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Optimal Growth under Flow-Based Collaterals

Daria Onori

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Daria Onori*

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Abstract

Some recent evidence on government finance statistics of European countries suggests that countries with public debt issues also show a low tax revenue-GDP ratio. In this paper we develop a small open economy model of endogenous growth in which the engine of growth is public spending. We assume that government can finance public expenditures by borrowing on imperfect international financial markets where her borrowing capacity is limited. In contrast to the existing literature, where debt is contrained by the stock of capital, the collaterals are based on GDP. The balanced growth path and the transitional dynamics are studied. First, we show that the economy may converge in a finite time to the regime with binding collateral constraint. Second, in such regime the steady state public expenditures-GDP ratio is greater than that of the models without collateral constraints and of the stock-based collaterals literature. Third, the model predictions are consistent with recent empirical literature: there exists a certain treshold of financial and institutional development and economic features that an economy needs to attain in order to benefit from financial liberalization. Finally, if the degree of financial markets imperfections is weak, technologically developed countries experience a higher long-run growth rate than that of the stock-based collaterals literature, otherwise the world interest rate need to be high enough.

Keywords: open economy; two-stage growth; public debt; flow collaterals; imperfect financial markets; multi-stage optimal control.
JEL Classification: CO2, C61, F34, F43, H63, O4, O16.

*Aix-Marseille University (Aix-Marseille School of Economics), CNRS & EHESS; Université catholique de Louvain, IRES; University of Rome "La Sapienza", Departement of Economics and Law, Faculty of Economics.

1 Introduction

The causes of the 2009-2010 European sovereign debt crisis are not just a matter of level of public debt, but a combination of different factors: the globalization of finance, easy credit conditions, the 2007-2012 global financial crisis, international trade imbalances, real-estate bubbles, the 2008-2012 global recession, fiscal policy choices related to government revenues and expenditures. The countries mainly involved by the crisis are Greece, Portugal, Ireland and Spain.[1] All of them have both debt and deficit issues. The worst situation is Greece. Among the causes of the crisis highlighted by the Greek Ministry of Finance in January 2010 there are low GDP growth rates, mainly due to the lack of competitiveness and to public spending directed to non-growth sectors; high government deficits, from 2004 to 2009 the nominal output has increased of 40%, primary expenditures have increased by 87% as compared to an increase of only 31% of tax revenues, which is below the expected level. Other outstanding problems are government debt level and the political budget compliance.

The countries' current account during the crisis, together with the fact that monetary (and currency) policies are not available to the individual states, contribute to explain the debt crisis in the Euro area. As the unique policy instrument left in each country's hands is the fiscal policy, we assume in the paper that this is the unique way to control the evolution of the level of public debt. We develop a small open economy model of endogenous growth in which government can finance productive public expenditures by levying income taxes and by borrowing on imperfect international financial markets. Foreign debt is constrained, as international lenders can seize only a fraction of collaterals in case of debt repudiation and the engine of growth is public spending.

The literature on endogenous growth, collateral constraint and financial markets is very limited and assumes that a country's debt is constrained by physical capital. Eaton and Gersovitz (1981) assume that borrowers who repudiate their debt face future exclusion from capital markets and lenders will establish a credit ceiling above which they will be unwilling to increase loans. If the ceiling is below the amount desired by the borrower then the borrower is rationed. Cohen and Sachs (1986) were the first to introduce a collateral constraint into an open economy growth model. The production function is assumed to be linear in physical capital and they find that the economy experiences a two-stage growth pattern. In the first stage of unconstrained borrowing debt grows more rapidly than the economy and the debt-GDP ratio rises. In the second stage borrowing is constrained and debt and GDP grow at the same rate and more slowly than in the first stage. Boucekkine and Pintus (2012) introduce

[1]Italy also has a high debt but Italy's budget position is better than the "crisis" economies. Moreover, despite Italian debt over GDP ratio increased, public debt is characterized by a long maturity and it is mostly held domestically.

imperfect contract enforcement in an open economy AK-model: debt contract has to depend upon past values of capital stock. The model can explain leapfrogging and growth reversals. They find that a history effect, due to the fact that the richer a country has been in the past the more it can borrow in international financial markets, may offset the growth-enhacing effect of foreign debt. Leapfrogging occurs for small delays when the growth effect dominates the history effect. Finally, the absence of leapfrogging implies growth reversals: the delay is so large that the history effect now dominates the growth effect.
However, the use of physical capital as a collateral has some important limits. First, it is difficult for creditors to seize the pledged capital because its value must be estimated and can decrease over time. Second, publicly or privatley owned capital usually take the form of physical assets (such as portions of territitory, monuments, infrastructures, etc.) which are of difficult appropriability for foreign creditors; this makes the use of public capital as a collateral unrealistic. Actually, the unique resource that a government can use as a viable collateral is its ability to raise revenues via taxation. Hence, the country's borrowing capacity should be related to the amount of obtainable fiscal revenues.
The analysis of recent government finance statistics of European countries underlines that countries with high debt problems also show the lowest tax revenues.
Tax revenue constitutes about 91% of total government revenue of total government revenue in the European Union. From 2007 to 2010 it has declined and now it has reached the lowest value of the period 1995-2010. In 2010 tax revenue in terms of GDP was the highest in Denmark, Belgium and Sweden (Figure 1.1) while, among the countries which joined the European Union before 2004, it was the lowest for Ireland (29.8%), Spain (32.9%), Greece (33.2%) and Portugal (34.8%). The drop of the tax revenue in terms of GDP reflects the economic and financial crisis.

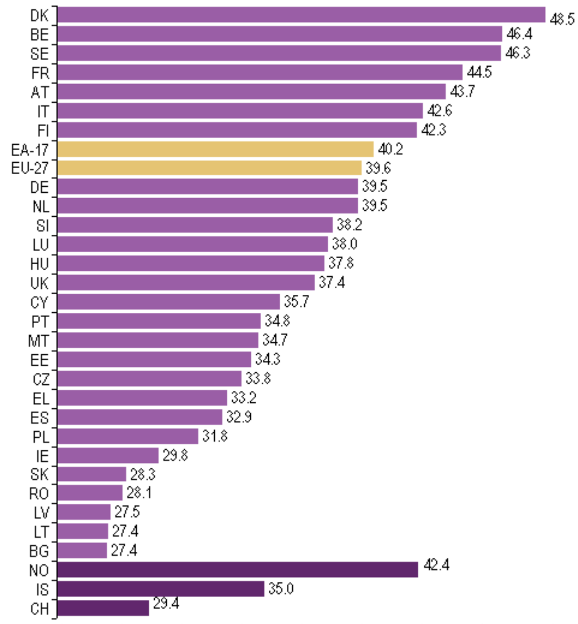

Figure 1.1: Ranking of total tax revenue as a % of GDP by Member States and EFTA countries, 2010 (Source: Eurostat)

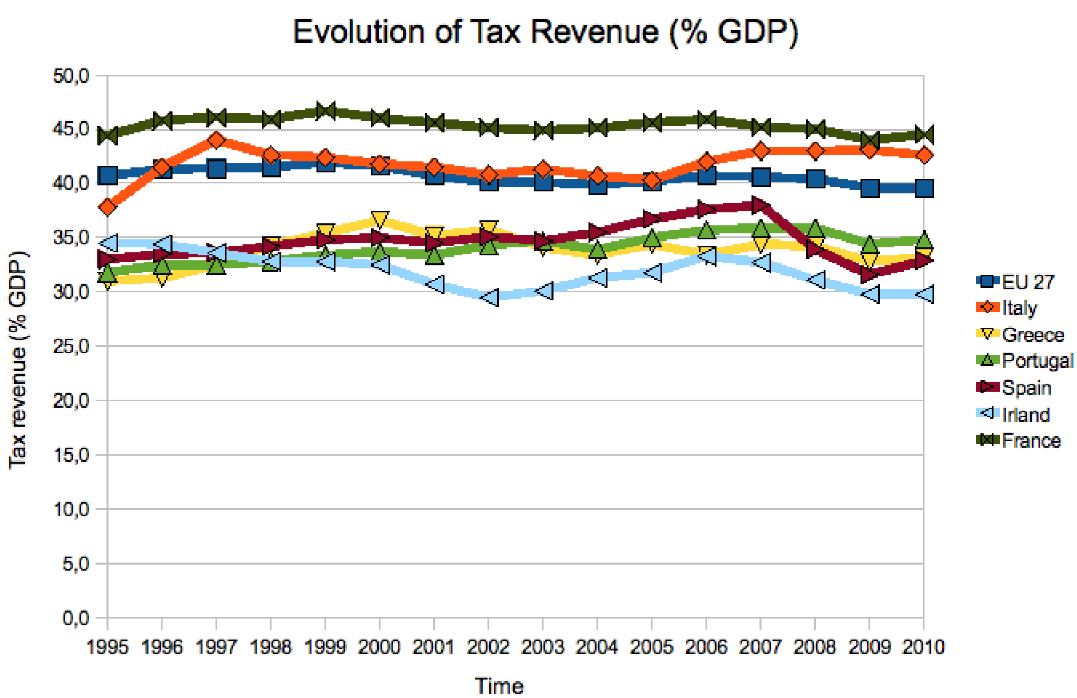

Figure 1.2: Evolution of Tax Revenue (Source: Eurostat)

Figure 1.2 shows the evolution of tax revenue-GDP ratio of the above mentioned countries (Greece, Portugal, Spain and Ireland). For sake of comparison we plotted also the average of the European Union, France, a country whose the tax revenue-GDP ratio is above the European average and whose budget position is not worrying, and Italy. Portugal, Spain and Greece have shown a similar and a low tax revenue which was substantially stable in the period before crisis. Then it dropped. Ireland registered a higher tax revenue at the beginning of the period but since then the ratio fell, staying at the lowest value among the sample of considered countries. Italy started below the European average, but subsequentely it hovered above the average so that this is the unique country whose tax revenue in terms of GDP has increased during the crisis period. France has stayed always above the average, showing a stable enough tax revenue. Figure 1.3 shows the evolution of public deficit for the period 1995-2011. In 2012 the largest deficit was registered by Irland, Portugal, Spain, Greece and in 2011 by Irland, Greece and Spain. By 2007 deficit has become a serious problem for the considered countries.

For the debt/GDP, the highest increases between 2007 and 2011 has been registered for Irland, Greece, Portugal and Spain. Figure 1.4 shows its evolution for the period 1995-2011. The dramatically increasing trend starts in 2007, as a result of economic and financial crisis.[2]

[2]Figure 1.4 shows the reason why debt positon of Italy is not dramatic: the evolution of debt is substantially stable.

