0 + L_t^1$, we define the social production function as:

$$
T(K_t, Y_t, L) = \Theta\left[\mu(K_t - \eta Y_t)^{-\rho} + (1-\mu)(L - Y_t)^{-\rho}\right]^{-\frac{\nu}{\rho}} \tag{4}
$$

Let us denote r_t the rental rate of capital, p_t the price of the investment good and w_t the wage rate, all in terms of the price of the consumption good. Using the envelope theorem we obtain the following three relationships:[12]

$$
\begin{aligned}
r(K_t, Y_t, L) = T_1(K_t, Y_t, L),\\
p(K_t, Y_t, L) = -T_2(K_t, Y_t, L),\\
w(K_t, Y_t, L) = T_3(K_t, Y_t, L).
\end{aligned}
\tag{5}
$$

where $T_1 = \partial T/\partial K_t$, $T_2 = \partial T/\partial Y_t$ and $T_3 = \partial T/\partial L$. Using equation (2) and the resource constraint we obtain the relative capital intensity difference b_t and the capital intensity in the

[11] CES technology do not always satisfy Inada conditions. However, as shown in our exposition, interior solutions for production are obtained.

[12] See details in Appendix 7.1.

consumption good sector a_t:

$$
\begin{aligned}
b\left(K_t, Y_t, L\right) &= \frac{L_t^1}{Y_t}\left(\frac{K_t^1}{L_t^1} - \frac{K_t^0}{L_t^0}\right) = \frac{\eta - K_t}{L - Y_t}, \\
a\left(K_t, Y_t, L\right) &= \frac{K_t^0}{L_t^0} = \frac{K_t - \eta Y_t}{L - Y_t}.
\end{aligned}
\tag{6}
$$

The sign of b_t is positive (resp. negative) if and only if the consumption good is labor (resp. capital) intensive. Since there may exist decreasing returns to scale in the consumption good sector, firms earn positive profit, π_c. From the first-order conditions we derive $\pi_c\left(K_t, Y_t, L\right) = T\left(K_t, Y_t, L\right)(1-\nu)$. In the following, we suppose that the owner of the firms is an infinitely-lived agent and receives positive profits. We further assume that the owner of the firms spend all the positive profits by consuming the consumption good. We define the GDP function as $T\left(K_t, Y_t, L\right) + p\left(K_t, Y_t, L\right) Y_t = w\left(K_t, Y_t, L\right) + r\left(K_t, Y_t, L\right) K_t + \pi_c\left(K_t, Y_t, L\right)$, we get the share of capital in the economy:

$$
s\left(K_t, Y_t, L\right) = \frac{r(K_t, Y_t, L) K_t}{T(K_t, Y_t, L) + p(K_t, Y_t, L) Y_t - \pi_c(K_t, Y_t, L)}
\tag{7}
$$

2.2 Preferences

Consider an infinite-horizon discrete time economy that is populated by overlapping generations of agents who live for two periods: young and old. There is no population growth and the population is normalized to one, i.e., $N = 1$. In the first period, young agents inelastically supply one unit of labor and receive an income w_t. They assign this income between the saving ϕ_t and the first period consumption C_t. In the second period, old agents are retired. The return on saving $R_{t+1}\phi_t$ give their income which they spend entirely in the second period consumption D_{t+1}. An agent born in period t has preferences defined over consumption of C_t and D_{t+1}. Intertemporal preferences of agent are described by the following CES utility function:

$$
U\left(C_t, D_{t+1}\right) = \left[C_t^{\frac{\gamma-1}{\gamma}} + \delta\left(\frac{D_{t+1}}{\Gamma}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}
\tag{8}
$$

where δ is the discount factor, γ is the elasticity of intertemporal substitution in consumption and Γ is a scaling constant parameter. Under perfect foresight and perfect competition w_t and R_{t+1} are considered as given. A young agent who is born at period t solves the following dynamic program:

$$
\max_{C_t, D_{t+1}} \left\{U\left(C_t, D_{t+1}\right) \mid w_t = C_t + \phi_t, D_{t+1} = R_{t+1}\phi_t\right\}
\tag{9}
$$

Solving the first order condition gives:

$$
C_t = \alpha\left(\frac{R_{t+1}}{\Gamma}\right) w_t, \quad \alpha\left(\frac{R_{t+1}}{\Gamma}\right) = \frac{1}{1 + \delta^{\gamma}\left(\frac{R_{t+1}}{\Gamma}\right)^{\gamma-1}}
\tag{10}
$$

where $\alpha\left(R_{t+1}/\Gamma\right) \in (0,1)$ is the propensity to consume of a young agent at period t. From the budget constraint (9), we obtain the saving function:

$$\phi_t = \left[1 - \alpha\left(\frac{R_{t+1}}{\Gamma}\right)\right] w_t \tag{11}$$

We assume that the saving is increasing with respect to the gross rate of return R_{t+1}.

Assumption 1. $\gamma > 1$.

This standard assumption states that the substitution effect following an increase in the gross rate of return R_{t+1} is greater than the income effect.[13] In the rest of the chapter we do not consider the non standard case where the saving is decreasing with respect to the gross rate of return R_{t+1}, i.e., $\gamma < 1$.[14]

2.3 Dynamic equilibrium

On a dynamic competitive equilibrium, total savings equal the production of the investment good: $\phi_t = p_t Y_t$. A perfect-foresight competitive equilibrium of an economy in the autarky regime is defined as:

Definition 1. *A sequence $\{K_t, Y_t\}_{t=0}^{\infty}$, with $K_{t=0}$ given, is a perfect-foresight competitive equilibrium if:*

i⌋ Producers and households are at their optimum: the FOC (5) and (10)-(11) are satisfied and $R_{t+1} = r_{t+1}/p_t$;

ii⌋ The labor market is given by $L = N = 1$;

iii⌋ The capital accumulation is determined by $p_t Y_t = \phi_t$ with (1);

iv⌋ The market clearing condition for the consumption good is given by $C_t + D_t = T\left(K_t, Y_t, 1\right)$.

We derive from definition 1 that the dynamics of an economy in the autarky regime is described by the evolution of the capital stock:

$$p\left(K_t, K_{t+1}, 1\right) K_{t+1} - w\left(K_t, K_{t+1}, 1\right)\left\{1 - \alpha\left[\frac{r(K_{t+1}, K_{t+2}, 1)}{\Gamma p(K_t, K_{t+1}, 1)}\right]\right\} = 0 \tag{12}$$

Using equations (1), the set of admissible (K_t, K_{t+1}) is defined as follows:

$$\Omega = \left\{(K_t, K_{t+1}) \in \mathbb{R}_+^2 \mid K_t \leq \overline{K},\ K_{t+1} \leq F^1\left(K_t, 1\right)\right\} \tag{13}$$

[13] This condition on the elasticity of intertemporal substitution in consumption is consistent with recent empirical evidence. Building on a natural experiment Kapoor and Ravi (2010) show that the elasticity of intertemporal substitution in consumption is higher than one. Using micro-level data, Mulligan (2002) and Vissing-Jorgensen and Antanasio (2003) find a similar result.

[14] If $\gamma = 1$, total savings are independent of the gross rate of return.

where the maximum admissible value of $\overline{K}$ is solution of $K - F^1(K, 1) = 0$.

2.4 Steady state and efficiency properties

A steady state $K_t = K_{t+1} = K_{t+2} = K^*$ is defined by:

$$p(K^*, K^*, 1) K^* - w(K^*, K^*, 1)\left\{1 - \alpha\left[\frac{r(K^*,K^*,1)}{\Gamma p(K^*,K^*,1)}\right]\right\} = 0 \tag{14}$$

In two-sector OLG models, any steady state depends on consumption and production behavior. Any variation of the elasticity of intertemporal substitution in consumption $\gamma(R/\Gamma)$ induces a change in the stationary capital-labor ratio and thus involves a modification of all the elasticities evaluated at the steady state. In the following, we consider a set of economic systems parametrized by $\gamma(R/\Gamma)$. In order to guarantee that the steady state remains unaltered when $\gamma(R/\Gamma)$ varies, the steady state K^* is normalized by using the scaling parameter Γ.[15] The nice feature of the normalization procedure is that it allows to provide a clear analysis of the local stability of the steady state via γ. Let us express $\xi = R/\Gamma$. Under assumption 1, $\alpha(\xi)$ is a monotone decreasing function with $lim_{\xi\to 0}\alpha(\xi) = \alpha_{sup}$, $lim_{\xi\to +\infty}\alpha(\xi) = \alpha_{inf}$ and $(\alpha_{inf}, \alpha_{sup}) \subseteq (0, 1)$. We define the inverse function of $\alpha(\xi)$:

$$\Phi_{K^*} = 1 - \frac{K^* p(K^*,K^*,1)}{w(K^*,K^*,1)} \tag{15}$$

By adopting a proper value for K^*, we may find a corresponding value for $\Phi_{K^*} \in (\alpha_{inf}, \alpha_{sup})$. Then, the following proposition holds:

Proposition 1. *Under assumption 1, let $K^* \in (0, \overline{K})$ be such that $\Phi_{K^*} \in (\alpha_{inf}, \alpha_{sup})$. There exists a unique value $\Gamma(K^*) > 0$ solution of (14) such that K^* is a steady state if and only if $\Gamma = \Gamma(K^*)$.*

Proof: See appendix 7.2. ■

In the rest of the chapter we make the following assumption so that the existence of a normalized steady state (NSS) is ensured in the autarky regime.

Assumption 2. $\Gamma = \Gamma(K^*)$.

Let us consider the dynamic efficiency properties of the competitive equilibrium around the NSS K^*. Drugeon et al. (2010) study the dynamic efficiency properties of equilibrium paths based on a NSS K^* represented by an under-accumulation of capital stock with respect to the Golden-Rule level. The Golden-Rule level of capital $\hat{K}$ is characterized from the total stationary consumption $C + D = T(\hat{K}, \hat{K}, 1)$. Denoting $R^A(\hat{K}, \hat{K}, 1) = -T_1(\hat{K}, \hat{K}, 1)/T_2(\hat{K}, \hat{K}, 1)$, $\hat{K}$ satisfies $R^A(\hat{K}, \hat{K}, 1) =$

[15] See Nourry and Venditti (2011).

1. Using equation (7) it follows that $w/rK = (1-s)/s$, we derive from (14) the stationary gross rate of return at the NSS:

$$R^A = \frac{s}{(1-\alpha)(1-s)} \tag{16}$$

If $R^A > 1$ (resp. $R^A < 1$), the NSS K^* is lower (resp. higher) than the Golden-Rule level, i.e., under- (resp. over-) accumulation of capital. Using the Golden-Rule level $R^A = 1$ and equation (16), we derive a condition on the propensity to consume of the young agent α to obtain a NSS K^* lower than the Golden-Rule level and to ensure the dynamic efficiency of equilibria. From, the proof of proposition 2 and 3 in Drugeon et al. (2010) we obtain the following:

Proposition 2. *Under assumptions 1-2, let* $\underline{\alpha} = (1-2s)/(1-s)$. *An intertemporal competitive equilibrium converging toward the NSS is dynamically efficient if* $\alpha \in (\underline{\alpha}, 1)$, *and dynamically inefficient if* $\alpha \in (0, \underline{\alpha})$.

If labor income of agents is higher than capital income, i.e., $s < 1/2$, we derive $\underline{\alpha} > 0$. In this case a young agents have enough labor income to save a sufficient amount. Under-accumulation of capital can be attained provided that the share of consumption of young agents is high enough, i.e., $\alpha > \underline{\alpha}$. The following assumption is made:

Assumption 3. $\alpha \in (\underline{\alpha}, 1/2)$ *and* $s \in (1/3, 1/2)$.

Assumption 3 states that we consider dynamically efficient paths and that we restrain the share of capital in the economy s in order to focus on realistic values. Indeed, Cecchi and Garcia-Peñalosa (2010) show that over the period 1960-2003, OECD countries were characterized by a share of capital between 0.35 and 0.5.

3 Local indeterminacy in the autarky regime

In this section we aim at characterizing the local stability of the normalized steady state of (12). Our model consists in one predetermined variable, the current capital stock, and one forward variable, the capital stock of the next period. Therefore, the dimension of the stable manifold is required to be two in order to obtain local indeterminacy. Local indeterminacy occurs when there exists a continuum of equilibrium paths converging to one steady state from the same initial value of the capital stock whereas local determinacy occurs when there is a unique converging equilibrium path for a given initial capital stock. In our setting, the existence of local indeterminacy occurs if the two characteristic roots associated with the linearization of the dynamical system (12) around the NSS have a modulus less than one. Let us introduce the elasticity of the rental rate of capital:

$$\varepsilon_{rk} = -\frac{T_{11}(K^*, K^*, 1)K^*}{T_1(K^*, K^*, 1)} \tag{17}$$

Drugeon (2004) points out that the elasticity of the rental rate of capital is negatively linked to the elasticities of capital-labor substitution. To proceed to the analysis of the local stability of the

NSS, we linearize the difference equation (12) (see Appendix 7.3.1). We follow the methodology of Grandmont et al. (1998) and study the variation of the trace $\mathscr{T}^A(\gamma)$ and the determinant $\mathscr{D}^A(\gamma)$ in the $(\mathscr{T}^A(\gamma), \mathscr{D}^A(\gamma))$ plane as one parameter of interest, i.e., γ, is made to vary continuously in its admissible range. This methodology allows to determine the occurrence of the local stability of the steady state and the local bifurcation. In the following we consider that the consumption good sector is more capital intensive than the investment good sector[16] and that the consumption good sector has decreasing returns to scale:[17]

Assumption 4. *$b < 0$ and $\nu \leq 1$.*

Then, we obtain this result:

Proposition 3. *Under assumptions 1-4, there exist $\underline{b} < \overline{b} < 0$, $\underline{\nu} < 1$, $\underline{\rho} > -1$, $\underline{\varepsilon}_{rk} > 0$, and $\gamma^{\mathcal{F}} > \gamma^{\mathcal{T}} > 1$ such that for $b \in (\underline{b}, \overline{b})$, $\nu \in (\underline{\nu}, 1]$, $\rho > \underline{\rho}$ and $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$, the following results prevail:*
i] the steady state is a sink, i.e., a locally indeterminate NSS, when $\gamma \in (\gamma^{\mathcal{T}}, \gamma^{\mathcal{F}})$;
ii] the steady state is a saddle, i.e., a locally determinate NSS, when $\gamma \in (1, \gamma^{\mathcal{T}}) \cup (\gamma^{\mathcal{F}}, +\infty)$.

Proof: See appendix 7.3.2. ■

This proposition extends the result of Nourry and Venditti (2011) with a non increasing returns to scale technology in the consumption good sector. In proposition 3, $\gamma^{\mathcal{T}}$ is generically a transcritical bifurcation[18] leading to the existence of a second steady state which is locally unstable (resp. saddle-point stable) in a right (resp. left) neighborhood of $\gamma^{\mathcal{T}}$, whereas $\gamma^{\mathcal{F}}$ is generically a flip bifurcation value giving rise to period-two cycles which are locally indeterminate (resp. unstable) in a right (resp. left) neighborhood of $\gamma^{\mathcal{F}}$ (See figure 1). The intuition of proposition 3 is the following. Suppose that agents expect that the rate of investment will rise at time t inducing a higher capital stock at time $t + 1$. This expectation will be self fulfilling provided that there is enough saving at time t, see equation (12). For agents to save a sufficient amount, they must first reduce their consumption at time t. This decrease of the current consumption lowers their level of utility and to compensate agents must increase their consumption at time $t + 1$. This configuration is obtained provided that γ is sufficiently high, i.e., $\gamma > \gamma^{\mathcal{T}}$. By contrast, if γ is too high, i.e., $\gamma > \gamma^{\mathcal{F}}$, the intertemporal substitution effect is large and thus the

[16]Using national accounting data (aggregate Input-Output tables) on the most developed countries, Takahashi et al. (2012) show that the aggregate consumption good sector is more capital intensive than the investment good sector. Baxter (1996) obtain also a similar result.

[17]This assumption on the returns to scale is compatible with empirical evidence. Basu and Fernand (1997), Burnside (1996) and Burnside et al. (1995) use disaggregated US data and show that returns to scale are approximately constant at the aggregate level but decreasing returns cannot be rejected at the industry level.

[18]When the bifurcation parameter γ crosses a $\gamma^{\mathcal{T}}$, one characteristic roots crosses 1. We cannot a priori differentiate between the transcritical, the pitchfork or the saddle-node bifurcation from the difference equation (12). Under Assumption 2, the existence of the NSS is always ensured and a saddle-node bifurcation cannot occur. Moreover, the pitchfork bifurcation requires non-generic conditions, see Ruelle (1989). In order to simplify the exposition we focus on the generic case and we relate the existence of one characteristic root going through 1 to a transcritical bifurcation.

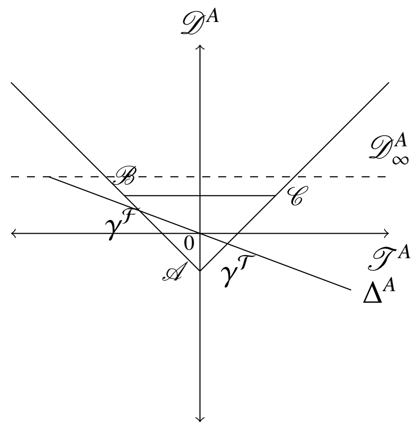

Figure 1: Local indeterminacy in the autarky regime.

expected increase in the rate of investment produces a relatively high amount of savings while the present consumption decreases. Meanwhile, the capital stock in the next period will rise at an important level. Since the consumption good sector is the most capital intensive sector, it implies through the Rybzcynski effect that there is a large increase in the production of the consumption good at time $t+1$. This rise in consumption good production may exceed the increase in future consumption demand. As a result, the initial expectation can be realized provided that γ has intermediate values, i.e., $\gamma \in \left(\gamma^{\mathcal{T}}, \gamma^{\mathcal{F}}\right)$.

4 The two-country model

In this section, we consider a two-country model, i.e., the north and the south, and discuss the two-country H-O model. First, we present the main assumptions of the model and derive the perfect foresight. Second we describe the existence of a NSS as well as the efficiency properties of the two-country H-O model. Finally, assuming that the north is the most patient country we examine the pattern of trade and analyze the local stability of the steady state.

4.1 Assumption of the two-country model

Consider a H-O world that consists of two countries, i.e., referred to N for the north and S for the south, which differ only with respect to their discount rate $\delta^N \neq \delta^S$. In other words, the two countries have different preferences and thus a different capital accumulation path. Capital and labor are immobile across countries. We assume free-trade, thus the relative price of the investment good is the same in the two countries, i.e., $p_t = p_t^N = p_t^S$. As the investment good is mobile, there are investment flows across countries. The population in both countries is normalized to one, i.e., $L^N = L^S = L = 1$. Let us introduce $K_t^W = K_t^N + K_t^S$ the world capital stock at time t, $Y_t^W = Y_t^N + Y_t^S$ the world production of investment good at time t and $L_t^W = L_t^N + L_t^S = 2$ the world labor force at time t. At each period the balance of trade is at the equilibrium for both countries. For an economy $i \in \{N, S\}$, the balance of trade is given by:

$$\chi_t^{i0} + p_t \chi_t^{i1} = 0 \tag{18}$$

where χ_t^{i0} is the net export of the consumption good in country i at time t and χ_t^{i1} is the net export of the investment good in country i at time t. Since the net export of goods of one country are the net export of the other country, we have:

$$\chi_t^{N0} + \chi_t^{S0} = 0, \;\; \chi_t^{N1} + \chi_t^{S1} = 0 \tag{19}$$

We suppose that the consumption good sector is capital intensive, i.e., $b^i < 0$, as in assumption 3 and there is no factor intensity reversal. In other words, the capital-labor ratio used in the consumption good sector, i.e., $a_t^i \equiv K_t^{i0}/L_t^{i0}$, is higher than the capital-labor ratio used in the investment good sector, i.e., $\eta \equiv K_t^{i1}/L_t^{i1}$, at each period (see figure 2). These two capital-labor ratios are crucial levels which describe the specialization pattern of a country $i \in \{N, S\}$ at time t. If $\eta < k_t^i < a_t^i$, country i diversifies in the production of the consumption good and the investment good. The range of capital-labor ratio between η and a_t^i is called the cone of diversification. When the north and the south have their capital-labor ratio in the cone of diversification, the H-O

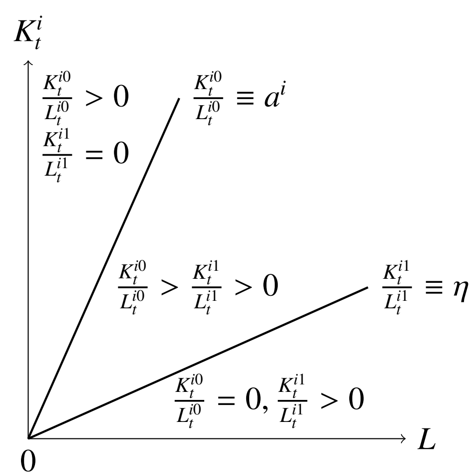

Figure 2: Cone of diversification of country $i \in \{N, S\}$ at time t.

trade equilibrium is characterized by an interior equilibrium in which the two countries produce both consumption and investment goods. In our model, the cone of diversification evolves over time since the capital-labor ratio changes overtime. The two countries produce both goods along a H-O free trade equilibrium if the two countries stay in the corresponding cone of diversification for each period t. Note that if the two countries are in the cone of diversification at the steady state, the north and the south stay in the cone of diversification in the neighborhood of the steady state. It allows us to provide an analysis of the local dynamics of a H-O world along which each country produces both goods. In section 4.4 we show that each country produces both goods at the steady state. If at least one country specializes, the dynamics of the world economy examined in the next section is different. To analyze the global behavior of the model, we need to address the model out of the cone of diversification. Caliendo (2010) presents a detailed analysis of the pattern of specialization in an infinitely-lived agents two-country three-sector (2 traded consumption goods and 1 nontraded investment good) H-O model inside and outside the cone of diversification. However, Caliendo (2010) does not investigate the local dynamics of the two-country model. Given that technologies are the same across countries and there is no intensity reversals, we derive from the first-order of the producer program (3) (details in appendix 7.1) the following:

Proposition 4. *Under assumptions 1-2 and let assume free-trade, i.e., $p_t = p_t^N = p_t^S$, then there exist Θ^{N*} such that factor prices equalize across countries, i.e., $r_t^N = r_t^S$ and $w_t^N = w_t^S$, if and only if $\Theta^N = \Theta^{N*}$ and $\Theta^S = 1$.*

Proof: See appendix 7.4. ■

In the rest of the chapter we assume that the restrictions of lemma 4 are satisfied in order to ensure that $r_t^N = r_t^S$ and $w_t^N = w_t^S$.

Assumption 5. $\Theta^S = 1$ *and* $\Theta^N = \Theta^{N*}$.

4.2 World dynamic equilibrium

Trade modifies the restriction that an economy faces. Indeed, an economy in the autarky regime can increase its own capital stock only by producing more investment goods but an economy in the trade regime may in addition import the investment good from another economy. For an economy $i \in \{N, S\}$ in the trade regime, the capital accumulation is now given by:

$$p_t y_t^i = p_t k_{t+1}^i + p_t \chi_t^{i1} = \phi_t^i \tag{20}$$

For our purpose, we study the dynamic behavior of the two countries at a world level. Such approach allows us to compare the dynamic behavior of an economy in the autarky regime with respect to the dynamic behavior of the world economy. We will thus characterize the world dynamic equilibrium. We define the world social production function $\tau\left(K_t^W, Y_t^W, 2\right)$ as:

$$\tau\left(K_t^W, Y_t^W, 2\right) = T\left(K_t^N, Y_t^N, 1\right) + T\left(K_t^S, Y_t^S, 1\right) \tag{21}$$

and solve the following optimal allocation of factors problem

$$\begin{aligned} \max_{K_t^i, L_t^i, i\in\{N,S\}} \quad & \tau\left(K_t^W, Y_t^W, 2\right) = T\left(K_t^N, Y_t^N, 1\right) + T\left(K_t^S, Y_t^S, 1\right) \\ s.t. \quad & K_t^N + K_t^S \leq K_t^w \\ & Y_t^N + Y_t^S \leq Y_t^w \end{aligned} \tag{22}$$

The first-order conditions gives

$$\begin{aligned} r\left(K_t^N, Y_t^N, 1\right) - r\left(K_t^S, Y_t^S, 1\right) = 0; \\ p\left(K_t^N, Y_t^N, 1\right) - p\left(K_t^S, Y_t^S, 1\right) = 0; \\ w\left(K_t^N, Y_t^N, 1\right) - w\left(K_t^S, Y_t^S, 1\right) = 0. \end{aligned} \tag{23}$$

From the envelope theorem we get:

$$\begin{aligned} r\left(K_t^W, Y_t^W, 2\right) = r\left(K_t^N, Y_t^N, 1\right) = r\left(K_t^S, Y_t^S, 1\right), \\ p\left(K_t^W, Y_t^W, 2\right) = p\left(K_t^N, Y_t^N, 1\right) = p\left(K_t^S, Y_t^S, 1\right), \\ w\left(K_t^W, Y_t^W, 2\right) = w\left(K_t^N, Y_t^N, 1\right) = w\left(K_t^S, Y_t^S, 1\right). \end{aligned} \tag{24}$$

We define a world perfect-foresight equilibrium as:

Definition 2 *.A sequence $\left\{K_t^W, Y_t^W\right\}_{t=0}^{\infty}$ with $K_0^W = K_0^N + K_0^S$ given, is a world perfect-foresight competitive equilibrium if:*

i⌋ Producers and households are at their optimum: the FOC (5) and (10)-(11) are satisfied and $R_{t+1}^W = r_{t+1}/p_t$;

ii⌋ Each period the balance of trade is at the equilibrium for both countries as in (18);

iii⌋ Using equations (19) and (24), the capital accumulation is determined by $p_t(Y_t^N + Y_t^S) = \phi_t^N + \phi_t^S$ with $K_{t+1}^N + K_{t+1}^S = Y_t^N + Y_t^S$;

iv⌋ The market clearing condition for the consumption good is given by $C_t^N + D_t^N + C_t^S + D_t^S = \tau(K_t^W, Y_t^W, 2)$.