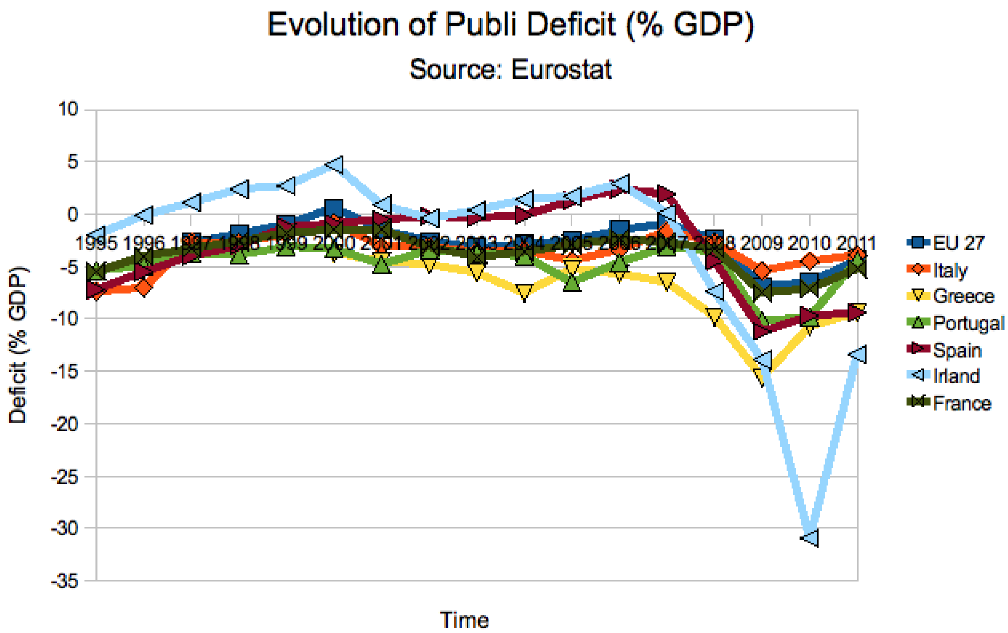

Figure 1.3: Evolution of Public Deficit (Source: Eurostat)

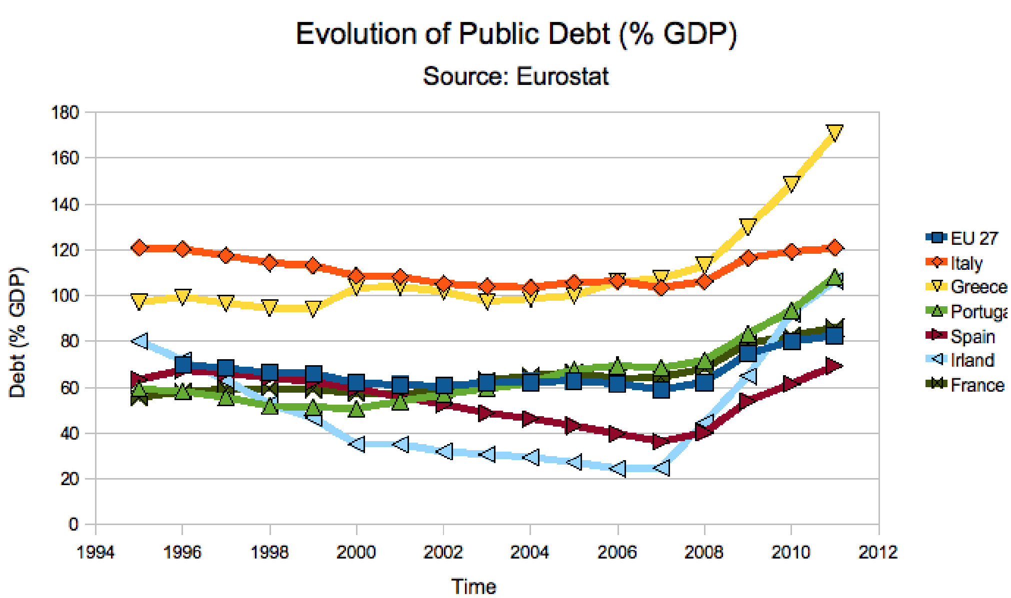

Figure 1.4: Evolution of Public Debt (Source: Eurostat)

Sovereign debt is particularly relevant for the issue of debt sustenability. In response to the economic and financial crisis governments increased the gross external debt. Figure 1.5 shows general government external debt for the period 2002-2011:

	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Ireland	:	:	58,7	61,5	62,2	63,1	73,2	59,8	52,4	60,9
Spain	43,3	38,8	47,3	48,1	50,7	47,1	46,3	46,3	43,8	38,6
Portugal	53,2	56,6	56,4	72,7	73,6	75,0	76,3	75,1	63,9	66,7
Italy	33,2	36,9	34,9	38,2	39,3	39,0	41,2	42,2	44,0	38,5
France	50,9	:	46,0	52,8	53,8	53,0	55,1	55,7	57,6	56,7

Figure 1.5: General government external debt (% of total). Source: Eurostat

Data on Greek sovereign debt are not available. Greece is in a very particular situation, given its economic and productive weakness and given that its total public debt is already very high. Its public external debt/GDP percentage was of 78.2% in 2007, 80.3% in 2008 and 91.4% in 2009 (Dias 2011).

The analysis of budget position of European countries highlights that countries with the lowest tax revenue experience the most serious problems of public debt: countries which have registered the most dramatic upward trend of debt in terms of GDP have had at the same time the lowest and decreasing tax revenue-GDP ratio.

These facts confirm that a government should use tax revenue to control the level of public debt: the higher the latter the greater is the borrowing capacity of a country.

We assume that public expenditures are the engine of growth. This can be justified by recalling that, as previously discussed, the unique control instrument available to European countries is fiscal policy, and by noticing that recently many critics of the Stability and Growth Pact have stressed the relevance of productive public goods for growth, hence suggesting that they should be removed from deficit computation.

The first model with productive public expenditures is Barro's (1990) seminal paper. Here public expenditures, which are financed by levying income taxes (budget balance assumption), increase the production of the economy, so that they are the engine of growth. The small open economy version is developed by Turnovsky (2009) in a model where a country can borrow without any limit from the rest of the world. We extend these two models to the case where the governement can finance public expenditures by taxation and by borrowing to the international financial markets and we assume that public debt is bounded by tax revenue: the higher the latter the higher is the borrowing capacity of the country. International financial markets are imperfect in the sense that creditors can just seize a fraction of the collaterals in the case of default.

The paper constitutes a novelty also from a technical point of view. In order to verify if this stock will meet the constraint and to determine endogenously the date at which it happens, we solve the model by using the multi-stage optimal control technique in which the standard optimal control problem is constructed as a dynamic programming problem, allowing to find the instant at which the economy changes the regime. Typical optimal switching problems that can be found in the literature involves regimes which are characterized by different technological or ecological states. Each regime is defined as a certain set of parameters values (see Tomiyama 1985 and Boucekkine et al. 2004). A more recent branch of the literature uses this technique to analyze the problem of irreversibility: when the state variable reaches a certain treshold, the economy switches to a different regime in which the parameters of the economy are different (see Boucekkine et al. 2013a, Boucekkine et al. 2013b, Prieur et al. 2011). In this paper we encompass another type of optimal switching time problem. The fact that at the initial date the constraint is not binding and then, if switching is optimal, it becomes binding modifies one of the state equation, thus definying two possible states of the economy.

The paper contains a number of new results and broad policy implications. First of all, the results of the regime where the constraint is not binding (non binding regime) are the same as the Barro-Turnovsky' model: the optimal public expenditures-GDP ratio is equal to the income share of public expenditures and the growth rate of consumption is equal to the difference between the exogenous world interest rate and the discount rate. Then, under certain conditions, namely if the growth rate of consumpion in the unconstrained regime is small enough, the economy will switch to the regime in which the constraint is binding. We show that the economy admits a balanced growth path (BGP) which is not immediately reached because of the presence of investment adjustment costs and we determine analitically the conditions for the BGP's stability. This state of the economy shows some important differences with respect to the Barro-Turnovsky and the stock-based literature's model: the public expenditures-GDP ratio is greater than the income share of public expenditures. This means that the country will use public debt also to finance the public expenditures-GDP ratio. This happens because a higher investment in public goods relaxes the borrowing constraint. We also show that if financial markets imperfections are not too weak, countries with a high TFP experience higher growth rates under the flow-based collateral assumption than under stock-based collaterals. Otherwise the international interest rate must be sufficiently high. When instead the TFP is low the flow-based growth rate is always smaller that the stock-based growth rate.
We finally find the endogenous switching date. This is a novelty in the literature: the above cited papers can determine the endogenous switching date since all the economy's regimes

admit closed form solutions, hence the date can be determined by replacing them in the matching conditions between the regimes. Our binding dynamic system is non linear, hence it cannot be solved in closed form. Furthermore, we are able to give analytical conditions for corner solutions and to our knowledge we are the first ones who simulated the inter-regime transition. First, we find that if there are no adjustment costs of capital, the economy immediately jumps to the regime where the constraint is binding. In other words, the presence of investment adjustment costs makes the inter-regime transition slowlier. Second, if the growth rate of consumption of the non binding regime is high, the economy will never converge to the binding regime, hence the date will be infinity. This translates into some treshold conditions on the relevant parameters of the model that the economy needs to meet in order to benefit from financial openess. In particular, if the economy is not financially and economically developed enough, i.e., if the international interest rate is too high, the installation costs parameter is too large and the productivity parameter is too low, the country will stay forever in autarky. Hence the model predictions are consistent with the recent empirical literature on financial integration. This literature shows that the benefits that a country can obtain from financial globalization depend on the country's situation. Kose et al. (2009) show that there exists a certain treshold of financial and institutional development and economic characteristics that an economy needs to attain in order to benefit from financial liberalization. Similary, Aizenman et al. (2011) show that economy's growth is lower in countries with high levels of past debt and weak institutions. We also performed a comparative statics analysis of the effect of the relevant economy's parameters on the switching date. This date turns out to be an increasing function of the degree of financial openess, the international interest rate, the adjustment costs; the switching date is also a decreasing function of the productivity parameter.[3]

The paper is structured as follow. In Section 2 we review Barro's (1990) and Turnovsky's (2009) models. In Section 3 we present and solve our model and we give the conditions for the switch from the non-binding to the binding regime. Section 4 compares our model with the literature. Section 5 analyzes the balanced growth path and Section 6 discusses the determination of the optimal switching date. Finally, Section 7 concludes. Technical details are contained in the Appendix.

[3]Aizenman et al. (2007) build a similar model to ours: countries are characterized by limited tax and debt capacity. However they do not tackle the problem of collaterals and imperfect financial markets. Furthermore, the results and the technical analysis are different. First, the economy is closed. Second, contrary to what we obtained in our paper, borrowing to finance public good is not efficient. Third, they do not take into account any problem of endogenous switching.

2 The literature

As we anticipated in the introduction, in this paper we extend Barro's (1990) model by assuming that government can finance public expenditures not only by taxation but also by borrowing on international financial markets. Turnovsky (2009, Ch. 2) has already extended some simple models of endogenous growth to the small open economy case, where the country can borrow or lend in the international financial markets. In Section 2.2 he developed Romer's (1986) model of learning by doing in a decentralized economy, while in Section 2.4 he modified Barro's (1990) model to the case of public good congestion. In this review, for the sake of comparison, we adapt the model of Section 2.2 by assuming that the engine of endogenous growth is productive public expenditures, as in Barro's paper, and we solve for the optimal policy case. The results in terms of model's properties do not change. In our subsequent analysis we start from this model and we assume, in addition, that international financial markets are imperfect: the representative country faces credit constraints. Moreover, public debt is bounded by fiscal revenue.