We derive from definition 2 that the dynamics of the world economy is represented by the evolution of the capital stock:

$$K_{t+1}^W - \frac{w\left(K_t^W, K_{t+1}^W, 2\right)\left\{2 - \alpha^N\left[\frac{r\left(K_{t+1}^W, K_{t+2}^W, 2\right)}{\Gamma^N p\left(K_t^W, K_{t+1}^W, 2\right)}\right] - \alpha^S\left[\frac{r\left(K_{t+1}^W, K_{t+2}^W, 2\right)}{\Gamma^S p\left(K_t^W, K_{t+1}^W, 2\right)}\right]\right\}}{p\left(K_t^W, K_{t+1}^W, 2\right)} = 0 \tag{25}$$

As in the autarky regime we determine the set of admissible paths:

$$\Omega^W = \left\{(K_t^W, K_{t+1}^W) \in \mathbb{R}_+^2 \mid K_t^W \leq \overline{K}^W,\ K_{t+1}^W \leq \tfrac{1}{2}\left[F^1\left(K_t^N, 1\right) + F^1\left(K_t^S, 1\right)\right]\right\} \tag{26}$$

where the maximal admissible value of $\overline{K}^W$ is solution of $K^W - \frac{1}{2}[F^1(K^N, 1) + F^1(K^S, 1)] = 0$. We define the GDP function as $\tau\left(K_t^W, K_{t+1}^W, 2\right) + p\left(K_t^W, K_{t+1}^W, 2\right)K_{t+1}^W + \pi_{c,t}^N + \pi_{c,t}^S = w\left(K_t^W, K_{t+1}^W, 2\right) + r\left(K_t^W, K_{t+1}^W, 2\right)K_t^W$, we get the share of capital in the world economy:

$$s^W\left(K_t^W, K_{t+1}^W, 1\right) = \frac{r\left(K_t^W, K_{t+1}^W, 2\right)K_t^W}{\tau\left(K_t^W, K_{t+1}^W, 2\right) + p\left(K_t^W, K_{t+1}^W, 1\right)K_{t+1}^W + \pi_c^N + \pi_c^S} \tag{27}$$

4.3 World steady state and efficiency properties

A world steady state $K_t^W = K_{t+1}^W = K_{t+2}^W = K^{W*}$ is defined by:

$$K^{W*} - \frac{w\left(K^{W*}, K^{W*}, 2\right)\left\{2 - \alpha^N\left[\frac{r\left(K^{W*}, K^{W*}, 2\right)}{\Gamma^N p\left(K^{W*}, K^{W*}, 2\right)}\right] - \alpha^S\left[\frac{r\left(K^{W*}, K^{W*}, 2\right)}{\Gamma^S p\left(K^{W*}, K^{W*}, 2\right)}\right]\right\}}{p\left(K^{W*}, K^{W*}, 2\right)} = 0 \tag{28}$$

As in the autarky regime, we consider that the world economy is parametrized by γ. In order to guarantee that the world steady state remains unaltered when γ varies, the steady state K^{W*} is normalized by using the scaling parameter Γ^N. Let us express $\xi^N = R/\Gamma^N$. Under gross substitutability, $\alpha^N(\xi^N)$ is a monotone decreasing function with $lim_{\xi^N \to 0}\alpha^N(\xi^N) = \alpha_{sup}^N$, $lim_{\xi^N \to +\infty}\alpha^N(\xi^N) = \alpha_{inf}^N$ and $(\alpha_{inf}^N, \alpha_{sup}^N) \subseteq (0, 1)$. Let us define the inverse function of $\alpha^N(\xi^N)$:

$$\Phi_{K^{W*}} = 2 - \alpha^S - \frac{K^{W*}p\left(K^{W*}, K^{W*}, 2\right)}{w\left(K^{W*}, K^{W*}, 2\right)} \tag{29}$$

By adopting a proper value for K^{W*}, we may find a corresponding value for $\Phi_{K^{W*}} \in (\alpha_{inf}^N, \alpha_{sup}^N)$. Then, the following Proposition holds:

Proposition 5. *Under assumptions 1 and 5, let* $K^{W*} \in (0, \overline{K}^W)$ *be such that* $\Phi_{K^{W*}} \in (\alpha_{inf}^N, \alpha_{sup}^N)$. *There exists a unique value* $\Gamma^N\left(K^{W*}\right) > 0$ *solution of (28) such that* K^{W*} *is a steady state if and only if* $\Gamma^N = \Gamma^N\left(K^{W*}\right)$.

Proof: See appendix 7.5. ■

We introduce the following assumption to guarantee the existence of a NSS in the trade regime.

Assumption 6. $\Gamma^N = \Gamma^N(K^{W*})$.

As in the autarky regime, the Golden-Rule level of capital $\hat{K}^W$ is characterized from the total stationary consumption $C^N + D^N + C^S + D^S = \tau(\hat{K}^W, \hat{K}^W, 2)$. Denoting $R^W(\hat{K}^W, \hat{K}^W, 2) = -\tau_1(\hat{K}^W, \hat{K}^W, 2)/\tau_2(\hat{K}^W, \hat{K}^W, 2)$, $\hat{K}^W$ satisfies $R^W(\hat{K}^W, \hat{K}^W, 2) = 1$. We derive from (28) the propensity to consume of young agents in the world economy α^W and the stationary gross rate of return R^W at the NSS:

$$\alpha^W = \frac{1}{2}\left(\alpha^N + \alpha^S\right), \; R^W = \frac{2s^W}{(2-\alpha^N-\alpha^S)(1-s^W)} \tag{30}$$

We derive following the same methodology used in the autarky regime (proposition 2) a condition on the propensity to consume of young agents α^W to obtain a NSS K^{W*} lower than the Golden-Rule level and to ensure the dynamic efficiency of equilibria:

Proposition 6. *Under assumptions 1 and 5-6, let* $\underline{\alpha}^W = (1-2s^W)/(1-s^W)$. *An intertemporal dynamic equilibrium converging towards a NSS is dynamically efficient if* $\alpha^W \in (\underline{\alpha}^W, 1)$, *and dynamically inefficient if* $\alpha^W \in (0, \underline{\alpha}^W)$.

Under free-trade and lemma 4 $p_t = p_t^N = p_t^S$ and $r_t = r_t^N = r_t^S$ hold. It implies that $R = R^N = R^S$. Thus if the NSS is dynamically efficient in the world economy, the NSS of the north and the south are dynamically efficient. Put differently, international trade with immobility of factors and factor price equalization does not modify the dynamic efficiency properties of a closed economy.[19]

4.4 Pattern of trade

Once we have the world dynamic equilibrium we need to consider the trade pattern. In our two-sector OLG models, during the dynamic transition the pattern of specialization may change overtime since the capital-labor ratio K_t^i, $i \in \{N, S\}$, evolves. As shown by the following proposition the two countries produce both goods at the NSS:

Proposition 7. *Under assumptions 1 and 6, let assume that* $\eta \in (0,1)$ *then each countries produce both goods at the steady state.*

Proof: See appendix 7.6. ■

This proposition allows to provide an analysis of the stability of the world equilibrium in the neighborhood of the NSS in which both countries produce both goods. We suppose in the following that the north and the south differ only with respect to their discount rate $\delta^N \neq \delta^S$ and that the north has a comparative advantage in the production of the capital intensive good. By using equations (10), (24) and (32), we can show that $\phi_t^N > \phi_t^S$. It implies that the north saves more than the south and thus the north is the most patient country. Then the following holds:

[19] The situation in which the steady state of one country is dynamically efficient and the steady state of the other country is dynamically inefficient can be achieve by relaxing one assumption, namely the immobility of factors across countries. Assuming capital immobility and labor mobility across countries lead to different gross rate of return of capital.

Proposition 8. *Under assumptions 1, 5 and 6, let assume that $\eta \in (0,1)$ and consider a world NSS equilibrium in which north and south differ only with respect to their discount rate, i.e., $\delta^N > \delta^S$. Then the north is the exporter of the capital intensive good while the south is the exporter of the labor intensive good.*

Proof: See appendix 7.7. ■

4.5 Local indeterminacy in the trade regime

We now consider the local stability results in the trade regime (see Appendix 7.8.1). As in the autarky regime, we study a capital intensive consumption good and show the following:

Proposition 9. *Under assumptions 1, 3-6, there exist $\underline{b}^S < \overline{b}^S < 0$, $\underline{b}^N < \overline{b}^N < 0$, $\underline{\nu} < 1$, $\underline{\rho}^W > -1$, $\underline{\varepsilon}_{rk}^W > 0$, and $\gamma^{W,\mathcal{F}} > \gamma^{W,\mathcal{T}} > 1$ such that for $b^S \in (\underline{b}^S, \overline{b}^S)$, $b^N \in (\underline{b}^N, \overline{b}^N)$, $\nu \in (\underline{\nu}^W, 1]$, $\rho > \underline{\rho}^W$ and $\varepsilon_{rk}^W > \underline{\varepsilon}_{rk}^W$, the following results prevail:*

i] the steady state is a sink, i.e., a locally indeterminate NSS, when $\gamma \in (\gamma^{W,\mathcal{T}}, \gamma^{W,\mathcal{F}})$;

ii] the steady state is a saddle, i.e., a locally determinate NSS, when $\gamma \in (1, \gamma^{W,\mathcal{T}}) \cup (\gamma^{W,\mathcal{F}}, +\infty)$.

Proof: See appendix 7.8.2. ■

The intuition of proposition 9 is similar to the intuition of proposition 3 in the autarky regime. $\gamma^{W,\mathcal{T}}$ represents generically a transcritical bifurcation resulting in the existence of a second steady state which is locally unstable (resp. saddle-point stable) in a right (resp. left) neighborhood of $\gamma^{W,\mathcal{T}}$, whereas $\gamma^{W,\mathcal{F}}$ is generically a flip bifurcation value giving rise to period-two cycles which are locally indeterminate (resp. unstable) in a right (resp. left) neighborhood of $\gamma^{W,\mathcal{F}}$. Proposition

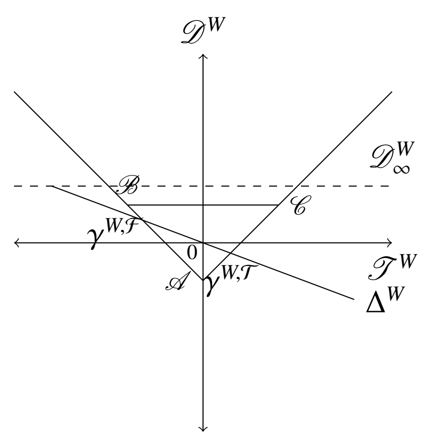

Figure 3: Local indeterminacy in the trade regime.

9 provides conditions under which local indeterminacy together with dynamic efficiency arise in

the world economy. These conditions on elasticities and shares are based on joint restrictions on the parameters of the north and the south. As our result on the occurrence of sunspot fluctuations of an economy in the autarky regime and the world economy differ, international trade may have various effects on the stability properties of each countries. In order to determine the impact of international trade we need to compare the conditions on elasticities and shares of the north and the south in the autarky regime (proposition 3) with respect to the conditions on elasticities and shares of the world economy (proposition 9). At this point, one may wonder whether or not international trade may have an impact on the local stability properties of the north and the south. To answer this question, we provide a numerical example in the next section.

5 The effect of international trade

In the present section we consider a numerical example in order to determine whether or not international trade has a (de)stabilazing effect. Our analysis will focus on the role played by the elasticity of intertemporal substitution in consumption γ. First, we present the numerical conditions such that local indeterminacy holds in the autarky regime for both countries (proposition 3). Second, based on the same set of parameters as the autarky regime, we provide numerical conditions such that local indeterminacy occurs in the trade regime (proposition 9). Finally, we discuss the effect of international trade on the local stability properties in the north and the south.