Hence it turns out to be useful to discuss Barro's (1990) and Turnovsky's (2009) models. This is the aim of the next two sections.

2.1 Barro's model

We consider a centralized economy. Government's purchases of goods and services, G, enter the production function as pure public goods. The production function is Cobb-Douglas:

$$Y = AG^{\alpha}K^{1-\alpha}, A > 0, 0 < \alpha < 1 \tag{2.1}$$

where K is the firm's stock of physical capital, A is the technology parameter and G is the flow of productive government expenditures. α represents the income share of public services. For a given G the economy faces diminishing returns to scale in the stock of capital and constant returns to scale in G and K. We normalize population to one troughout the paper: labor is supplied inelastically and population growth rate is exogenous. Capital can be interpreted in a broad sense by including human capital.

Government finances public expenditures by levying income taxes. Its budget constraint is

$$G = \tau Y, 0 < \tau < 1 \tag{2.2}$$

By assuming that the agent's intertemporal utility is given by $U(C) = \int_0^{+\infty} \ln Ce^{-\rho t}dt$, where $0 < \rho < 1$ is the discount factor, the benevolent planner maximizes the stream of consumption with respect to consumption and public expenditures subject to the economy aggregate resource constraint $\dot{K} = AG^{\alpha}K^{1-\alpha} - C - G$. We assume that capital does not depreciate for the sake of simplicity. We will mantain this assumption throughout the paper, but the results do not change if the depreciacion rate is taken positive. The necessary condition with respect to public goods gives the efficiency condition

$$\frac{\partial Y}{\partial G} = 1$$

which states that at the optimum the marginal benefit of increasing G of one unit must be equal to the unitary social marginal cost. This gives the optimal public expenditures-GPD ratio:

$$\left(\frac{G}{Y}\right)^{\text{Barro}} = \alpha$$

The growth rate of the centralized economy turns out to be equal to

$$g^{\text{Barro}} = \left[(1-\alpha)\,A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} - \rho\right]$$

2.2 Turnovsky's model

The production function is the same as in Barro's model. Here the economy is small and open, hence the interest rate is exogenous. We shall call it $\hat{r}$. The economy has unconstrained access to international financial markets. In order to be consistent with our model we assume that the country is a net borrower on world capital markets.[4] We denote by B the country's debt, so that the economy's aggregate resource constraint (i. e. the economy's current account) is given by

$$\dot{B} = C + I + G + \hat{r}B + \frac{h}{2}\frac{I^2}{K} - Y \tag{2.3}$$

where K is capital and I is investment. Capital accumulation involves convex investment adjustment costs of the form $\Phi(I, K) = I\left(1 + \frac{h}{2}\frac{I}{K}\right), h > 0$. This is the traditional Hayashi's (1982) formulation, where the adjustment costs is proportional to the rate of investment in terms of units of installed capital. The presence of investment adjustment costs is a necessary assumption of small open economy models for two reasons. First, in a world where the country has unlimited access to international financial market, hence it is not constrained by internal savings, the investment rate would be infinity in the absence of adjustment costs (see Turnovsky 2002). Second, without adjustment costs, this model would show the strong knife-edge condition that the internal marginal productivity of capital would be constrained to be equal to the exogenous interest rate (see Appendix A). Finally we assume that capital

[4] Turnovsky (2009) assumes that the country may accumulate foreign bonds. This difference does not affect the properties of the model.

does not depreciate so that

$$\dot{K} = I \tag{2.4}$$

The planner has to choose consumption, public expenditures, investment, debt, capital to maximize intertemporal utility subject to (2.3) and (2.4). The optimality conditions with respect to C, I and G are

$$\frac{1}{C}e^{-\rho t} + \lambda_1 = 0 \tag{2.5}$$

$$1 + h\frac{I}{K} = q \tag{2.6}$$

$$\frac{G}{Y} = \alpha \tag{2.7}$$

Equation (2.7) is the same as Barro's (1990) model. $q \equiv -\frac{\lambda_2}{\lambda_1}$ is the market value of capital in terms of the price of bonds.[5] Hence condition (2.6) allows us to write the growth rate of capital as

$$\frac{\dot{K}}{K} = \frac{q-1}{h}$$

Hence capital will increase/decrease according whether $q \gtrless 1$. Optimization with respect to B and K gives

$$\lambda_1 \hat{r} = -\dot{\lambda}_1 \tag{2.8}$$

which equates the marginal value of consumption to the exogenous costs of debt, and

$$\frac{\dot{q}}{q} + \frac{(q-1)^2}{2hq} + (1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \frac{1}{q} = \hat{r} \tag{2.9}$$

which equates the return of domestic capital to the exogenous cost of bonds.

Finally the transversality conditions must hold

$$\lim_{t \to +\infty} \lambda_1 B = 0, \quad \lim_{t \to +\infty} \lambda_2 K = 0.$$

Equations (2.5) and (2.8) determine the growth rate of consumption

$$\frac{\dot{C}}{C} = \hat{r} - \rho = \Psi \tag{2.10}$$

which is given by the difference between the international interest rate and the discount rate. For the endogenous model to make sense, $\hat{r} > \rho$ must be assumed. By solving this equation we obtain the level of consumption: $C(t) = C(0)e^{\Psi t}$, where the initial level of consumption

[5] λ_1 and λ_2 are the two co-state variables of the optimization problem.

C(0) must be determined.

The growth rate of capital is determined by q, whose dynamics is given by (2.9). By setting $\dot{q} = 0$ we obtain two solutions

$$q_1 = (1 + \hat{r}h) - \sqrt{(1 + \hat{r}h)^2 - \left[1 + 2h(1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}\right]}$$

$$q_2 = (1 + \hat{r}h) + \sqrt{(1 + \hat{r}h)^2 - \left[1 + 2h(1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}\right]}$$

A necessary and sufficient condition for the capital to converge to its path of steady growth requires that these solutions must be real, which happens when $\hat{r}\left[1 + \frac{\hat{r}h}{2}\right] \geq (1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$. The interpretation is related to the presence of adjustment costs: the smaller is h, the smaller the marginal value of capital $(1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$ must be in order to guarantee the existence of a balanced growth path. A smaller h implies a greater rate of return to capital due to the difference between the valuation of new capital and the value of the resources it utilizes, per unit of installed capital, which is given by the term $\frac{(q-1)^2}{2hq} = \frac{(qI - \Phi)}{qK}$ in the arbitrage equation (2.9). This has to be consistent with the term $(1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \frac{1}{q}$ of equation (2.9), which depends on the domestic return to capital.

The smaller equilibrium value, q_1, is an unstable root, while q_2 is stable. However any stable arm converging to q_2 violates the transversality condition $\lim_{t\to\infty} \lambda_2(t) K(t) \iff \lim_{t\to\infty} -q\lambda_1(t) K(t)$, where $K(t) = K(0) e^{\int_0^t \frac{q(s)-1}{h} ds}$ and $\lambda_1(t) = \lambda_1(0) e^{-\hat{r}t}$. In fact the stable adjustment path is $q(t) = q_2 + (q(0) - q_2) e^{\mu t}$, where $\mu < 0$ is the stable eigenvalue . Hence $\int_0^t \frac{q(s)-1}{h} ds = \frac{q_2 - 1}{h} t + \frac{q(0) - q_2}{\mu} (e^{\mu t} - 1)$. By substituting this expression into the transversality condition we verify that it is violated. In the case of the unstable root q_1, there is no transitional dynamics and the transversality condition is satisfied for $q = q_1$. Hence q is always at its steady state value q_1. This implies that capital is always on its balanced growth path, with a growth rate given by

$$\Lambda = \frac{q_1 - 1}{h} \tag{2.11}$$

By substituting (2.10), (2.11) and q_1 into the economy's current account, we can solve for the stock of debt:

$$\begin{aligned} B(t) = & -\tfrac{C(0)}{\rho} e^{(\hat{r}-\rho)t} - \theta K(0) e^{\frac{q_1-1}{h} t} \\ & + \left(B(0) + \tfrac{C(0)}{\rho} + \theta K(0)\right) e^{-\hat{r}t} \end{aligned}$$

where $\theta \equiv \frac{\left[\frac{q_1-1}{h} + A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} + \frac{(q_1-1)^2}{2h} - A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}\right]}{\left(\hat{r} - \frac{q_1-1}{h}\right)}$. In order to ensure national intertemporal solvency, the transversality condition $\lim_{t\to\infty} \lambda_1(t) B(t) = \lim_{t\to\infty} \lambda_1(0) e^{-\hat{r}t} B(t) = 0$ must be satisfied and this hold if and only if

$$C(0) = -\rho(B(0) + \theta K(0))$$

If $C(0) > 0$, it must be $B(0) + \theta K(0) < 0$. Given the definition of θ and given that the transversality condition implies $\hat{r} > \frac{q_1 - 1}{h}$, a necessary condition is $\frac{q_1-1}{h} + A^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}} + \frac{(q_1-1)^2}{2h} - A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} < 0$.

The equilibrium stock of public debt is hence given by

$$B(t) = \ (B(0) + \theta K(0))\, e^{(\hat{r}-\rho)t} - \theta K(0)\, e^{\frac{q_1-1}{h}t} \tag{2.12}$$

Turnovsky's model has some important implications. First, the small open economy economy implies that the growth rate of domestic consumption can be different from the growth rate of domestic output (capital), which is in sharp contrast with the closed economy version of the model, where all the variables grow at the same rate. Second, the growth rate of the stock of debt varies over time and asymptotically converges to $[\Lambda, \Psi]$. In particular if $\Lambda < \Psi$, $\lim_{t\to+\infty} B(t) = -\infty$ and the country would accumulate assets. On the contrary, if $\Lambda > \Psi$, $\lim_{t\to+\infty} B(t) = +\infty$ the country would accumulate a forever increasing debt. Finally, the fact that the economy can borrow from the rest of the world does not influence the part of domestic output invested on public expenditures, which is equal to the income share of public goods, α, as in the Barro's (1990) model.

3 The model

In this section we describe our economy. The production function is still given by (2.1).

The individual agent's budget constraint includes convex investment installation costs:

$$\dot{K} = (1-\tau)\, Y - \frac{h}{2}\frac{I^2}{K} - C$$

where $0 < \tau < 1$ is the output tax rate.

The governement sets taxes and expenditures accordingly with its budget constraint

$$\dot{B} + \tau Y = G + \hat{r} B \tag{3.1}$$

We assume here that the governement is a net borrower on international financial markets, hence $B \geq 0$.

We assume that the debt is bounded by tax revenue, hence public debt is constrained by

$$0 \leq B \leq \lambda \tau Y = \phi Y \tag{3.2}$$

where λ represents the credit multiplier. The $\lambda = 0$ case corresponds to autarky,[6] while $\lambda > 0$ depicts international finacial integration. If $\lambda < 1$ we are in presence of credit market frictions because international lenders can just seize a fraction of collaterals in case of debt repudiation. We shall concentrate on this last case. The parameter ϕ is the product between the degree of financial imperfection and the tax rate. Hence $\phi < 1$.

Finally the economy's resource constraint (the economy's current account) can be espressed as[7]

$$\dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - Y \tag{3.3}$$

which states the economy accumulates debt to finance its total expenditures on consumption, public purchases, private capital, interest payments net of produced output.[8]

Finally the intertemporal utility is the same as the one assumed in Barro-Turnovsky's model and we assume no depreciation of capital, hence $\dot{K} = I$.

The social planner must choose the level of consumption, public expenditures, investment, the stock of public debt and capital which solve the problem

$$\begin{array}{ll} \max_{\{C,G,I\}} & U(C) = \int_0^{+\infty} \ln C e^{-\rho t} dt \\ \text{s. t.} & \dot{K} = I \\ & \dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - AG^{\alpha}K^{1-\alpha} \\ & B \leq \phi Y, B \geq 0, \forall t \\ & K(0), B(0) \text{ given} \end{array}$$

The conditions (3.2) define three economic regimes. First, the stock of public debt B can be equal to zero (autarky case) or positive. In the latter case the constraint (3.2) can be non binding or binding. The identification of the optimal autarky cases is quite straightforward. We will rather focus on the two other regimes.

[6] In such a case the constraint (3.2) is binding.

[7] In order to obtain this espression we aggregated the consumer's budget constraint $\dot{K} = (1-\tau)Y - \frac{h}{2}\frac{I^2}{K} - C$ and combined it with the governement's budget constraint.

[8] This is equal to Turnovsky's (2009) resource constraint.

In order to find endogenously the date at which the constraint becomes binding, we solve the model by using the two-stage optimal control technique. As stated in the Introduction, our paper constitutes a novelty from a technical point of view. Typical optimal switching time problems that can be found in the literature involve technological or ecological states of the world that are exogenously given. For example Tomiyama (1985) and Boucekkine et al. (2004) index the successive regimes by a finite number of discrete parameter values: a newly available technological regime may exhibit a higher productivity parameter. A more recent branch of the literature applies the optimal switching problem to the analysis of irreversibilities (see Boucekkine et al. 2013a, Boucekkine et al. 2013b, Prieur et al. 2011): for example when the state variable (the stock of pollution) exceeds a certain treshold value, the natural decay rate goes down permanently. One can reformulate this as an optimal timing problem because the date at which the stock variable reaches the treshold becomes a control variables.

In this paper we tackle a different type of optimal regime switching problems. The economy starts from a situation in which the stock of public debt is such that the collateral constraint is not binding and it becomes binding at a certain date (this is proved in Proposition 2) which has to be optimally determined. This modifies one of the state equations. For this reason the problem can be set as an optimal switching time problem.[9] The two possible states of the economy can be summarized as follows

$$\begin{cases} \dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - AG^{\alpha}K^{1-\alpha} & \text{if } B < \phi Y \quad t \in [0, T] \\ \\ \dot{B} = C + I + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \frac{h}{2}\frac{I^2}{K} - \frac{1}{\phi}B & \text{if } B = \phi Y \quad t \in (T, +\infty) \end{cases} \tag{3.4}$$

Hence the problem can be reformulated as follows

$$\max_{\{C,G,T\}} \quad \int_0^{+\infty} \ln C e^{-\rho t} dt$$

$$\text{s. t.} \qquad \dot{K} = I$$

$$(3.4)$$

where $K(0)$ and $B(0)$ are given.

A natural approach is to decompose the problem into two sub-problems, for a given time

[9] Of course the problem can be solved by using the traditional method of Lagrange, but this does not allow to determine the endogenous switching time. This methodology is contained in Appendix C.

variable, to solve each of them, and then to identify the optimal timing T. In our case since

$$V(K(0), B(0)) = \int_0^T \ln Ce^{-\rho t}dt + \int_T^{+\infty} \ln Ce^{-\rho t}dt$$

one can view the problem sequentially starting from the second regime, for $t \in (T, +\infty)$ and then solving for the first regime. In the binding regime the problem becomes:

$$\begin{array}{ll} \max_{\{C,I\}} & V_2 = \int_T^{+\infty} \ln Ce^{-\rho t}dt \\ \text{s. t.} & \dot{K} = I \\ & \dot{B} = C + I + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \frac{h}{2}\frac{I^2}{K} - \frac{1}{\phi}B \\ & \text{for a given } T, K(T), B(T) \end{array} \tag{3.5}$$

The corresponding hamiltonian is

$$H^{(2)} = \ln Ce^{-\rho t} + \lambda_1^{(2)} \left[C + I + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \frac{h}{2}\frac{I^2}{K} - \frac{1}{\phi}B\right] + \lambda_2^{(2)} I$$

where the upper-script "(2)" means that the constraint binds, $\lambda_1^{(2)}$ is the co-state variable associated to public debt and $\lambda_2^{(2)}$ is the co-state variable associated to capital when the economy is in the second regime.
Then we solve

$$\begin{array}{ll} \max_{\{C,G,I\}} & V_1 = \int_0^T \ln Ce^{-\rho t}dt + V_2^*(T, B(T), K(T)) \\ \text{s. t.} & \dot{K} = I \\ & \dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - AG^{\alpha}K^{1-\alpha} \\ & K(0), B(0) \text{ given, } T \text{ free} \end{array} \tag{3.6}$$

and the corresponding hamiltonian is

$$H^{(1)} = \ln Ce^{-\rho t} + \lambda_1^{(1)} \left[C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - AG^{\alpha}K^{1-\alpha}\right] + \lambda_2^{(1)} I$$

where the upper-script "$^{(1)}$" means that the regime is non binding.

This recursive scheme works exactly as a dynamic programming problem: the Bellman principle applies in the two sub-interval involved by the timing problem instead of discrete periods of time.

We will discuss the Pontryagin optimality condition in the subsequent sections.[10] Here we shall focus on the optimality conditions with respect to the timing variable T. They are summarized by the following theorem:

Theorem 1. *Let* $0 < T < +\infty$ *the optimal timing. Then*

$$\lambda_1^{(1)}(T) = \lambda_1^{(2)}(T) \tag{3.7}$$

$$\lambda_2^{(1)}(T) = \lambda_2^{(2)}(T) \tag{3.8}$$

$$H^{(1)}(T) = H^{(2)}(T) \tag{3.9}$$

A detailed proof is contained in Appendix B. Some comments are in order here. First, the above conditions can be read as continuity or matching conditions at the junction point: condition (3.9) imposes the continuity of the hamiltonian at the optimal junction time, (3.7) and (3.8) the continuity of the co-state variables. Here both the co-state variables are continuous because the state variables do not jump at the switching time: we did not impose any treshold hence they do not have to satisfy any pure state constraint and both the state variables can be freely chosen. Second, one can interpret the matching conditions (3.7)-(3.9) as first order optimal timing conditions with respect to the switching date T. In fact condition (3.9) can be rewritten as $H^{(1)}(T) - H^{(2)}(T) = 0$ and it can be interpreted as the marginal gain from extending the first regime at the expense for the regime associated to the next interval. Since there is no direct switching cost, the marginal cost is zero. Hence this condition equalizes the marginal benefit to the marginal cost of delaying the switching time.

3.1 Preliminary analysis of economic regimes

In this section we will solve the model by starting with the Pontryagin problem of the binding regime and the by proceeding backward to the non binding case.

3.1.1 The binding regime

For a given T, $B(T)$ and $K(T)$, the Pontryagin necessary conditions of problem (3.5) are

[10] The problem is concave hence each the two sub-intervals problems is well behaved.

$$\frac{1}{C}e^{-\rho t} + \lambda_1 = 0 \tag{3.10}$$

$$\lambda_1 \left[1 + h\frac{I}{K}\right] + \lambda_2 = 0 \tag{3.11}$$

$$\lambda_1 \left[\frac{1}{\alpha}\left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K}\right)^{\frac{1-\alpha}{\alpha}} + \hat{r} - \frac{1}{\phi}\right] = -\dot{\lambda}_1 \tag{3.12}$$

$$-\lambda_1 \left[\frac{h}{2}\frac{I^2}{K^2} + \frac{1-\alpha}{\alpha}\left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K}\right)^{\frac{1}{\alpha}}\right] = -\dot{\lambda}_2 \tag{3.13}$$

$$\lim_{t\to\infty} \lambda_1 B = 0 \qquad \lim_{t\to\infty} \lambda_2 K = 0$$

By using the shadow value of public debt λ_1 as the numeraire and defining $q = -\frac{\lambda_2}{\lambda_1}$, we find the growth rate of capital $\frac{I}{K} = \frac{q-1}{h}$ from equation (3.11) and by combining it with (3.12) and (3.13) we obtain the arbitrage equation

$$\frac{\dot{q}}{q} = \frac{1}{\alpha}\left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K}\right)^{\frac{1-\alpha}{\alpha}} + \hat{r} - \frac{1}{\phi} - \frac{1}{q}\frac{1-\alpha}{\alpha}\left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K}\right)^{\frac{1}{\alpha}} - \frac{(q-1)^2}{2hq}$$

We find it convenient to rewrite the arbitrage equation as a function of $\left(\frac{G}{Y}\right)$ in order to compare the results with Barro's (1990) and Turnovsky's (2009) models. From the constraint we know that the ratio $\left(\frac{B}{K}\right)$ is linked to the ratio $\left(\frac{G}{K}\right)$ in the following way: $\frac{B}{K} = \phi A \left(\frac{G}{K}\right)^{\alpha}$. Moreover, by using the production function $\frac{G}{K} = A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}}$. Hence the arbitrage equation becomes

$$\frac{\dot{q}}{q} = \frac{1}{\alpha}\left(\frac{1}{\phi}\right)\left(\frac{G}{Y}\right) + \hat{r} - \frac{1}{\phi} - \frac{1}{q}\frac{1-\alpha}{\alpha}A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} - \frac{(q-1)^2}{2hq} \tag{3.14}$$

Log-differentiating equation (3.10) and using equation (3.12) we derive the growth rate of consumption as a function of $\left(\frac{G}{Y}\right)$:

$$\frac{\dot{C}}{C} = \hat{r} - \frac{1}{\phi} + \frac{1}{\alpha}\frac{1}{\phi}\left(\frac{G}{Y}\right) - \rho \tag{3.15}$$

which is equal to the growth rate of capital and output since the constraint is binding.