5.1 The autarky regime

Let us consider the following set of parameters: $\Theta^S = 1$, $\Theta^N = 0.9988$, $\mu = 0.99996, \eta = 0.1002, \rho = 9, \nu = 0.99, \gamma = 1.6$, $K^{N*} = 0.7501$ and $K^{S*} = 0.7409$. We get an efficient steady state in the north $\alpha^N \simeq 0.1589$ and the south $\alpha^S \approx 0.1581$ with $\underline{\alpha}^N \approx 0.093$ and $\underline{\alpha}^S \approx 0.0825$. The corresponding share of capital in the north is $s^N \approx 0.4755$ and the south is $s^S \approx 0.4784$. We find the relative capital intensity difference in the north $b^N \approx -2.601$ and in the south $b^S \approx -2.597$. We obtain the elasticity of the rental rate of capital in the north $\varepsilon_{rk}^N \approx 2.6117$ and the south $\varepsilon_{rk}^S \approx 2.5926$. The existence of local indeterminacy is based on a condition on technology, i.e., b^i, ε_{rk}^i and ν, and on a condition on preferences, i.e., γ. The conditions on technology in proposition 3 are satisfied in the north and the south for any $b^N \in (-5.292, -1.925)$, $b^S \in (-5.322, -1.915)$, $\varepsilon_{rk}^N > 2.4579$, $\varepsilon_{rk}^S > 2.3523$, $\nu \in (0.084, 1)$ and $\rho > 1.009$. The conditions on preference in proposition 3 are satisfied in the north for any $\gamma \in (1.5864, 1.7682)$ and the south for any $\gamma \in (1.5974, 1.7818)$. The condition on γ are represented in figure 4. Figure 4 shows that the bifurcation parameters for the north and the south are different. In other words, the local stability properties of the two countries in the autarky regime are different. For example when $\gamma \in (\gamma^{N,\mathcal{T}}, \gamma^{N,\mathcal{F}})$ the NSS of the north is locally indeterminate while the NSS of the south can either be locally determinate or indeterminate.

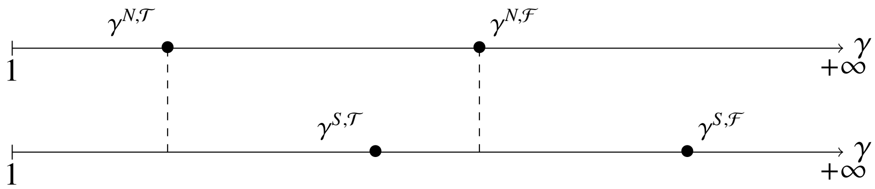

Figure 4: Bifurcation parameters of the north and the south.

5.2 The trade regime

Now, we follow the same approach as in the autarky regime. We consider the same set of parameters: $\Theta^S = 1$, $\Theta^N = 0.9988$, $\mu = 0.99996, \eta = 0.1002, \rho = 9, \gamma = 1.6$, $K^{N*} = 0.7501$, $K^{S*} = 0.7409$, $Y^{N*} = 0.7409$ and $Y^{S*} = 0.7501$. As in the autarky regime, the existence of local indeterminacy is based on a condition on technology, i.e., b^i, ε^i_{rk} and ν, and a condition on preferences, i.e., γ. We find the relative capital intensity difference in the north $b^N \approx -2.5998$ and in the south $b^S \approx -2.5998$. We obtain the elasticity of the rental rate of capital in the north $\varepsilon^N_{rk} \approx 2.6117$ and the south $\varepsilon^S_{rk} \approx 2.5926$. The conditions on technology in proposition 9 are satisfied in the world economy for any $b^N \in (-5.296, -2.071)$, $b^S \in (-5.317, -2.071)$, $\varepsilon^W_{rk} > 2.586$ and $\nu > 0.081$. The conditions on preferences in proposition 9 are satisfied in the world economy for any $\gamma \in (1.5867, 1.7696)$. Figure 5 gathers the bifurcation parameters of the north and the south in the autarky regime and the world economy. As already mentioned above, the local

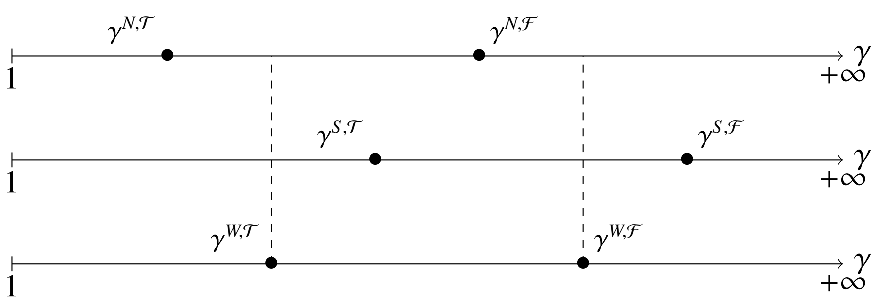

Figure 5: Bifurcation parameters of the north, the south and the world economy.

stability properties of the two countries are not the same. Moreover, the bifurcation parameters of the world economy are different from the bifurcation parameters of the two countries. In particular, when $\gamma \in (\gamma^{W,\mathcal{T}}, \gamma^{W,\mathcal{F}})$ the NSS of the world economy is locally indeterminate while the NSS of the north and the south can either be locally determinate or indeterminate.

5.3 Result and economic intuition

In figure 6, we represent the different values of γ defined in figure 5 in only one line. Then for a given value of γ, we can deduce the local stability properties of the two countries and thus the effect of international trade. We derive from figure 6 the main result of the chapter:

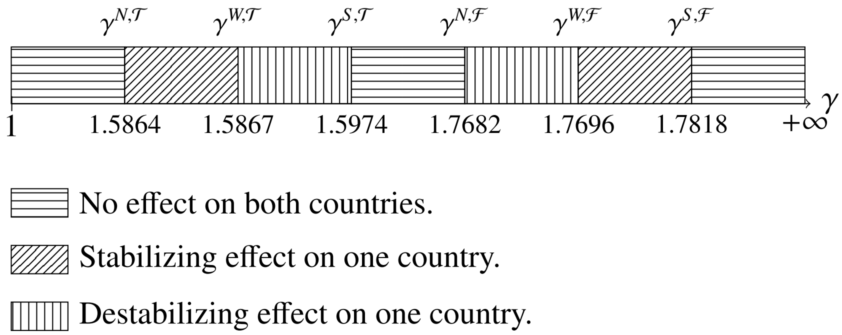

Figure 6: The effect of international trade on the local stability properties of the north and the south.

Result 1. *Contagion of cycles from one country to another*

Under Assumptions 1, 3-6, there exists a set of parameters $(\eta, \mu, \rho, \gamma, \nu, \gamma^{N,\mathcal{F}}, \gamma^{W,\mathcal{F}})$*, with* $i \in \{N, S\}$*, and allocations* (K^{N*}, K^{S*}) *such that the autarky equilibrium of the north is saddle-point stable with oscillatory convergence and the south is characterized by period-two cycles, while period-two cycles holds for both countries in the trade regime.*

The intuition when $\gamma \in (\gamma^{N,\mathcal{F}}, \gamma^{W,\mathcal{F}})$ is the following. Assume that all agents expect a higher current rate of investment inducing an increase of the future capital stock. This expectation will be self fulfilling provided that the amount of saving is sufficiently high. Agents save enough if they reduce their current consumption.

Let us first consider the expectation of agents in the autarky equilibrium. On the one hand, agents in the south save enough since a high enough γ, i.e., $\gamma < \gamma^{S,\mathcal{F}}$, provides agents to offset the decrease of their current consumption by an increase of their future consumption. On the other hand, agents in the north save an important amount since the intertemporal substitution effect is relatively large, i.e. $\gamma > \gamma^{N,\mathcal{F}}$. It implies through the Rybzcinsky effect that the production of the future consumption is important. This increase of the prouction exceed the rise in the future consumption demand. As a result, the initial expectation is fulfilled for agents in the south while the initial expectation is not fulfilled for agents in the north.

The initial expectation can be achieved in the north if there is a decrease of the supply of the consumption good.

Let us now consider the free-trade equilibrium. In the trade regime, the two countries exchange goods. In particular, the north exports the consumption good and the south imports the consumption good. It implies that the relative price of the investment good increase in the south and thus the price of the consumption decrease with respect to autarky. As in the autarky equilibrium, agents in the two countries must decrease their current consumption to save enough.

In the south, since the consumption good is less expensive with respect to the autarky equilibrium agents may increase their consumption consumption by importing. It implies that there is less production of the future consumption good that stays in the north. As a result, the initial expectation can be realized as an equilibrium for both countries in the trade regime.

Remark 1 *Based on figure 6 and the numerical example, international trade can have two other effects under different conditions on γ. On the one hand, opening to trade may rule out sunspot fluctuations that exist in the autarky regime. In the other hand, the two countries in the trade regime replicate either local determinacy or local indeterminacy that exist in the autarky regime.*

6 Concluding remarks

This chapter studies the effect of international trade on the local stability properties of trading countries in a two-country two-sector Heckscher-Ohlin overlapping generations model. The two countries are characterized by CES life cycle utility, a non increasing returns to scale CES technology in the consumption good sector and a constant returns to scale Leontief technology in the investment good sector. Using a numerical simulation, we show that international trade can be a source of macroeconomic volatility. Indeed, considering a free-trade equilibrium in which one country is a net exporter of the capital intensive consumption good and the other country is a net exporter of the labor intensive investment good, we provide a numerical example which show that period-two cycles can occur in the free-trade equilibrium although one country is characterized by saddle-point stability in the autarky equilibrium.

7 Appendix

7.1 The partial derivatives of $T(K_t, Y_t, 1)$ in the autarky regime

Let us introduce the following notations: $T_g = T_g(K_t, Y_t, 1)$, $T_{gh} = T_{gh}(K_t, Y_t, 1)$, with $g = 1, 2, 3$ and $h = 1, 2$. The first and second partial derivatives of the social production function $T(K_t, Y_t, 1)$ are given by the following two lemmas:

Lemma 1. *The first partial derivatives of $T(K_t, Y_t, 1)$ satisfy the following:*

$$T_1 = \Theta\mu\nu(K_t - \eta Y_t)^{-(1+\rho)}\left[\mu(K_t - \eta Y_t)^{-\rho} + (1-\mu)(1-Y_t)^{-\rho}\right]^{-\frac{\nu+\rho}{\rho}};$$
$$T_2 = -T_1\left[\eta + \left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t - \eta Y_t}{1-Y_t}\right)^{1+\rho}\right];$$
$$T_3 = \left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t - \eta Y_t}{1-Y_t}\right)^{1+\rho} T_1.$$

Proof: The first partial derivatives of $T\,(K_t, Y_t, 1)$ directly follow from computations of (4). ■

Lemma 2. *The second partial derivatives of* $T\,(K_t, Y_t, 1)$ *satisfy the following:*

$$T_{11} = \frac{T_1\left[(\nu-1)-(1+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{\rho}\right]}{(K_t-\eta Y_t)\left[1+\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{\rho}\right]};$$

$$T_{21} = -T_{11}\left[\eta + \frac{(\nu+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{1+\rho}}{(\nu-1)-(1+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{\rho}}\right];$$

$$T_{22} = T_{11}\left\{\eta^2 + \frac{(\nu+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{1+\rho}\left[2\eta(\nu+\rho)-(1+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)+(\nu-1)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{1+\rho}\right]}{(\nu-1)-(1+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{\rho}}\right\};$$

$$T_{31} = \frac{T_{11}(\nu+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{1+\rho}}{(\nu-1)-(1+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{\rho}};$$

$$T_{32} = -\frac{T_{11}\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{1+\rho}\left[\eta(\nu+\rho)-(1+\rho)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)+(\nu-1)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{1+\rho}\right]}{(\nu-1)-(1+\rho)\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t-\eta Y_t}{1-Y_t}\right)^{\rho}}.$$

Proof: The second partial derivatives of $T\,(K_t, Y_t, 1)$ directly follow from computations of lemma 1 and the fact that $T_2 = -T_1\,(K_t, Y_t, 1)\,\eta - T_3\,(K_t, Y_t, 1)$. The strict concavity of the production functions implies that the determinant of the Hessian matrix of $T\,(K_t, Y_t, 1)$ satisfies $|H(K, Y, 1)| = T_{11}(K, Y, 1)T_{22}(K, Y, 1) - T_{21}(K, Y, 1)^2 > 0$. ■

We can write the second partial derivatives of $T(K^*, K^*, 1)$ at the NSS as:

Lemma 3. *At the NSS, the first and second partial derivatives of* $T\,(K^*, K^*, 1)$ *satisfy the following:*

$$T_2 = -\frac{T_1[s\eta(1-b)+(1-s)a]}{s(1-b)};$$

$$T_3 = \frac{T_1(1-s)a}{s(1-b)};$$

$$T_{11} = \frac{T_1\left[(\nu-1)(1-b)s-(1+\rho)(1-s)\right]}{K^*(1-\eta)(1-sb)};$$

$$T_{21} = -\frac{T_{11}\left[\eta(\nu-1)(1-bs)-b(\nu+\rho)(1-s)\right]}{(\nu-1)(1-b)s-(1+\rho)(1-s)};$$

$$T_{22} = \frac{T_{11}\left\{\eta\left[\eta(\nu-1)(1-bs)-b(\nu+\rho)(1-s)\right]+a(1-s)\left[\frac{\nu-1}{R}+b(1+\rho)\right]\right\}}{(\nu-1)(1-b)s-(1+\rho)(1-s)};$$

$$T_{31} = \frac{T_{11}(\nu+\rho)(1-s)a}{(\nu-1)(1-b)s-(1+\rho)(1-s)};$$

$$T_{32} = -\frac{T_{11}a(1-s)\left[\frac{\nu-1}{R}+(1+\rho)b\right]}{(\nu-1)(1-b)s-(1+\rho)(1-s)}.$$