We find it convenient to rewrite the dynamical system made up of the aggregate resource constraint, (3.14) and (3.15) by defining $b \equiv \frac{B}{K}$ and $c \equiv \frac{C}{K}$, so that

$$\begin{cases} \frac{\dot{q}}{q} = \frac{1}{\alpha}\left(\frac{1}{\phi}\right)\left(\frac{G}{Y}\right) + \hat{r} - \frac{1}{\phi} - \frac{1}{q}\frac{1-\alpha}{\alpha}A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} - \frac{(q-1)^2}{2hq} \\ \frac{\dot{c}}{c} = \hat{r} - \frac{1}{\phi} + \frac{1}{\alpha}\frac{1}{\phi}\left(\frac{G}{Y}\right) - \rho - \frac{q-1}{h} \\ \frac{\dot{b}}{b} = \frac{1}{b}\left[c + \frac{q-1}{h} + A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} + \frac{(q-1)^2}{2h} + \hat{r}b - A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{\alpha}{1-\alpha}}\right] - \frac{q-1}{h} \end{cases} \quad (3.16)$$

3.1.2 The non binding regime

For a given T, $B(T)$ and $K(T)$, the first order necessary conditions of problem (3.6) writes as[11]

$$\frac{1}{C}e^{-\rho t} + \lambda_1 = 0 \quad (3.17)$$

$$\lambda_1\left(1 + h\frac{I}{K}\right) + \lambda_2 = 0 \quad (3.18)$$

$$\frac{G}{K} = (A\alpha)^{\frac{1}{1-\alpha}} \iff \frac{G}{Y} = \alpha \quad (3.19)$$

$$\lambda_1\hat{r} = -\dot{\lambda}_1 \quad (3.20)$$

$$-\lambda_1\left[\frac{h}{2}\frac{I^2}{K^2} + A(1-\alpha)\left(\frac{G}{K}\right)^{\alpha}\right] = -\dot{\lambda}_2 \quad (3.21)$$

By defining $q = -\frac{\lambda_2}{\lambda_1}$ we obtain from equation (3.18) $\frac{I}{K} = \frac{q-1}{h}$.

By using (3.21), (3.20) and the definition of q we get the arbitrage equation

$$\frac{\dot{q}}{q} = \hat{r} - \frac{(q-1)^2}{2hq} - (1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\frac{1}{q} \quad (3.22)$$

Combining equation (3.17) with equation (3.20) we obtain the growth rate of consumption

$$\frac{\dot{C}(t)}{C(t)} = \hat{r} - \rho \quad (3.23)$$

hence

$$C(t) = C(0)\exp\left[(\hat{r} - \rho)t\right] \quad (3.24)$$

where $C(0)$ has to be determined.

Equation (3.17) detemines the trajectory of

$$\lambda_1(t) = -\frac{1}{C(t)}e^{-\rho t}. \quad (3.25)$$

[11] This regime is the same as Turnovsky's (2009) model with exception of the transversality conditions: in our case the boundary conditions are given by the continuity conditions of Theorem 1.

By defining $c = \frac{C}{K}$, $b = \frac{B}{K}$ we can write the reduced system as

$$\begin{cases} \frac{\dot{c}}{c} = (\hat{r} - \rho) - \frac{q-1}{h} \\ \frac{\dot{q}}{q} = \hat{r} - \frac{(q-1)^2}{2hq} - (1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \frac{1}{q} \\ \frac{\dot{b}}{b} = \frac{1}{b} \left[c + \frac{q-1}{h} + A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} + \frac{(q-1)^2}{2h} + \hat{r} b - A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \right] - \frac{q-1}{h} \end{cases} \tag{3.26}$$

3.2 Optimal switching

3.2.1 The convergence to the binding regime

In this section we study the conditions for having the convergence to the binding regime. We suppose that the economy is in the non binding regime and we assume that the system will be unconstrained forever, that is $T = +\infty$. Hence, the model turns out to be the same as Turnovsky's (2009) that we discussed in Section 2. In this section we extend his analysis in order to show when the corner solution $T = +\infty$ is possible.

We have already shown that the dynamics of $B(t)$, given by equation (2.12), depends on the difference between the growth rate of consumption $\hat{r} - \rho$ and the growth rate of capital $\frac{q_1-1}{h}$, whose sign depends on the value of q_1. If $\hat{r} < (1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$, $q_1 > 1$ and the growth rate of capital is positive. It is negative in the opposite case.

Proposition 2. *Suppose that $\hat{r} - \rho < \frac{q_1-1}{h}$. Then there exists a date $T < +\infty$ at which the economy hits the constraint.*

If $\hat{r} - \rho < \frac{q_1-1}{h}$,[12] $B(t)$ is an increasing function and $\lim_{t\to+\infty} B(t) = +\infty$. Hence in this case $B(t)$ hits the constraint at a finite time.

In the opposite case

Proposition 3. *When $\hat{r} - \rho \geq \frac{q_1-1}{h}$, $\lim_{t\to+\infty} B(t) = -\infty$: the economy will end up in autarky whithin a finite time.*

The case $\hat{r} - \rho \geq \frac{q_1-1}{h}$ is more complex.

If $q_1 < 1$, $B(t)$ decreases exponentially and $\lim_{t\to+\infty} B(t) = -\infty$.

If $q_1 > 1$, $\lim_{t\to+\infty} B(t) = -\infty$, but $B(t)$ can be either always decreasing or first increasing and then decreasing, depending on the values of $\frac{B(0)}{K(0)}$. At $t = 0$, $B'(0)$ is equal to:

$$B'(0) = \theta K(0) \left[\hat{r} - \rho - \frac{q_1 - 1}{h} \right] + (\hat{r} - \rho) B(0)$$

which is negative if $\frac{B(0)}{K(0)} < -\theta \left[1 - \frac{(q_1-1)/h}{\hat{r}-\rho} \right]$.

[12]Notice that this case is possible only if $q_1 > 1$.

The other condition on $\frac{B(0)}{K(0)}$ is given by the collateral constraint: $\frac{B(0)}{K(0)} < \phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$.

If $\phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} < -\theta \left[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}\right]$, then $B(t)$ is always decreasing and never hits the constraint.

If $\phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \geq -\theta \left[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}\right]$, then when $\frac{B(0)}{K(0)} < -\theta \left[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}\right]$, $B(t)$ is decreasing and never hits the constraint. When $\frac{B(0)}{K(0)} \geq -\theta \left[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}\right]$ and $\frac{B(0)}{K(0)}$ is close to the collateral constraint, $B(t)$ is first increasing then decreasing, hence it may hit the constraint twice.

One might think that when $\hat{r} - \rho \geq \frac{q_1-1}{h}$ the binding regime starts at $t = 0$. However we have checked numerically that $\hat{r} - \rho \geq \frac{q_1-1}{h}$ is incompatible with the binding regime optimal from $t = 0$. A detailed discussion is contained in Section 6.

3.2.2 The existence of the non binding regime: the role of investment adjustment costs

We have previously shown in Section 2 that the assumption of investment adjustment costs is necessary to rule out the irrealistic knife-edge condition that the exogenous interest rate must be equal to the marginal return of domestic capital. But this assumption has another important implications in our two regimes model. In the absence of capital adjustment costs, if the knife-edge condition is not met, the regime with slack constraint does not exist because the country would choose to invest or disinvest at an infinite rate if the internal productivity is, respectively, greater or smaller than the interest rate (see Cohen and Sachs 1986). Since we assume that the country is a net debtor, then the internal return must be greater than the world interest rate, so that the economy switches to the binding regime since the initial date. A proof of this result is included in Appendix D. Hence the presence of adjustment costs of investment enriches the inter-regime transitional dynamics.

4 Optimal public expenditures, growth and collateral constraints

4.1 The optimal public expenditure - GDP ratio and the collaterals

In this section we will compare the optimal (G/Y) of the flow-based collaterals model with that of the Barro (1990) and Turnovsky's (2009) models that we discussed in Section 2.

Proposition 4. *The optimal public expenditures-GDP ratio of the model with the flow-based collateral constraint is greater than the Barro (1990) and Turnovsky's (2009) solution.*

The following equation determines the optimal (G/Y) ratio of our model:[13]

$$\frac{\partial Y}{\partial G} - \frac{\eta}{\lambda_1}\frac{\partial Y}{\partial G} = 1 \tag{4.1}$$

where $\eta > 0$ is the Lagrange multiplier attached to the equality constraint (3.2) and it is equal to $\eta = \frac{1}{\phi}\left[1 - \frac{1}{\alpha}\left(\frac{G}{Y}\right)\right]\lambda_1$. Being $\eta > 0$ and $\lambda_1 < 0$, equation (4.1) implies that in the binding regime (G/Y) must be greater than α, i. e. greater than the Barro (1990) and Turnovsk's (2009) value. When the constraint is binding, the country uses public debt also to finance the public expenditures-GDP ratio. The consequent increase in public expenditures will in turn relax the borrowing contraint. This is why the public expenditures-GDP ratio is greater than that of the literature's models. In Turnovsky (2009) debt is not constrained. Equation (4.1) states that the social planner will choose the optimal (G/Y) by equating the marginal benefit of increasing G to the marginal cost, which, by the economy's resource constraint, is equal to one. In the literature's models the corresponding expression is

$$\frac{\partial Y}{\partial G} = 1$$

Hence in the model with flow-based collateral constraint the marginal benefit is equal to the marginal benefit of incresing G on production augmented by the term $-\frac{\eta}{\lambda_1}\frac{\partial Y}{\partial G}$ which represents the marginal benefit of relaxing the collateral constraint.

4.2 The growth rate and the nature of the collateral constraint

The aim of this section is to compare our model's balanced growth path properties with those of the literature's models where the collateral constraint is stock-based, so that public borrowing is limited by ϕK.

In order to simplify the analysis, we set $h = 0$. As it has be shown in Section 3, in the absence of capital adjustment costs the economy is in the binding regime from $t = 0$.

The optimality conditions of the model with physical capital as collateral are included in Appendix E.

We start by comparing the optimal (G/Y). The results are contained in the following proposition:

Proposition 5. *The optimal public expenditures-GDP ratio of the model with flow-based collateral constraint is greater than that of the stock-based model.*

[13]Equation (4.1) is obtained by solving the optimal control problem by attaching a Lagrange multiplier to the equality constraint (3.2). This allows to carry out an economic interpretation, see Appendix C. This condition is independent from the presence of adjustment costs for investment.

The proof and the economic intuition are the same as that of Proposition 4.

We now tackle the comparison of the long-run growth rates in what follows. We shall set $\alpha = 1/2$ in order to find the analytical expression of the flow-based optimal (G/Y):

$$\left(\frac{G}{Y}\right)_{\text{flow-based}} = \frac{1 - \sqrt{1 - (1 - \phi\hat{r})\,\phi A^2}}{\phi A^2} \tag{4.2}$$

Hence the growth rate turns out to be equal to

$$g_{\text{flow-based}} = r - \frac{1}{\phi} + 2\frac{1}{\phi}\frac{1 - \sqrt{1 - (1 - \phi\hat{r})\,\phi A^2}}{\phi A^2} - \rho \tag{4.3}$$

The growth rate of the stock-based economy is given by (E.1). When $\alpha = 1/2$ it is equal to

$$g_{\text{stock-based}} = \frac{\phi}{1-\phi}\left(\frac{A^2}{4\phi} - \hat{r}\right) - \rho \tag{4.4}$$

In order to study the difference between (4.3) and (4.4) we fix $\hat{r}$ and ϕ and let A vary.

The admissible values of A are bounded. Proposition 5 requires a public expenditures-output ratio greater than α for optimality and this defines the lower bound of A: $A_{\min} \equiv 2\sqrt{r}$. The upper bound is the value of A for which the discriminant of (4.2) is positive: $A_{\max} \equiv \frac{1}{\sqrt{(1-\phi\hat{r})\phi}}$. In the following analysis we assume $\hat{r} < 1/2$.