Proof: Consider equation (6). The second partial derivatives of $T\,(K^*, K^*, 1)$ directly follow from computations of lemma 1 and lemma 2 and the fact that

$$\frac{1-s}{s(1-b)} = \frac{T_3}{T_1 K^*(1-b)} = \left(\frac{1-\mu}{\mu}\right)\left(\frac{K^*-\eta K^*}{1-K^*}\right)^{\rho}$$

Using the fact that $T_{22} = -T_{21}\eta - T_{32}$ and lemma 3 we derive:

$$\mid H \mid = -\frac{(1-s)T_{11}^2 a\left[(1+\rho)\left[\eta(\nu-1)(1-bs)-b(\nu+\rho)(1-s)\right]-\left(\frac{\nu-1}{R}+(1+\rho)b\right)\left[(\nu-1)(1-b)s-(1+\rho)(1-s)\right]\right]}{(\nu-1)(1-b)s-(1+\rho)(1-s)}$$

Then we obtain $\mid H \mid > 0$ if $\nu < 1$ and $\mid H \mid = 0$ if $\nu = 1$. ■

7.2 Proof of proposition 1

From the set of admissible paths defined by (13), we have $K^* \in \left(0, \overline{K}\right)$. K^* is a solution of (14) if:

$$\frac{1}{1+\delta^\gamma\left(\frac{r(K^*,K^*,1)}{\Gamma p(K^*,K^*,1)}\right)^{\gamma-1}} = 1 - \frac{K^* p(K^*,K^*,1)}{w(K^*,K^*,1)} \equiv \Phi_{K^*} \in (0,1) \tag{31}$$

Let us express $\xi = R/\Gamma$. Under assumption 1, $\alpha(\xi)$ is a monotone decreasing function with $lim_{\xi\to 0}\alpha(\xi) = \alpha_{sup}$, $lim_{\xi\to +\infty}\alpha(\xi) = \alpha_{inf}$ and $(\alpha_{inf}, \alpha_{sup}) \subseteq (0,1)$. It follows that $\alpha(\xi)$ admits an inverse function defined over $(\alpha_{inf}, \alpha_{sup})$. Let $K^* \in \left(0, \overline{K}\right)$ be such that $\Phi_{K^*} \in (\alpha_{inf}, \alpha_{sup})$. We then derive that K^* is a steady state if and only if $\Gamma = \Gamma(K^*)$. Using lemma 3, the scaling parameter Γ is a unique solution of (31) and defined by

$$\Gamma(K^*) = R\left\{\frac{\delta^\gamma\left[(1-s)a(1-K^*)-s\eta(1-b)K^*\right]}{K^*\left[s\eta(1-b)+(1-s)a\right]}\right\}^{\frac{1}{\gamma-1}} \tag{32}$$

■

7.3 Local stability in the autarky regime

7.3.1 Characteristic polynomial in autarky regime

From (10), the derivative of $\alpha(R/\Gamma)$ is given by:

$$\frac{d\alpha(R/\Gamma)}{d(R/\Gamma)} = -\frac{\alpha(\gamma-1)(1-\alpha)\Gamma}{R} \tag{33}$$

Consider equations (4), (6), (17) and (33) evaluated at the NSS . Let us rewrite the difference equation (12) as:

$$G(K_t, K_{t+1}, K_{t+2}) = K_{t+1} - \frac{w(K_t,K_{t+1})}{p(K_t,K_{t+1})}\left\{1 - \alpha\left[\frac{r(K_{t+1},K_{t+2})}{\Gamma p(K_t,K_{t+1})}\right]\right\} = 0 \tag{34}$$

We obtain from the linearization of (34) around the NSS the following:

$$\begin{pmatrix} K_{t+2} - \tilde{K} \\ K_{t+1} - \tilde{K} \end{pmatrix} = J\begin{pmatrix} K_{t+1} - \tilde{K} \\ K_t - \tilde{K} \end{pmatrix} \tag{35}$$

where J is the Jacobian matrix of the difference equation (12) and is given by

$$J = \begin{pmatrix} -\frac{\partial G/\partial K_{t+1}}{\partial G/\partial K_{t+2}} & -\frac{\partial G/\partial K_t}{\partial G/\partial K_{t+2}} \\ 1 & 0 \end{pmatrix} \tag{36}$$

The characteristic polynomial is defined by

$$\mathscr{P}(\lambda) = \lambda^2 - \lambda \mathscr{T}^A(\gamma) + \mathscr{D}^A(\gamma)$$

with the trace $\mathscr{T}^A(\gamma)$ and the determinant $\mathscr{D}^A(\gamma)$:

$$\mathscr{T}^A(\gamma) = \frac{1+\varepsilon_{rk}\left\{R^A\left[\frac{T_{32}(1-\alpha)}{T_{11}K^*}+\frac{T_{22}}{T_{11}}\right]+\alpha(\gamma-1)\left[1+\frac{T_{22}R^A}{T_{11}}\right]\right\}}{\alpha(\gamma-1)\left(-\frac{T_{21}\varepsilon_{rk}}{T_{11}}\right)} \tag{37}$$

$$\mathscr{D}^A(\gamma) = \frac{R^A\left[1+\alpha(\gamma-1)+(1-\alpha)\frac{T_{31}}{T_{21}K^*}\right]}{\alpha(\gamma-1)} \tag{38}$$

where the first and second partial derivatives of $T(K_t, Y_t, L)$ given in lemma 1 and 2. ■

7.3.2 Proof of proposition 3

Under assumption 2, the NSS remains constant as γ varies within $(1, \infty)$. We can analyse the variation of the trace $\mathscr{T}^A(\gamma)$ and the determinant $\mathscr{D}^A(\gamma)$ in the $(\mathscr{T}^A(\gamma), \mathscr{D}^A(\gamma))$ plane. The relationship between $\mathscr{T}^A(\gamma)$ and $\mathscr{D}^A(\gamma)$ is given by a half-line $\Delta^A\left(\mathscr{T}^A\right)$ which is characterized from the consideration of its extremities. The starting point is the couple $(\lim_{\gamma\to+\infty}\mathscr{T}^A \equiv \mathscr{T}^A_\infty, \lim_{\gamma\to+\infty}\mathscr{D}^A \equiv \mathscr{D}^A_\infty)$, while the end point is the couple $(\lim_{\gamma\to 1}\mathscr{T}^A \equiv \mathscr{T}^A_1, \lim_{\gamma\to 1}\mathscr{D}^A \equiv \mathscr{D}^A_1)$. Solving $\mathscr{T}^A$ and $\mathscr{D}^A$ with respect to $\alpha(\gamma-1)$ yields to the following linear relationship:

$$\mathscr{D}^A = \Delta^A\left(\mathscr{T}^A\right) = \mathscr{S}^A\mathscr{T}^A + \mathscr{D}^A_\infty - \mathscr{S}^A\mathscr{T}^A_\infty \tag{39}$$

where the slope $\mathscr{S}^A$, $\mathscr{D}^A_\infty$ and $\mathscr{T}^A_\infty$ are given by:

$$\mathscr{S}^A = -\frac{R^A\frac{T_{21}K^*}{T_{11}}\varepsilon_{rk}\left[1+(1-\alpha)\frac{T_{31}}{T_{21}K^*}\right]}{1+\varepsilon_{rk}R^A\left[\frac{T_{32}(1-\alpha)}{T_{11}K^*}+\frac{T_{22}}{T_{11}}\right]},\ \mathscr{D}^A_\infty = R^A,\ \mathscr{T}^A_\infty = -\frac{1+\frac{T_{22}R^A}{T_{11}}}{\frac{T_{21}}{T_{11}}} \tag{40}$$

Let us now prove proposition 3. Assume that $b < 0$, $\nu \leq 1$, $\alpha \in (\underline{\alpha}, 1/2)$ and $s \in (1/3, 1/2)$. Let us first consider the starting point $(\mathscr{T}^A_\infty, \mathscr{D}^A_\infty)$. From equation (40) we have $\mathscr{D}^A_\infty > 1$ and we need to compute the sign of $\mathscr{T}^A_\infty$. Using lemma 3 the numerator of $\mathscr{T}^A_\infty$ is defined as:

$$1 + \frac{T_{22}R^A}{T_{11}} = 1 + \frac{\eta^2R^A(1-bs)(\nu-1)+a(1-s)(\nu-1)-R^Ab(1-s)(b(1+\rho)+\eta(\nu-1))}{(\nu-1)(1-b)s-(1+\rho)(1-s)}$$

while the denominator is given by

$$-\frac{T_{21}}{T_{11}} = \frac{\eta(\nu-1)-b\left[[s\eta+(1-s)]\nu-s\eta+\rho(1-s)\right]}{(\nu-1)(1-b)s-(1+\rho)(1-s)}$$

Under $\nu \in (\underline{\nu}, 1)$ and $b < b_0$, $-T_{21}/T_{11} < 0$ and $1 + T_{22}R^A/T_{11} > 0$ implying that $\mathscr{T}^A_\infty < 0$, where $\underline{\nu}$ and b_0 are specified by

$$\underline{\nu} = \frac{s\eta}{s\eta+1-s},\ b_0 = \frac{\eta(\nu-1)}{[s\eta+(1-s)]\nu-s\eta+\rho(1-s)}$$

To establish the precise location of the starting point $(\mathscr{T}^A_\infty, \mathscr{D}^A_\infty)$ we need to compute $\mathscr{P}^A_\infty(1) = 1 - \mathscr{T}^A_\infty + \mathscr{D}^A_\infty$ and $\mathscr{P}^A_\infty(-1) = 1 + \mathscr{T}^A_\infty + \mathscr{D}^A_\infty$. Since $\mathscr{T}^A_\infty < 0$ when $\nu \in (\underline{\nu}, 1)$ and $b < b_0$, we get $\mathscr{P}^A_\infty(1) > 0$. Using equation (40) and the fact that $T_{22} = -T_{21}\eta - T_{32}$, $\mathscr{P}^A_\infty(-1)$ is given by:

$$\mathscr{P}^A_\infty(-1) = \frac{R^A\left(\frac{T_{21}}{T_{11}} - \frac{T_{22}}{T_{11}}\right) + \frac{T_{21}}{T_{11}} - 1}{\frac{T_{21}}{T_{11}}} \tag{41}$$

By applying lemma 3, we obtain after straightforward computations the following:

$$\mathscr{P}^A_\infty(-1) = \frac{(\nu-1)[(1-b)s + a(1-s) + \eta(1-bs)(1+R^A+R^A\eta) - R^A b\eta(1-s)] - (1+\rho)(1-s)}{\eta(\nu-1)(1-bs) - b(\nu+\rho)(1-s)} - \frac{b(1-s)[R^A b(1+\rho) + (\nu+\rho)(1+R^A)]}{\eta(\nu-1)(1-bs) - b(\nu+\rho)(1-s)}$$

$\mathscr{P}^A_\infty(-1) < 0$ if and only if $b < -(\nu+\rho)(1+R^A)/R^A(1+\rho) \equiv \overline{b}$, with $\overline{b} < b_0$, and $\nu \in (\underline{\nu}, 1)$. As a result, the starting point is in the left area outside the triangle ABC (see figure 1). To determine the precise location of the end point $(\mathscr{T}^A_1, \mathscr{D}^A_1)$, it is sufficient to determine that $\Delta^A(\mathscr{T}^A)$ is pointing upward or downward. We thus study the sign of $\mathscr{D}^{A'}(\gamma)$:

$$\mathscr{D}^{A'}(\gamma) = -\frac{R^A\left[1 + \frac{(1-\alpha)T_{31}}{T_{21}K^*}\right]}{\alpha(\gamma-1)^2} \tag{42}$$

where

$$1 + \frac{(1-\alpha)T_{31}}{T_{21}K^*} = \frac{(\nu-1)(1-b)s - (1+\rho)(1-s) - (\nu+\rho)(1-s)(b\alpha+1-\alpha)}{\eta(\nu-1) - b\left[s\eta(\nu-1) + (\nu+\rho)(1-s)\right]} \tag{43}$$

From lemma 3, we have $T_{11} < 0$. Under $b \in (\underline{b}, b_0)$, with $\underline{b} = -(1-\alpha)/\alpha$, and $\nu \in (\underline{\nu}, 1)$ we get $\mathscr{D}^{A'}(\gamma) > 0$ and thus $\Delta^A(\mathscr{T}^A)$ is pointing downward. Local indeterminacy may arise as shown by Δ^A in figure 1 if only if $\mathscr{T}^A > 0$ when $\mathscr{D}^A = -1$. Using the expressions of the trace $\mathscr{T}^A$ and the determinant $\mathscr{D}^A$ allows to show that when $\mathscr{D}^A = -1$, $\mathscr{T}^A > 0$ if and only if

$$1 + \varepsilon_{rk} \underbrace{\left\{R^A\left[\frac{T_{32}(1-\alpha)}{T_{11}K^*} + \frac{T_{22}}{T_{11}}\right] + \alpha(\gamma-1)\left[1 + \frac{T_{22}R^A}{T_{11}}\right]\right\}}_{\equiv\Lambda} < 0 \tag{44}$$

where

$$\alpha(\gamma-1) = -\frac{R^A\left(1 + \frac{T_{31}(1-\alpha)}{T_{21}K^*}\right)}{\left(1+R^A\right)} \tag{45}$$

Using equations (44)-(45), lemma 3, and the fact that $T_{22} = -T_{21}\eta - T_{32}$, we can write Λ as:

$$\Lambda = \frac{R^A}{1+R^A}[\underbrace{-\left(1 + \frac{T_{21}\eta}{T_{11}}\right) + \frac{T_{31}(1-\alpha)R^A\eta}{T_{11}K^*} + \frac{T_{32}(1-\alpha)R^A}{T_{11}K^*}}_{\equiv\chi_0}$$
$$\underbrace{-\frac{T_{32}}{T_{11}}\left[1 + \frac{T_{31}(1-\alpha)R^A}{T_{21}K^*}\right]}_{\equiv\chi_1}$$

$$\underbrace{-\frac{T_{31}(1-\alpha)}{T_{11}K^*}+\frac{T_{32}(1-\alpha)}{T_{11}K^*}}_{\equiv\chi_2}+\underbrace{\frac{T_{31}(1-\alpha)}{T_{11}K^*}-\frac{T_{31}(1-\alpha)}{T_{21}K^*}}_{\equiv\chi_3}]$$

Since $T_{21}/T_{11} > 0$ if $b < b_0$ and $\nu \in (\underline{\nu}, 1)$, $T_{31}/T_{11} < 0$ and $T_{32}/T_{11} < 0$, we obtain $\chi_0 < 0$. From lemma 3, χ_1, χ_2 and χ_3 are defined as:

$$\chi_1 = \frac{(\eta-b)(1-s)\left(\frac{\nu-1}{R}+(1+\rho)b\right)\left[\eta(\nu-1)(1-bs)-(\nu+\rho)(s+b(1-2s))\right]}{[\eta(\nu-1)(1-bs)-b(\nu+\rho)(1-s)][(\nu-1)(1-b)s-(1+\rho)(1-s)]}$$

$$\chi_2 = -\frac{(1-\alpha)(1-s)(1-b)\left[\nu+\rho+b(1+\rho)+\frac{\nu-1}{R^A}\right]}{\left[(\nu-1)(1-b)s-(1+\rho)(1-s)\right]}$$

$$\chi_3 = -\frac{(\nu+\rho)(1-\alpha)(1-s)(1-b)\left[\eta(\nu-1)(1-bs)-b(\nu+\rho)(1-s)+(\nu-1)(1-b)s-(1+\rho)(1-s)\right]}{\left[\eta(\nu-1)(1-bs)-b(\nu+\rho)(1-s)\right]\left[(\nu-1)(1-b)s-(1+\rho)(1-s)\right]}$$

$\chi_1,\chi_2 < 0$ if and only if $b \in (\underline{b}, \overline{b})$ and $\nu \in (\underline{\nu}, 1)$. $\chi_3 < 0$ if and only if $b \in (\underline{b}, \overline{b})$, $\nu \in (\underline{\nu}, 1)$ and $\rho > \underline{\rho}$ where $\underline{\rho} = s(1+\eta)/(1-s)$. We then have $\Lambda < 0$. It follows that when $\alpha \in (\underline{\alpha}, 1/2)$, $s \in (1/3, 1/2)$, $b \in (\underline{b}, \overline{b})$, $\nu \in (\underline{\nu}, 1)$, $\rho > \underline{\rho}$ and $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$, $\mathscr{T}^A > 0$ when $\mathscr{D}^A = -1$. The bound $\underline{\varepsilon}_{rk}$ is defined as $\underline{\varepsilon}_{rk} = -1/\Lambda$. The bifurcation values $\gamma^{\mathcal{T}}$ and $\gamma^{\mathcal{F}}$ are respectively defined as the solutions of $\mathscr{P}^A(1) = 1 - \mathscr{T}^A + \mathscr{D}^A = 0$ and $\mathscr{P}^A(-1) = 1 + \mathscr{T}^A + \mathscr{D}^A = 0$ and given by:

$$\gamma^{\mathcal{T}} = \frac{-1-R^A\left[\frac{T_{32}}{T_{11}K^*}(1-\alpha)+\frac{T_{22}}{T_{11}}\right]+\varepsilon_{rk}\left\{\alpha+\frac{T_{22}}{T_{11}}R^A\alpha+\frac{T_{21}}{T_{11}}\left(1+R^A\right)\alpha-\frac{T_{21}}{T_{11}}R^A\left[1+(1-\alpha)\frac{T_{31}}{T_{21}K^*}\right]\right\}}{\alpha\varepsilon_{rk}\left[1+\frac{T_{22}}{T_{11}}R^A+\frac{T_{21}}{T_{11}}\left(1+R^A\right)\right]}$$

$$\gamma^{\mathcal{F}} = \frac{-1-R^A\left[\frac{T_{32}}{T_{11}K^*}(1-\alpha)+\frac{T_{22}}{T_{11}}\right]+\varepsilon_{rk}\left\{\alpha+\frac{T_{22}}{T_{11}}R^A\alpha-\frac{T_{21}}{T_{11}}\left(1+R^A\right)\alpha+\frac{T_{21}}{T_{11}}R^A\left[1+(1-\alpha)\frac{T_{31}}{T_{21}K^*}\right]\right\}}{\alpha\varepsilon_{rk}\left[1+\frac{T_{22}}{T_{11}}R^A-\frac{T_{21}}{T_{11}}\left(1+R^A\right)\right]}$$

Results follow. ■

7.4 Proof of proposition 4

Using Lemma 1 $r_t^N = r_t^S$ if and only if $\Theta^S = 1$ and $\Theta^N = \Theta^{N*}$, with:

$$\Theta^{N*} = \left(\frac{K_t^S-\eta Y_t^S}{K_t^N-\eta Y_t^N}\right)^{\nu-1}\left[\frac{1+\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t^N-\eta Y_t^N}{1-Y_t^N}\right)^{\rho}}{1+\left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t^S-\eta Y_t^S}{1-Y_t^S}\right)^{\rho}}\right]^{\frac{\nu+\rho}{\rho}}$$

Under free-trade we derive: $p_t = p_t^N = p_t^S$. We obtain from Lemma 1 that $p_t = r_t^N\eta + w_t^N = r_t^S\eta + w_t^S$. It follows that if $\Theta^S = 1$ and $\Theta^N = \Theta^{N*}$ we obtain $w_t^N = w_t^S$. ■

7.5 Proof of proposition 5

From the set of admissible paths defined by (26), we have $K^{W*} \in (0, \overline{K}^W)$. K^{W*} is a solution of (25) if:

$$\frac{1}{1+(\delta^N)^\gamma\left(\frac{r\left(K^{W*},K^{W*},2\right)}{\Gamma^N p\left(K^{W*},K^{W*},1\right)}\right)^{\gamma-1}} = 2 - \alpha^S - \frac{K^{W*}p\left(K^{W*},K^{W*},1\right)}{w\left(K^{W*},K^{W*},1\right)} \equiv \Phi_{K^{W*}} \in (0,1) \tag{46}$$

Let us express $\xi^N = R/\Gamma^N$. Under assumption 1, $\alpha^N(\xi^N)$ is a monotone decreasing function with $lim_{\xi\to 0}\alpha^N(\xi^N) = \alpha^N_{sup}$, $lim_{\xi\to+\infty}\alpha^N(\xi^N) = \alpha^N_{inf}$ and $(\alpha^N_{inf}, \alpha^N_{sup}) \subseteq (0, 1)$. It follows that $\alpha^N(\xi^N)$ admits an inverse function defined over $(\alpha^N_{inf}, \alpha^N_{sup})$. Let $K^{W*} \in (0, \overline{K}^W)$ be such that $\Phi_{K^{W*}} \in (\alpha^N_{inf}, \alpha^N_{sup})$. We then derive that K^{W*} is a steady state if and only if $\Gamma^N = \Gamma^N(K^{W*})$ with $\Gamma^N(K^{W*})$ defined by:

$$\Gamma^N(K^{W*}) = R\left\{\frac{(\delta^N)^\gamma\left[(1-s^N)a^N\left(1-K^{W*}-K^{N*}\right)-s^N\eta(1-b^N)(K^{W*}+K^{N*})\right]}{(K^{W*}+K^{N*})\left[s^N\eta(1-b^N)+(1-s^N)a^N\right]}\right\}^{\frac{1}{\gamma-1}} \tag{47}$$

■

7.6 Proof of proposition 7

Let us consider the capital constraint $K^i = K^{i0} + K^{i1}$ and the labor constraint $1 = L^{i0} + L^{i1}$ in a country $i \in \{N, S\}$. We obtain $k^i = L^{i0}k^{i0} + (1 - L^{i0})k^{i1}$, where $k^{ij} = \frac{K^{ij}}{L^{ij}}$ with $i \in \{N, S\}$ and $j \in \{0, 1\}$. L^{i0} is given by:

$$L^{i0} = \frac{k^i - k^{i1}}{k^{i0} - k^{i1}}$$

From equation (6), we have $k^{i0} - k^{i1} = -b^i$ and $k^i = (\eta - b^i)/(1 - b^i)$. Substituting in L^{i0} we derive:

$$L^{i0} = \frac{1-\eta}{1-b^i} \tag{48}$$

Let $\eta \in (0, 1)$, from equation (6) evaluated at the NSS we derive $a^i > 0$ and $b^i < 1$. To obtain diversification at the NSS, we need that $L^{i0} \in (0, 1)$. Straightforward computations of (48) show that if $\eta \in (0, 1)$ we obtain $L^{i0} \in (0, 1)$. ■

7.7 Proof of proposition 8

Using equations (10) and (24), $\alpha^S > \alpha^N$ if and only if $(\delta^N)^\gamma(\Gamma^N)^{1-\gamma} > (\delta^S)^\gamma(\Gamma^S)^{1-\gamma}$. Replace Γ^N and Γ^S, defined in equation (32), we obtain that $K^N > K^S$ and $\phi^N > \phi^S$, meaning that the north is the most patient country. Let express k^S and y^S in terms of k^N and y^N so that

$$K^N = K^S + dK^S,\ Y^N = Y^S + dY^S,\ \ dK^S, dY^S > 0 \tag{49}$$

From Definition 2, the world market condition for investment good at the NSS is:

$$K^N + K^S = Y^N + Y^S \tag{50}$$

Using equation (49), we get the following relationship:

$$2K^N - dK = 2Y^N - dK,\ 2K^S + dK = 2Y^S + dK \tag{51}$$

Let consider the full employment condition:

$$\begin{pmatrix} \frac{L^{0i}}{Y_0^i} & \frac{L^{1i}}{Y^i} \\ \frac{K^{0i}}{Y_0^i} & \frac{K^{1i}}{Y^i} \end{pmatrix} \begin{pmatrix} Y_0^i \\ Y^i \end{pmatrix} = \begin{pmatrix} 1 \\ K^i \end{pmatrix} \tag{52}$$

Differentiating this system yields to:

$$\begin{pmatrix} \frac{L^{0i}}{Y_0^i} & \frac{L^{1i}}{Y^i} \\ \frac{K^{0i}}{Y_0^i} & \frac{K^{1i}}{Y^i} \end{pmatrix} \begin{pmatrix} dY_0^i \\ dY^i \end{pmatrix} + \underbrace{\begin{pmatrix} d\frac{L^{0i}}{Y_0^i} & d\frac{L^{1i}}{Y^i} \\ d\frac{K^{0i}}{Y_0^i} & d\frac{K^{1i}}{Y^i} \end{pmatrix}}_{=A} \begin{pmatrix} Y_0^i \\ Y^i \end{pmatrix} = \begin{pmatrix} 0 \\ dK^i \end{pmatrix} \tag{53}$$

The envelope theorem implies that $A = 0$ in (53):

$$\begin{pmatrix} \frac{L^{0i}}{Y_0^i} & \frac{L^{1i}}{Y^i} \\ \frac{K^{0i}}{Y_0^i} & \frac{K^{1i}}{Y^i} \end{pmatrix} \begin{pmatrix} dY_0^i \\ dY^i \end{pmatrix} = \begin{pmatrix} 0 \\ dK^i \end{pmatrix} \tag{54}$$

Let rewrite dY_0^i in terms of dY^i, $dY_0^i = -\frac{Y_0^i L^{1i}}{Y^i L^{0i}} dY^i$, we obtain the Rybczynscki effect $dY^i/dK^i = b^i$. b^i is defined in equation (6). Then equation (51) can be express as:

$$K^N - Y^N = -\frac{1-b^N}{2b^N} dK,\ K^S - Y^S = \frac{1-b^S}{2b^S} dK \tag{55}$$

Let $\eta \in (0,1)$, from equation (6) evaluated at the NSS we derive $a^i > 0$ and $b^i < 1$. It implies that $K^S \gtrless Y^S$ and $K^N \lessgtr Y^N$ if and only if $b^i \gtrless 0$. It follows directly that if $b^i \in (0,1)$ the north exports the capital intensive investment good and the south exports the labor intensive consumption good and if $b^i < 0$ the north exports the capital intensive consumption good and the south exports the labor intensive investment good. ∎

7.8 Local stability in the trade regime

7.8.1 Characteristic polynomial in trade regime

Let us introduce the following notations:

$$\tau\left(K^W, Y^W, 2\right) = T\left(K^N, Y^N, 1\right) + T\left(K^S, Y^S, 1\right);$$
$$\tau_{jh} = \tau_{jh}\left(K^W, Y^W\right),\ j \in \{1,2,3\},\ h \in \{1,2\};^{20}$$
$$\mid H^i \mid = \mid H^i\left(K_t^i, Y_t^i\right) \mid,\ i \in \{N, S\}.$$

In order to obtain a tractable formulation of the characteristic polynomial, we have to define the second partial derivatives of the world social production function $\tau(K_t^W, Y_t^W, 2)$.