The difference between (4.3) and (4.4) is

$$g_{\text{flow-based}} - g_{\text{stock-based}} = \frac{8(1-\phi) - 4\phi A^2(1-\phi) - \phi^2 A^4 - 4(1-\phi)\,2\sqrt{1 - (1-\phi\hat{r})\,\phi A^2} + 4\phi^2\hat{r}A^2}{(1-\phi)\,\phi^2 A^2} \tag{4.5}$$

So that its sign depends on the numerator. In particular $g_{\text{flow-based}} > g_{\text{stock-based}}$ if and only if

$$8(1-\phi) - 4\phi A^2(1-\phi) - \phi^2 A^4 > 4(1-\phi)\,2\sqrt{1 - (1-\phi\hat{r})\,\phi A^2} - 4\phi^2\hat{r}A^2$$

The left hand side and the right hand side define two decreasing and concave functions of A, , respectively, $f(A)$ and $g(A)$. At $A_{\min}$, $f(A) = g(A)$. The existence of an intersection between the two fonctions depends on the difference between the first derivative at $A_{\min}$ and on the difference between the two functions at $A_{\max}$. We start by analyzing the latter. At $A_{\max}$ (4.5) rewrites as

$$g_{\text{flow-based}} - g_{\text{stock-based}} = \frac{4(1-\phi)(1-2\hat{r}\phi)(1-\phi\hat{r}) - 1 + 4\hat{r}\phi(1-\hat{r}\phi)}{(1-\hat{r}\phi^2)}$$

The numerator is zero for $\hat{r}_1 = \frac{1}{2\phi}$ and $\hat{r}_2 = \frac{-3+4\phi}{2\phi(-1+2\phi)}$. The coefficient of $\hat{r}^2$ is $4\phi^2\hat{r}^2(1-2\phi)$ which is non-negative for $\phi \leq 1/2$. In this a case $1 < \hat{r}_1 < \hat{r}_2$ and $g_{\text{flow-based}} > g_{\text{stock-based}}, \forall \hat{r} \in (0,1)$.

When instead it is $\phi > 1/2$, the situation is more complicated. In particular when $1/2 < \phi \leq 3/4$, $\hat{r}_2 \leq 0$ and $\hat{r}_1 > 1/2$,[14] hence the flow-based growth rate is always greater than the stock-based one.

When $\phi > 3/4$, $\hat{r}_2 > 0$ and $g_{\text{flow-based}} > g_{\text{stock-based}}$ for $\hat{r}_2 < \hat{r} < \hat{r}_1$. These results are synthetized in the following proposition:

Proposition 6. *If financial markets imperfections are not too weak, countries with a high TFP experience a greater long-run growth rate under flow-based collaterals. Otherwise, the flow-based growth rate is greater than the stock-based growth rate if the international interest rate is high enough.*

In the neighborhood of $A_{\min}$ we will study the sign of difference $f'(A_{\min}) - g'(A_{\min}) = 16\hat{r}^{\frac{3}{2}}\phi^3\frac{2\hat{r}-1}{1-2\hat{r}\phi}$. Given the assumption on the international interest rate and for all the values of ϕ in the $(0,1)$ interval, $f'(A_{\min}) < g'(A_{\min})$, so that it is $g_{\text{flow-based}} < g_{\text{stock-based}}$.

We can hence state

Proposition 7. *Countries with low technological level experience a higher growth rate under stock-based collaterals.*

The economic intuition behind Propositions 6 and 7 is the following. When A is low, that is when A is smaller than $(2-\phi)/(1-\phi\hat{r})$, the public debt-capital ratio of the flow-based model, which is given by rewriting the collateral constraint as $(B/K) = \phi A^{\frac{1}{1-\alpha}}(G/Y)^{\frac{\alpha}{1-\alpha}}$, is smaller than that of the stock-based collaterals model, $(B/K) = \phi$. Hence the borrowing capacity of the flow-based economy is smaller. This is the reason why for low values of A the growth rate of the flow-based model is smaller than that of the stock-based model. However an increase in A does not have any effect on the stock-based constraint, while it relaxes the borrowing contraint in the flow-based case through a direct effect on the GDP and an indirect effect on the optimal public expenditures-GDP ratio (*collateral constraint effect*). Hence for a high A the growth rate of the flow-based model turns out to be greater than that of the stock-based model.

5 Balanced growth path analysis

In this section we study the balanced growth path of the economy (BGP hereafter) in the case where the economy ends up in the binding regime.

[14] $\hat{r}_1$ is a decreasing function of ϕ which varies between 1 and $1/2$ when $\phi \in [1/2, 1]$.

Definition. *The endogenous variables* $(c, G/Y, q)$ *of system* (3.16) *define a BGP if they grow at a positive constant rate.*

In BGP $\frac{\dot{c}}{c} = 0$, hence from (3.16) $q = h\left[\hat{r} - \frac{1}{\phi} + \frac{1}{\alpha}\frac{1}{\phi}\left(\frac{G}{Y}\right) - \rho\right] + 1$. By plugging it into the $\dot{q} = 0$ equation and rearranging terms we obtain the equation which determines the equilibrium (G/Y) ratio

$$\begin{aligned} -\tfrac{1-\alpha}{\alpha}A^{\frac{1}{1-\alpha}}\phi\left(\tfrac{G}{Y}\right)^{\frac{1}{1-\alpha}} + \tfrac{h}{2}\left(\tfrac{1}{\phi}\right)\tfrac{1}{\alpha^2}\left(\tfrac{G}{Y}\right)^2 + \left[h\left(\hat{r}\phi - 1\right) + \phi\right]\tfrac{1}{\phi}\tfrac{1}{\alpha}\left(\tfrac{G}{Y}\right) \\ + \tfrac{1}{\phi}\left[\tfrac{h}{2}\left(\hat{r}\phi - 1\right)^2 + \phi\left(\hat{r}\phi - 1\right) - \tfrac{h}{2}\left(\rho\phi\right)^2\right] = 0 \end{aligned} \tag{5.1}$$

Note that by setting $\phi = 0$ we obtain Barro's (1990) and Turnovsky's (2009) optimal public expenditures-GDP ratio: $(G/Y) = \alpha$. In the following proposition we give the conditons for the existence of some solution for the empirical appealing case $\alpha \leq \frac{1}{2}$.[15]

Proposition 8. *(Necessary Conditions). The dynamical system admits*

(1) One solution iff $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$*;*

(2) Two solutions iff $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$*.*

Proof. Consider equation (5.1). We define $f\left(\frac{G}{Y}\right) \equiv \frac{h}{2}\left(\frac{1}{\phi}\right)\frac{1}{\alpha^2}\left(\frac{G}{Y}\right)^2 + \left[h\left(\hat{r}\phi - 1\right) + \phi\right]\frac{1}{\phi}\frac{1}{\alpha}\left(\frac{G}{Y}\right) + \frac{1}{\phi}\left[\frac{h}{2}\left(\hat{r}\phi - 1\right)^2 + \phi\left(\hat{r}\phi - 1\right) - \frac{h}{2}\left(\rho\phi\right)^2\right]$ and $g\left(\frac{G}{Y}\right) \equiv \frac{1-\alpha}{\alpha}A^{\frac{1}{1-\alpha}}\phi\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}}$. $f\left(\frac{G}{Y}\right)$ is a convex parabola, while $g\left(\frac{G}{Y}\right)$ is a convex increasing function such that $g(0) = 0$ and $\lim_{\frac{G}{Y}\to+\infty} g\left(\frac{G}{Y}\right) = +\infty$. The assumption $\alpha < \frac{1}{2}$ guarantees that $\lim_{\frac{G}{Y}\to+\infty} \frac{f\left(\frac{G}{Y}\right)}{g\left(\frac{G}{Y}\right)} = 0$.

We study the intersections of $f\left(\frac{G}{Y}\right)$ with the positive $\left(\frac{G}{Y}\right) - axis$ because this helps in determining the number of intersections between $g\left(\frac{G}{Y}\right)$ and $f\left(\frac{G}{Y}\right)$ (the assumption $\alpha < \frac{1}{2}$ limits the number of possible intersections to two). The discriminant of the second degree equation defined by $f\left(\frac{G}{Y}\right) = 0$ is positive, hence it can have either two negative solutions, one positive and one negative solution or two positive solutions:

	$\frac{h}{2}\left(\frac{1}{\phi}\right)\frac{1}{\alpha^2}$	$\left[h\left(\hat{r}\phi - 1\right) + \phi\tau\right]\frac{1}{\phi}\frac{1}{\alpha}$	$\frac{1}{\phi\tau}\left[\frac{h}{2}\left(\hat{r}\phi - 1\right)^2 + \phi\left(\hat{r}\phi - 1\right) - \frac{h}{2}\left(\rho\phi\right)^2\right]$	$f\left(\frac{G}{Y}\right) = 0$
1	+	+	+	2 negative solutions
2	+	+	-	1 positive and one negative
3	+	-	-	1 positive and one negative
4	+	-	+	2 positive solutions

[15] This range of values is consistent with the empirical estimations existing in the literature: Erberts's (1986) estimation for the income share of public expenditures is around 0.43, Aschauer (1989) finds that it is 0.39, Ai and Cassou (1995) find estimations which range between 0.15 and 0.26, finally Chatterjee et al. (2003) assume a value of 0.2.

We define $D \equiv \left[h\left(\hat{r}\phi-1\right)+\phi\right]\frac{1}{\phi}\frac{1}{\alpha}$ and $E \equiv \frac{1}{\phi}\left[\frac{h}{2}\left(\hat{r}\phi-1\right)^2+\phi\left(\hat{r}\phi-1\right)-\frac{h}{2}\left(\rho\phi\right)^2\right]$. The following cases are possible:

1. $D \geq 0$ if $h \leq \frac{\phi}{1-\hat{r}\phi}$ and $E > 0$ if $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Since $\frac{\phi}{1-\hat{r}\phi} < \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$ the two conditions are incompatible.

2. $D \geq 0$ if $h \leq \frac{\phi}{1-\hat{r}\phi}$, $E \leq 0$ if $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Hence to have case 2 we must require $h \leq \frac{\phi}{1-\hat{r}\phi}$.

3. $D < 0$ if $h > \frac{\phi}{1-\hat{r}\phi}$, $E \leq 0$ if $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Hence $\frac{\phi}{1-\hat{r}\phi} < h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

4. $D < 0$ if $h > \frac{\phi}{1-\hat{r}\phi}$, $E > 0$ if $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. This implies $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

Cases 2 and 3 can be summarized by requiring $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. □

Both the soultions satisfy the transversality condition: $\lim_{t\to+\infty} -q_i\lambda_1^{(2)}\left(t\right)K\left(t\right) = -q_i\lambda_1^{(2)}\left(T\right)\exp\left[-\rho\right]$, $i = 1, 2$, where q_i is the BGP value of the Tobin q.

We can prove that case 2 of Proposition 8 implies just a false multiplicity. This is shown in the following proposition.