Lemma 4. *Along a free-trade equilibrium, the second partial derivatives of* $\tau\left(K_t^W, Y_t^W, 2\right)$ *satisfy the following:*

$$\begin{aligned}
\tau_{11} &= \tfrac{1}{\Xi}\left[T_{11}^N \mid H^S \mid + T_{11}^S \mid H^N \mid\right] \\
\tau_{12} &= \tfrac{1}{\Xi}\left[T_{12}^N \mid H^S \mid + T_{12}^S \mid H^N \mid\right] \\
\tau_{22} &= \tfrac{1}{\Xi}\left[T_{22}^N \mid H^S \mid + T_{22}^S \mid H^N \mid\right] \\
\tau_{31} &= \tfrac{1}{\Xi}\left[T_{31}^N \mid H^S \mid + T_{31}^S \mid H^N \mid\right] \\
\tau_{32} &= \tfrac{1}{\Xi}\left[T_{32}^N \mid H^S \mid + T_{32}^S \mid H^N \mid\right]
\end{aligned}$$

where

$$\mid H^i \mid = T_{11}^i T_{22}^i - \left(T_{21}^i\right)^2 > 0$$
$$\Xi = \left[T_{11}^N + T_{11}^S\right] \times \left[T_{22}^N + T_{22}^S\right] - \left[T_{12}^N + T_{12}^S\right] > 0$$

Proof: The proof is based upon similar arguments of Nishimura and Yano (1993). The associated Lagrangian of the world social production program defined in (22) is:

$$\begin{aligned}
\mathscr{L} = {} & T^N\left(K_t^N, Y_t^N, L_t^N\right) + T^S\left(K_t^S, Y_t^S, L_t^S\right) + \lambda_0\left(K_t^W - K_t^N - K_t^S\right) \\
& + \lambda_1\left(Y_t^W - Y_t^N - Y_t^S\right) + \lambda_2\left(L_t^W - L_t^N - L_t^S\right)
\end{aligned}$$

We obtain the following first order conditions:

$$\begin{array}{l}
T_1^N - \lambda_0 = 0,\ \ T_1^S - \lambda_0 = 0 \\
T_2^N - \lambda_1 = 0,\ \ T_2^S - \lambda_1 = 0 \\
T_3^N - \lambda_2 = 0,\ \ T_3^S - \lambda_2 = 0
\end{array} \tag{56}$$

[20] Where $\tau_{11} = \partial\tau^2/\partial^2 K^W$, $\tau_{12} = \tau_{21} = \partial\tau^2/\partial K^W \partial Y^W$, $\tau_{22} = \partial\tau^2/\partial^2 Y^W$, $\tau_{31} = \partial\tau^2/\partial K^W \partial L^W$ and $\tau_{32} = \partial\tau^2/\partial L^W \partial Y^W$.

Totally differentiate (56) and the resource constraint, $K^W - K^N - K^S = 0$, $Y^W - Y^N - Y^S = 0$ and $L_t^W - L_t^N - L_t^S = 0$, we obtain:

$$\underbrace{\begin{pmatrix} T_{11}^N & T_{12}^N & T_{13}^N & 0 & 0 & 0 & -1 & 0 & 0 \\ T_{21}^N & T_{22}^N & T_{23}^N & 0 & 0 & 0 & 0 & -1 & 0 \\ T_{31}^N & T_{32}^N & T_{33}^N & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & T_{11}^S & T_{12}^S & T_{13}^S & -1 & 0 & 0 \\ 0 & 0 & 0 & T_{21}^S & T_{22}^S & T_{23}^S & 0 & -1 & 0 \\ 0 & 0 & 0 & T_{31}^S & T_{32}^S & T_{33}^S & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}}_{\equiv A} \begin{pmatrix} dK^N \\ dK^S \\ dY^N \\ dY^S \\ dL^N \\ dL^S \\ d\lambda_0 \\ d\lambda_1 \\ d\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ dK^W \\ dY^W \\ dL^W \end{pmatrix} \tag{57}$$

Let denote Ξ denotes the determinant of the matrix A. We assume that is non singular. From the first-order conditions we have $\lambda_0 = \partial\tau/\partial K^W$, $\lambda_1 = \partial\tau/\partial Y^W$ and $\lambda_2 = \partial\tau/\partial L^W$. In order to obtain τ_{12}, we need to find $d\lambda_0/dy^w$ in the system (57). We derive:

$$\frac{d\lambda_0}{dy^W} = \frac{1}{\Xi} \begin{pmatrix} T_{11}^N & T_{12}^N & T_{13}^N & 0 & 0 & 0 & -1 & 0 & 0 \\ T_{21}^N & T_{22}^N & T_{23}^N & 0 & 0 & 0 & 0 & 0 & 0 \\ T_{31}^N & T_{32}^N & T_{33}^N & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_{11}^S & T_{12}^S & T_{13}^S & -1 & 0 & 0 \\ 0 & 0 & 0 & T_{21}^S & T_{22}^S & T_{23}^S & 0 & 0 & 0 \\ 0 & 0 & 0 & T_{31}^S & T_{32}^S & T_{33}^S & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -dY^W \end{pmatrix} \tag{58}$$

τ_{11}, τ_{22}, τ_{31} and τ_{32} are obtained in a similar way. ∎

Let us denote:

$$\Phi^N = \tfrac{1-\alpha^N}{2-\alpha^N-\alpha^S}, \Phi^S = \tfrac{1-\alpha^S}{2-\alpha^N-\alpha^S}. \tag{59}$$

Consider the derivatives of α^i with respect to (R/Γ) given in equation (33) and $\varepsilon_{rk}^W = -\tau_{11}K^{W*}/\tau_1$. Totally differentiate equation (25) evaluated at the NSS gives the characteristic $\mathscr{P}^W(\lambda^W) = (\lambda^W)^2 - \lambda^W \mathscr{T}^W(\gamma) + \mathscr{D}^W(\gamma)$ with $\mathscr{T}^W(\gamma)$ is the trace and $\mathscr{D}^W(\gamma)$ is the determinant:

$$\mathscr{T}^W(\gamma) = \frac{1+\varepsilon_{rk}^W\left\{R^W\left[\frac{\tau_{32}\left(2-\alpha^N-\alpha^S\right)}{2\tau_{11}K^{W*}}+\frac{\tau_{22}}{\tau_{11}}\right]+\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)(\gamma-1)\left[1+\frac{\tau_{22}R^W}{\tau_{11}}\right]\right\}}{\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)(\gamma-1)\left(-\frac{\tau_{21}\varepsilon_{rk}^W}{\tau_{11}}\right)} \tag{60}$$

$$\mathscr{D}^W(\gamma) = \frac{R^W\left[1+\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)(\gamma-1)+\frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)}{2\tau_{21}K^{W*}}\right]}{\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)(\gamma-1)} \tag{61}$$

where the second partial derivatives of $\tau(K^W, Y^W, 2)$ are defined in lemma 4.

7.8.2 Proof of proposition 9

Under assumption 6, the NSS remains constant as γ varies, we can analyse the variation of the trace $\mathscr{T}^W(\gamma)$ and the determinant $\mathscr{D}^W(\gamma)$ in the $(\mathscr{T}^W(\gamma), \mathscr{D}^W(\gamma))$ plane will be studied as γ evolves continuously within $(1, +\infty)$. Solving $\mathscr{T}^W$ and $\mathscr{D}^W$ with respect to $\left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma - 1)$ yields to the following linear relationship:

$$\mathscr{D}^W = \Delta^W\left(\mathscr{T}^W\right) = \mathscr{S}^W\mathscr{T}^W + \mathscr{D}^W_\infty - \mathscr{S}^W\mathscr{T}^W_\infty \tag{62}$$

where the slope $\mathscr{S}^W$, $\mathscr{D}^W_\infty$ and $\mathscr{T}^W_\infty$ are given

$$\mathscr{S}^W = -\frac{\frac{\tau_{21}\varepsilon^W_{rk}R^W}{\tau_{11}}\left[1+\frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)}{\tau_{31}K^{W*}}\right]}{1+\varepsilon^W_{rk}R^W\left[\frac{\tau_{32}\left(2-\alpha^N-\alpha^S\right)}{2\tau_{11}K^{W*}}+\frac{\tau_{22}}{\tau_{11}}\right]},\ \mathscr{D}^W_\infty = R^W,\ \mathscr{T}^W_\infty = -\frac{1+\frac{\tau_{22}R^W}{\tau_{11}}}{\frac{\tau_{12}}{\tau_{11}}} \tag{63}$$

Let us consider $b^N < 0$, $b^S < 0$, $\nu \leq 1$, $\alpha^N \in (\underline{\alpha}^N, 1/2)$, $\alpha^S \in (\underline{\alpha}^S, 1/2)$, $s^N \in (1/3, 1/2)$ and $s^S \in (1/3, 1/2)$. From equation (61) we have $\mathscr{D}^W_\infty > 1$ and we need to compute the sign of $\mathscr{T}^W_\infty$. Using lemma 3 the numerator of $\mathscr{T}^W_\infty$ is given by

$$1 + \frac{\tau_{22}R^W}{\tau_{11}} = 1 + \frac{T^N_{11}|H^S|A_0+T^S_{22}|H^N|A_1}{T^N_{11}|H^S|+T^S_{11}|H^N|} > 0$$

with

$$A_0 = \frac{\eta^2R^W(1-b^Ns^N)(\nu-1)+(\eta-b^N)(1-s^N)(\nu-1)-R^Wb^N(1-s^N)(b^N(1+\rho)+\eta(\nu-1))}{(\nu-1)(1-b^N)s^N-(1+\rho)(1-s^N)}$$

$$A_1 = \frac{\eta^2R^W(1-b^Ss^S)(\nu-1)+(\eta-b^S)(1-s^S)(\nu-1)-R^Wb^S(1-s^S)(b^S(1+\rho)+\eta(\nu-1))}{(\nu-1)(1-b^S)s^S-(1+\rho)(1-s^S)}$$

while the denominator is given by

$$-\frac{\tau_{21}}{\tau_{11}} = \frac{T^N_{11}|H^S|\left[\frac{\eta(\nu-1)(1-b^Ns^N)-b^N(\nu+\rho)(1-s^N)}{(\nu-1)(1-b^N)s^N-(1+\rho)(1-s^N)}\right]+T^S_{11}|H^N|\left[\frac{\eta(\nu-1)(1-b^Ss^S)-b^S(\nu+\rho)(1-s^S)}{(\nu-1)(1-b^S)s^S-(1+\rho)(1-s^S)}\right]}{T^N_{11}|H^S|+T^S_{11}|H^N|}$$

From the proof of proposition 3 we know that under $\nu \in \left(\underline{\nu}^W \equiv \max\left\{\underline{\nu}^N, \underline{\nu}^S\right\}, 1\right)$, $b^N < b^N_0$ and $b^S < b^S_0$, $-\tau_{21}/\tau_{11} < 0$ and $1 + \tau_{22}R^W/\tau_{11} > 0$ implying that $\mathscr{T}^W_\infty < 0$, where ν^i_0 and b^i_0 are given by

$$\underline{\nu}^i = \frac{s^i\eta}{s^i\eta^i+1-s^i},\ b^i_0 = \frac{\eta^i(\nu^i-1)}{[s^i\eta^i+(1-s^i)]\nu^i-s^i\eta^i+\rho^i(1-s^i)}$$

To determine the precise location of the starting point $(\mathscr{T}_\infty^W, \mathscr{D}_\infty^W)$ we need to compute $\mathscr{P}_\infty^W(1) = 1 - \mathscr{T}_\infty^W + \mathscr{D}_\infty^W$ and $\mathscr{P}_\infty^W(-1) = 1 + \mathscr{T}_\infty^W + \mathscr{D}_\infty^W$. Since $\mathscr{T}_\infty^W < 0$ when $\nu \in (\underline{\nu}^W, 1)$, $b^N < b_0^N$ and $b^S < b_0^S$, we get $\mathscr{P}_\infty^W(1) > 0$. Using equation (63), $\mathscr{P}_\infty^W(-1)$ is given by:

$$\mathscr{P}_\infty^W(-1) = \frac{R^W\left(\frac{\tau_{21}}{\tau_{11}} - \frac{\tau_{22}}{\tau_{11}}\right) + \frac{\tau_{21}}{\tau_{11}} - 1}{\frac{\tau_{21}}{\tau_{11}}} \tag{64}$$

Under $\nu \in (\underline{\nu}^W, 1)$, $b^N < b_0^N$ and $b^S < b_0^S$ we get that the denominator of $\mathscr{P}_\infty^W(-1)$, i.e., $\tau_{21}/\tau_{11} > 0$. We need to study the sign of the numerator of $\mathscr{P}_\infty^W(-1)$. By lemma 3 and the fact that $T_{22}^i = -T_{21}^i\eta - T_{32}^i$, we obtain the numerator of $\mathscr{P}_\infty^W(-1)$:

$$\mathscr{P}_\infty^W(-1) = -1 - \left(\frac{|H^S| T_{11}^N B_0}{\tau_{11}\left[(\nu-1)(1-b^N)s^N - (1+\rho)(1-s^N)\right]} + \frac{|H^N| T_{11}^S B_1}{\tau_{11}\left[(\nu-1)(1-b^S)s^S - (1+\rho)(1-s^S)\right]}\right)$$

with

$$\begin{aligned} B_0 = (\nu-1)&\left[\eta\left(1 + R^W + R^W\eta\right)\left(1 - b^N s^N\right) + \frac{\left(\eta - b^N\right)\left(1 - s^N\right)}{R^N}\right] + \\ b^N\left(1 - s^N\right)&\left[\left(\eta - b^N\right)(1+\rho) - \left(1 + R^W + R^W\eta\right)(\nu+\rho)\right] \end{aligned}$$

$$\begin{aligned} B_1 = (\nu-1)&\left[\eta\left(1 + R^W + R^W\eta\right)\left(1 - b^S s^S\right) + \frac{\left(\eta - b^S\right)\left(1 - s^S\right)}{R^S}\right] + \\ b^S\left(1 - s^S\right)&\left[\left(\eta - b^S\right)(1+\rho) - \left(1 + R^W + R^W\eta\right)(\nu+\rho)\right] \end{aligned}$$

$\mathscr{P}_\infty^W(-1) < 0$ if and only if $b^N < \eta - (1 + R^W + R^W\eta)(\nu+\rho)/(1+\rho) \equiv \underline{b}^N$, $b^S < \eta - (1 + R^W + R^W\eta)(\nu+\rho)/(1+\rho) \equiv \underline{b}^S$ and $\nu \in (\underline{\nu}^W, 1)$. To determine the precise location of the end point $(\mathscr{T}_1^W, \mathscr{D}_1^W)$, it is sufficient to determine that $\Delta^W(\mathscr{T}^W)$ is pointing upward or downward. $\Delta^W(\mathscr{T}^W)$ is pointing downward or upward depending of the sign of $\mathscr{D}^{W\prime}$:

$$\mathscr{D}^{W\prime}(\gamma) = -\frac{R^W\left[1 + \frac{\tau_{31}\left(2 - \alpha^N - \alpha^S\right)}{2\tau_{21}K^{W*}}\right]}{\left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma-1)^2} \tag{65}$$

where

$$1 + \frac{\tau_{31}(2 - \alpha^N - \alpha^S)}{2\tau_{21}K^{W*}} = \frac{C_0 + C_1}{T_{21}^N|H^S| + T_{21}^S|H^N|}$$

with

$$C_0 = \frac{-T_{11}^N|H^S|\left[\eta(\nu-1)(1-b^N)s^N - (\nu+\rho)(1-s^N)[b^N + \chi^N(1-b^N)(1-\alpha^N)]\right]}{(\nu-1)(1-b^N)s^N - (1+\rho)(1-s^N)}$$

$$C_1 = \frac{-T_{11}^S|H^N|\left[\eta(\nu-1)(1-b^S)s^S - (\nu+\rho)(1-s^S)[b^S + \chi^S(1-b^S)(1-\alpha^S)]\right]}{(\nu-1)(1-b^S)s^S - (1+\rho)(1-s^S)}$$

with $\chi^i = \frac{K^{i*}}{K^{W*}}$. Let us define

$$\underline{b}^i = -\frac{\chi^i(1-\alpha^i)}{1-\chi^i(1-\alpha^i)}$$

Under $b^N \in (\underline{b}^N, b_0^N)$, $b^S \in (\underline{b}^S, b_0^S)$ and $\nu \in (\underline{\nu}^W, 1)$ we get $\mathscr{D}^{W\prime}(\gamma) > 0$ and thus $\Delta^W(\mathscr{T}^W)$ is pointing downward. In this configuration local indeterminacy occurs if only if $\mathscr{T}^W > 0$ when $\mathscr{D}^W = -1$. Using the expressions of the trace $\mathscr{T}^W$ and the determinant $\mathscr{D}^W$ allows to show that when $\mathscr{D}^W = -1$, $\mathscr{T}^W > 0$ if and only if

$$1 + \varepsilon_{rk}^W \Lambda^W < 0$$

where

$$\Lambda^W = \frac{R^W\left[\tau_{32}\left(2-\alpha^N-\alpha^S\right)+2\tau_{22}K^{W*}\right]}{2K^{W*}} + \left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma-1)\left[1 + \frac{\tau_{22}R^W}{\tau_{11}}\right]$$

and

$$\left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma-1) = -\frac{R^W\left(1+\frac{\tau_{31}(2-\alpha^N-\alpha^S)}{2\tau_{21}K^{W*}}\right)}{\left(1+R^W\right)}$$

We can write Λ^W as:

$$\begin{aligned}\Lambda^W = \frac{R^W}{1+R^W}[&\underbrace{-\left(1+\frac{\tau_{21}\eta}{\tau_{11}}\right) + \frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)R^W\eta}{2\tau_{11}K^{W*}} + \frac{\tau_{32}\left(2-\alpha^N-\alpha^S\right)R^W}{2\tau_{11}K^{W*}}}_{\equiv\chi_0^W} \\ &\underbrace{-\frac{\tau_{32}}{\tau_{11}}\left[1+\frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)R^W}{2\tau_{21}K^{W*}}\right]}_{\equiv\chi_1^W}\underbrace{-\frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)}{2\tau_{11}K^{W*}} + \frac{\tau_{32}\left(2-\alpha^N-\alpha^S\right)}{2\tau_{11}K^{W*}}}_{\equiv\chi_2^W} \\ &+\underbrace{\frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)}{2\tau_{11}K^{W*}} - \frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)}{2\tau_{21}K^{W*}}}_{\equiv\chi_3^W}]\end{aligned}$$

Applying lemma 3 and lemma 4 we obtain

$$\begin{aligned}\chi_1^W = -\frac{\tau_{31}}{\tau_{11}\tau_{21}}[&\frac{T_{11}^N|H^S|\left[-(\nu-1)(1-b^Ns^N+(\nu+\rho)[a^Ns^N+b^N(\nu+\rho)(1-s^N)\right]}{(\nu-1)(1-b^N)s^N-(1+\rho)(1-s^N)} \\ &+\frac{T_{11}^S|H^N|\left[-(\nu-1)(1-b^Ss^S+(\nu+\rho)[a^Ss^S+b^S(\nu+\rho)(1-s^S)\right]}{(\nu-1)(1-b^S)s^S-(1+\rho)(1-s^S)}]\end{aligned}$$

$$\chi_2^W = -\frac{(2-\alpha^N-\alpha^S)}{2\tau_{11}K^{W*}}\left\{\frac{|H^S|a^N(1-s^N\left[\frac{\nu-1}{R^N}+(1+\rho)b^N-\nu-\rho\right]}{(\nu-1)(1-b^N)s^N-(1+\rho)(1-s^N)} + \frac{|H^N|a^S(1-s^S\left[\frac{\nu-1}{R^S}+(1+\rho)b^S-\nu-\rho\right]}{(\nu-1)(1-b^S)s^S-(1+\rho)(1-s^S)}\right\}$$

$$\begin{aligned}\chi_3^W = \frac{(2-\alpha^N-\alpha^S)\tau_{31}}{2\tau_{11}K^{W*}}[&\frac{|H^S|\left[\eta(\nu-1)(1-b^Ns^N)-b^N(\nu+\rho)(1-s^N)+(\nu-1)(1-b^N)s^N-(1+\rho)(1-s^N)\right]}{\left[(\nu-1)(1-b^N)s^N-(1+\rho)(1-s^N)\right]\left[\eta(\nu-1)(1-b^Ns^N-b^N(\nu+\rho)(1-s^N)\right]} \\ &+\frac{|H^N\left[\eta(\nu-1)(1-b^ss^s)-b^S(\nu+\rho)(1-s^S)+(\nu-1)(1-b^S)s^S-(1+\rho)(1-s^S)\right]}{\left[(\nu-1)(1-b^S)s^S-(1+\rho)(1-s^S)\right]\left[\eta(\nu-1)(1-b^Ss^S-b^S(\nu+\rho)(1-s^S)\right]}]\end{aligned}$$

Since $\tau_{21}/\tau_{11} > 0$ if $b < b_0^N$, $b < b_0^S$ and $\nu \in (\underline{\nu}^W, 1)$, $\tau_{31}/\tau_{11} < 0$ and $\tau_{32}/\tau_{11} < 0$, we obtain $\chi_0^W < 0$. χ_1^W,$\chi_2^W < 0$ if and only if $b^N \in (\underline{b}^N, \overline{b}^N)$, $b^S \in (\underline{b}^S, \overline{b}^S)$ and $\nu \in (\underline{\nu}^W, 1)$. $\chi_3 < 0$ if and only if $b^N \in (\underline{b}^N, \overline{b}^N)$, $b^S \in (\underline{b}^S, \overline{b}^S)$ and $\nu \in (\underline{\nu}^W, 1)$ and $\rho > \underline{\rho}^W$ where $\underline{\rho}^W = \max\left\{s^N(1+\eta)/(1-s^N), s^S(1+\eta)/(1-s^S)\right\}$. We then have $\Lambda^W < 0$. Under $b^N \in (\underline{b}^N, \overline{b}^N)$,

$b^S \in (\underline{b}^S, \overline{b}^S)$, $\nu \in (\underline{\nu}^W, 1)$ we have $\Lambda^W < 0$. It follows that when $\alpha^N \in (\underline{\alpha}^N, 1/2)$, $\alpha^S \in (\underline{\alpha}^S, 1/2)$, $s^N \in (1/3, 1/2)$, $s^S \in (1/3, 1/2)$, $b^N \in (\underline{b}^N, \overline{b}^N)$, $b^S \in (\underline{b}^S, \overline{b}^S)$, $\nu \in (\underline{\nu}^W, 1)$, $\rho > \underline{\rho}^W$ and $\varepsilon_{rk}^W > \underline{\varepsilon}_{rk}^W$, $\mathscr{T}^W > 0$ when $\mathscr{D}^W = -1$. The bound $\underline{\varepsilon}_{rk}^W$ is defined as $\underline{\varepsilon}_{rk}^W = -1/\Lambda^W$. The bifurcation values $\gamma^{W,\mathcal{T}}$ and $\gamma^{W,\mathcal{F}}$ are respectively defined as the solutions of $\mathscr{P}^W(1) = 1 - \mathscr{T}^W + \mathscr{D}^W = 0$ and $\mathscr{P}^W(-1) = 1 + \mathscr{T}^W + \mathscr{D}^W = 0$ and given by:

$$\gamma^{W,\mathcal{T}} = \frac{-1+\varepsilon_{rk}^W\left\{\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)\left[1+\frac{\tau_{22}}{\tau_{11}}R^W+\frac{\tau_{21}}{\tau_{11}}\left(1+R^W\right)\right]-\frac{\tau_{22}}{\tau_{11}}R^W-\frac{R^W\left(2-\alpha^N-\alpha^S\right)}{2\tau_{11}K^{W*}}(\tau_{31}+\tau_{32})\right\}}{\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)\left[1+\frac{\tau_{22}}{\tau_{11}}R^W+\frac{\tau_{21}}{\tau_{11}}\left(1+R^W\right)\right]\varepsilon_{rk}^W}$$

$$\gamma^{W,\mathcal{F}} = \frac{-1+\varepsilon_{rk}^W\left\{\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)\left[1+\frac{\tau_{22}}{\tau_{11}}R^W-\frac{\tau_{21}}{\tau_{11}}\left(1+R^W\right)\right]-\frac{\tau_{22}}{\tau_{11}}R^W+\frac{R^W\left(2-\alpha^N-\alpha^S\right)}{2\tau_{11}K^{W*}}(\tau_{31}-\tau_{32})\right\}}{\left(\alpha^N\Phi^N+\alpha^S\Phi^S\right)\left[1+\frac{\tau_{22}}{\tau_{11}}R^W-\frac{\tau_{21}}{\tau_{11}}\left(1+R^W\right)\right]\varepsilon_{rk}^W}$$

Results follow. ■

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