Proposition 9. *(Uniqueness of the Balanced Growth Path). The balanced growth path is unique.*

Proof. A solution of equation (5.1) is a balanced growth path if the growth rate (3.15) is positive, that is

$$\hat{r}\phi - 1 + \frac{1}{\alpha}\left(\frac{G}{Y}\right) \geq \rho\phi$$

This allows to write the following inequality

$$\begin{aligned} & \tfrac{h}{2}\left(\tfrac{1}{\phi}\right)\tfrac{1}{\alpha^2}\left(\tfrac{G}{Y}\right)^2 + h\left(\hat{r}\phi-1\right)\tfrac{1}{\phi}\tfrac{1}{\alpha}\left(\tfrac{G}{Y}\right) + \tfrac{1}{\phi}\left[\tfrac{h}{2}\left(\hat{r}\phi-1\right)^2 - \tfrac{h}{2}\left(\rho\phi\right)^2\right] + \rho\phi \\ \leq \; & \tfrac{h}{2}\left(\tfrac{1}{\phi}\right)\tfrac{1}{\alpha^2}\left(\tfrac{G}{Y}\right)^2 + \left[h\left(\hat{r}\phi-1\right)+\phi\right]\tfrac{1}{\phi}\tfrac{1}{\alpha}\left(\tfrac{G}{Y}\right) + \tfrac{1}{\phi}\left[\tfrac{h}{2}\left(\hat{r}\phi-1\right)^2 + \phi\left(\hat{r}\phi-1\right) - \tfrac{h}{2}\left(\rho\phi\right)^2\right] \\ & = \tfrac{1-\alpha}{\alpha}A^{\frac{1}{1-\alpha}}\phi\left(\tfrac{G}{Y}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

By defininig $l\left(\frac{G}{Y}\right)$ the first function, $g\left(\frac{G}{Y}\right)$ the second one and $f\left(\frac{G}{Y}\right)$ the third one, each solution of equaion (5.1) must be such that

$$l\left(\frac{G}{Y}\right) \leq g\left(\frac{G}{Y}\right) = f\left(\frac{G}{Y}\right)$$

in order to be a BGP. In what follows we are going to give a geometrical proof. We can show that

- $l\left(0\right) > g\left(0\right)$
- $l\left(\frac{G}{Y}\right)$ and $g\left(\frac{G}{Y}\right)$ cross just once, $\left(\frac{G}{Y}\right)|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} = \alpha\left(\rho\phi - \hat{r}\phi + 1\right) > 0$,

- $\left(\frac{G}{Y}\right)|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} > \left(\frac{G}{Y}\right)_1 |_{g\left(\frac{G}{Y}\right)=0} = \frac{h\alpha-\alpha\phi-h\hat{r}\phi-\sqrt{\alpha^2\phi^2+h^2\alpha^2\rho^2\phi^2}}{h}$,
- $\left(\frac{G}{Y}\right)|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} > \left(\frac{G}{Y}\right)_2 |_{g\left(\frac{G}{Y}\right)=0} = \frac{h\alpha-\alpha\phi-h\hat{r}\phi+\sqrt{\alpha^2\phi^2+h^2\alpha^2\rho^2\phi^2}}{h}$.

These results exclude the smallest solution of case 2 of Proposition 8, as it is shown by the figure below (Figure 5.1):

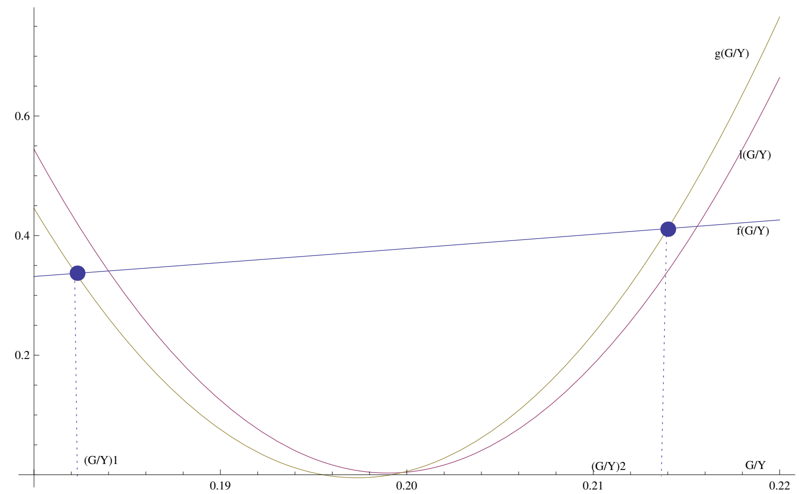

Figure 5.1: Existence and Uniqueness of the BGP

In the figure, $\left(\frac{G}{Y}\right)_1$ and $\left(\frac{G}{Y}\right)_2$ are the solutions of equation (5.1). For $\left(\frac{G}{Y}\right)_1$, $l\left(\left(\frac{G}{Y}\right)_1\right) > g\left(\left(\frac{G}{Y}\right)_1\right) = f\left(\left(\frac{G}{Y}\right)_1\right)$, hence the growth rate is negative. $\left(\frac{G}{Y}\right)_1$ is not a BGP. For $\left(\frac{G}{Y}\right)_2$, if, as in the figure, $l\left(\left(\frac{G}{Y}\right)_2\right) < g\left(\left(\frac{G}{Y}\right)_2\right) = f\left(\left(\frac{G}{Y}\right)_2\right)$ it is a BGP ($l\left(\frac{G}{Y}\right)$ should cross $g\left(\frac{G}{Y}\right)$ before it crosses $f\left(\frac{G}{Y}\right)$). It should be remarked that the parameter A, which rules the position of $f\left(\frac{G}{Y}\right)$ on the graphic, should not be to low, otherwise we should reject also the larger solution of case 2) in Proposition 8.[16] □

5.1 The special case $\alpha = \frac{1}{2}$

In order to obtain analytical solutions for the ratio (G/Y), we shall focus on the $\alpha = \frac{1}{2}$ case.[17] Equation (5.1) becomes

$$\begin{gathered}\left(\frac{2h}{\phi} - A^2\phi\right)\left(\frac{G}{Y}\right)^2 + 2\left[h\left(\hat{r}\phi-1\right)+\phi\right]\frac{1}{\phi}\left(\frac{G}{Y}\right) \\ +\frac{1}{\phi}\left[\frac{h}{2}\left(\hat{r}\phi-1\right)^2+\phi\left(\hat{r}\phi-1\right)-\frac{h}{2}\left(\rho\phi\right)^2\right]=0\end{gathered} \tag{5.2}$$

[16] Numerical analysis has shown that the result is valid for values of A commonly used in the simulations. The numerical results are available from the author upon request.

[17] This case is perfectly equivalent to the case $\alpha < 1/2$, hence the conditions for the existence of a BGP do not change.

In what follows we give the necessary conditions for the existence of the solutions of this equation, under the assumption $\left(\frac{2h}{\phi} - A^2\phi\right) > 0$ (the other case is symmetric: it is sufficient to multiply by -1 equation (5.2)).

Corollary 10. *(Necessary Conditions).* Assume that the discriminant of equation (5.2) is non-negative.

The dynamical system admits
(1) One solution if $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$;
(2) Two solutions if $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.
The proof is identical to the one of Proposition 8.

Corollary 11. *(Uniqueness of the Balanced Growth Path). The balanced growth path is unique.*

The proof is exactly the same as the one of Proposition 9 and the unique admissible solution is

$$\left(\frac{G}{Y}\right)^* = \frac{-\left[h\left(\hat{r}\phi - 1\right) + \phi\right]\frac{1}{\phi} + \sqrt{\frac{1}{2}(2 + 2h^2\rho^2 + A^2(h - 2(1 + h\hat{r})\phi + (r(2 + h\hat{r}) - h\rho^2)\phi^2))}}{\left(\frac{2h}{\phi} - A^2\phi\right)} \tag{5.3}$$

The next step is to study of the stability of the BGP. System (3.16) allows transitional dynamics. This is due to the presence of investment adjustment costs: the economy does not reach immediately its steady state because adjustment costs do not allow capital to immediately jump to its BGP. Proposition 12 contains the local stability properties of $\left(\frac{G}{Y}\right)_1$:

Proposition 12. *If* $A^2 > -\frac{2}{h-2(1+\hat{r}h)\phi+(\hat{r}(2+h\hat{r})-h\rho^2)\phi^2}$*, the balanced growth path solution,* $\left(\frac{G}{Y}\right)^*$*, is saddle-point stable.*

Proof. The dynamical system is

$$\begin{cases} \dot{c} = \left(\hat{r} - \frac{1}{\phi} + 2\left(\frac{1}{\phi}\right)^2 A^{-2}b - \rho - \frac{q-1}{h}\right)c \\ \dot{q} = \left(\hat{r} - \frac{1}{\phi}\right)q + 2\left(\frac{1}{\phi}\right)^2 A^{-2}bq - \left(\frac{1}{\phi}\right)^2 A^{-2}b^2 - \frac{(q-1)^2}{2h} \\ \dot{b} = c + \frac{q-1}{h} + \left(\frac{1}{\phi}\right)^2 A^{-2}b^2 + \hat{r}b + \frac{(q-1)^2}{2h} - \frac{1}{\phi}b - \frac{q-1}{h}b \end{cases}$$

and the Jacobian matrix, J_1, writes as

$$\begin{bmatrix} \hat{r}-\frac{1}{\phi}+2\left(\frac{1}{\phi}\right)^2 A^{-2}b_1-\rho-\frac{q_1-1}{h} & -\frac{1}{h}c_1 & 2\left(\frac{1}{\phi}\right)^2 A^{-2}c_1 \\ 0 & 2\left(\frac{1}{\phi}\right)^2 A^{-2}b_1+\hat{r}-\frac{1}{\phi}-\frac{q_1-1}{h} & 2\left(\frac{1}{\phi}\right)^2 A^{-2}q_1-2\left(\frac{1}{\phi}\right)^2 A^{-2}b_1 \\ 1 & \frac{1}{h}+\frac{q_1-1}{h}-\frac{1}{h}b_1 & 2\left(\frac{1}{\phi}\right)^2 A^{-2}b_1+\hat{r}-\frac{1}{\phi}-\frac{q_1-1}{h} \end{bmatrix}$$

where

$$b_1=\phi A^2\frac{-\left[h\left(\hat{r}\phi-1\right)+\phi\right]\frac{1}{\phi}+\sqrt{\frac{1}{2}(2+2h^2\rho^2+A^2(h-2(1+h\hat{r})\phi+(r(2+h\hat{r})-h\rho^2)\phi^2))}}{\left(\frac{2h}{\phi}-A^2\phi\right)}$$

$$q_1=h\left[\hat{r}-\frac{1}{\phi}+2\left(\frac{1}{\phi}\right)^2 A^{-2}b_1^2-\rho\right]+1$$

$$c_1=\frac{q_1-1}{h}b_1+\frac{1}{\phi}b_1-\frac{q_1-1}{h}-\left(\frac{1}{\phi}\right)^2 A^{-2}b_1^2-\hat{r}b_1-\frac{\left(q_1-1\right)^2}{2h}$$

$$\det(J_1)=\begin{array}{c} -\frac{1}{\sqrt{2}A^2h(\phi)^2}\left[\rho\sqrt{2+2h^2\rho^2+A^2\left(h-2\left(1+\hat{r}h\right)\phi+\left(\hat{r}\left(2+h\hat{r}\right)-h\rho^2\right)\left(\phi\right)^2\right)}\right. \\ \left.\left(-2h\rho+\sqrt{2}\sqrt{2+2h^2\rho^2+A^2\left(h-2\left(1+\hat{r}h\right)\phi+\left(\hat{r}\left(2+h\hat{r}\right)-h\rho^2\right)\left(\phi\right)^2\right)}\right)\right] \end{array}$$

$$\mathrm{tr}(J_1)=2\rho$$

The trace is positive.

The determinant is negative if $A^2>-\frac{2}{h-2(1+\hat{r}h)\phi+(\hat{r}(2+h\hat{r})-h\rho^2)\phi^2}$.[18] □

5.2 Comparative statics

The aim of this section is to run a comparative statics analysis. We fixed $\alpha\leq 1/2$[19], $h=15$ (Barro and Sala-i-Martin 2003, Chatterjee et al. 2003), $\hat{r}=0.035$ and $\rho=0.03$ (these values allow to meet the conditions needed to move from the non binding to the binding regime. See Section 3, Proposition 2). We set $A=1$ in order to obtain the a value for the capital-output ratio consistent with the real one, which is around 2.[20]

Numerical Finding 1: *The optimal (G/Y) and the economy's growth rate are increas-*

[18] The negative determinant implies that the product of the three eigenvalues is negative, implying that the number of negative eigenvalues is either one or three. However, the positive trace, being the sum of the three eigenvalues, rules out the latter case, thus implying one negative eigenvalue.

[19] We runned the simulations for $\alpha=0.2$ (Chatterjee et al. 2003) and $\alpha=1/2$.

[20] We repeated the excercice for other values of the parameter and the results are robust.

ing in ϕ.

Higher ϕ implies higher growth because a greater financial openess relaxes the borrowing constraint (growth enhancing effect of foreign borrowing).

Finding 2. *The optimal (G/Y) is decreasing in the world interest rate $\hat{r}$.*
An increase in $\hat{r}$ moves the function $g\left(\frac{G}{Y}\right)$ of Proposition 9 up in the graph, so that $(G/Y)_2$ decreases.
Numerical Finding 3: *The growth rate is decreasing in the world interest rate $\hat{r}$.*
This result is rather intuitive since an increase in the international interest rate makes borrowing more costly. The planner finances public expenditures by debt. Hence the optimal public expenditures-GDP ratio reduces and, since public spending is the engine of growth, the growth rate reduces as well.
Finding 4: *The optimal (G/Y) and the economy's growth rate are increasing in the technology parameter A.*

An increase in A moves the $f\left(\frac{G}{Y}\right)$ curve of Proposition 9 up in the graph, so that $(G/Y)_2$ increases. Given that (3.15) is increasing in (G/Y), the growth rate increases as well.
Finally,

Numerical Finding 5. *The optimal (G/Y) and the economy's growth rate are decreasing in the installation cost parameter h.*
Higher capital adjustment costs reduce the public expenditures-GDP ratio and the growth rates since more resources are spent in installation.

The numerical analysis also shows that the growth rate of GDP is smaller than that of Turnovsky (2009). We remind that in his model the country can borrow as much as it wants because there is no collateral constraint and the growth rate of GDP coincides with the growth rate of capital, which can be different from the growth rate of consumption. This is fixed by the difference between the exogenous international interest rate and the discount rate, and it is independent from any technological parameter. The growth rate in Turnovsky turns out to be greater than that of our model due to the absence of growth-limiting borrowing constraints, hence growth is enhanced.

6 Determination of the optimal switching date

6.1 Transitional dynamics

In this section we simulate the two-regime model in order to find the endogenous switching date. The existence of transitional dynamics in the binding regime does not allow us to find analytical solutions for the date T. Our strategy is to apply the matching condition of Theorem 1 in order to link two two-point boundary value problems, one for each regime. The final boundary problems are commonly used for the simulation of the transition process of growth models. We follow the method proposed by Trimborn et al. (2008a, 2008b).

The conditions (3.7) and (3.8) of Theorem 1 require the equality of the co-state variables between the two regimes. Hence at T the following condition must hold:

$$q^{(1)}(T) = q^{(2)}(T) = q(T) \tag{6.1}$$

We replace conditions (3.9), which is difficult to implement when a regime can not be solved in closed form, by two other conditions. Given that at T the state variables can not jump, it must be $b^{(1)}(T) = b^{(2)}(T) = b(T)$. This value can be obtained by the collateral constraint $\frac{B}{K} = b = \phi A \left(\frac{G}{K}\right)^{\alpha} = \phi A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{\alpha}{1-\alpha}}$. b must also be continuous at T so that the same is true for the public expenditures-GDP ratio. In the non binding regime $\left(\frac{G}{Y}\right) = \alpha$, hence when $b^{(1)}(t)$ hits the constraint it must be

$$b^{(1)}(T) = b^{(2)}(T) = b(T) = \phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \tag{6.2}$$

As a consequence the value of $b(T)$ is known and $\left(\frac{G}{Y}\right)$ is continuous at T.

The last condition stems from cequation (3.7), the equality between the shadow value of public debt, and from the first order conditions (3.10) and (3.17). These together implies that in this model consumption is continuous at T, so that

$$c^{(1)}(T) = c^{(2)}(T) = c(T) \tag{6.3}$$

Equation (6.3) also stems from the continuity of $\left(\frac{G}{Y}\right)$ at T and from the expression of the growth rate of consumption in the binding regime: $\frac{\dot{C}}{C} = \hat{r} - \frac{1}{\phi}\left(1 - \frac{1}{\alpha}\frac{G}{Y}\right) - \rho$.

The adopted algorithm is the relaxation algorithm proposed by Trimborn (2008a). Our numerical procedure starts with the simulation of the binding regimes. The inputs are given by the boundary conditions,[21] the value of $b(T)$, given by condition (6.2), and a guess for the switching date T. The output is made up by $q(T)$ and $c(T)$. Condition (6.2) shows also that the values of the variables at T do not depend on the choice of T. This choice will only affect the duration of the transition to the steady state.

The output of the binding regime is then the input of the non binding regime, provided by the initial value of the state variable, $b(0)$, and its value at T, $b(T)$. As the first regime does not admit any steady state, the simulation of the dynamical system (3.26) consists in

[21] The transversality conditions are replaced by the steady state values of the variables.

finding the value of T such that (6.1), (6.2) and (6.3) are satisfied.

We assume the same parameters' values of the baseline calibration of Section 5. We set $\alpha = 1/2$ because in this case we know what conditions the parameters need to meet for the BGP to be stable (cfr. Proposition 12) and $b(0) = 0.0145$.[22] This parameterisation satisfies the convergence conditions stated in Proposition 2. The figures below (Figure 6.1, Figure 6.2 and Figure 6.3) show the transition of the economy from the non binding to the binding regime. The first panel represents the non-binding regime, the second one the binding regime.[23] The switching date is equal to 12 periods:

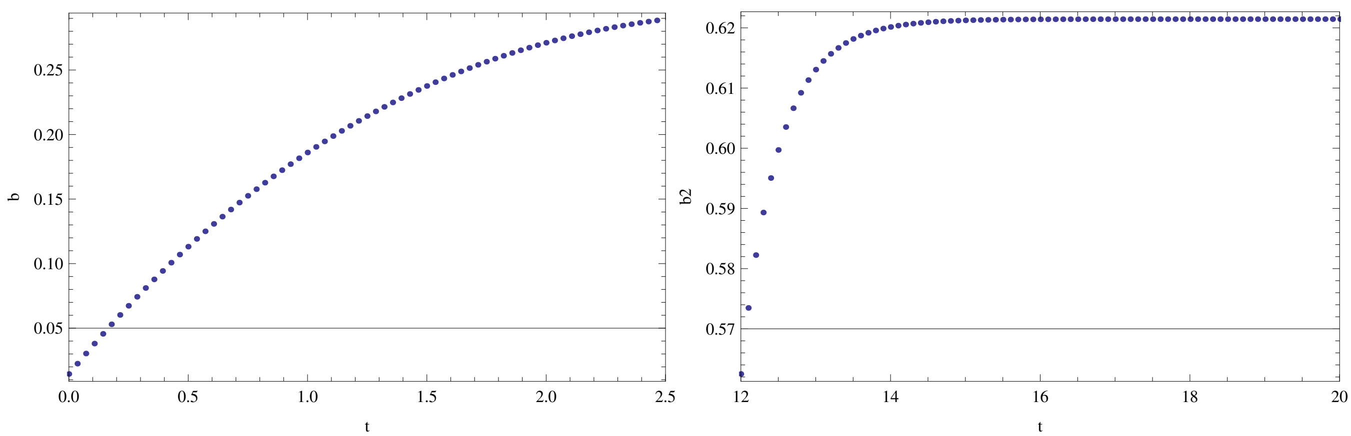

Figure 6.1: The transition inter regime: the public debt-capital ratio

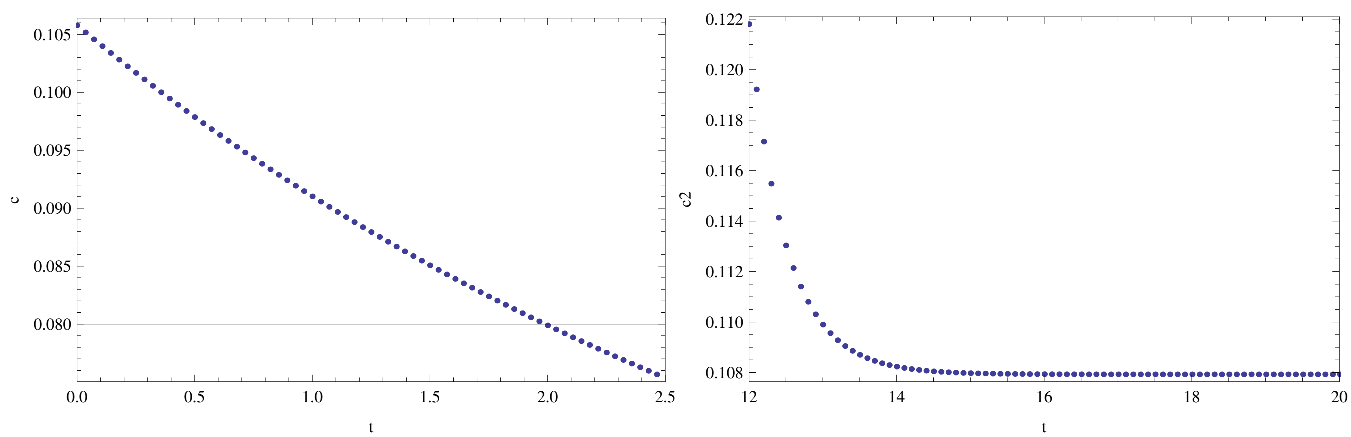

Figure 6.2: The transition inter regime: the consumption-capital ratio

[22] The chosen value of initial conditions affects the switching date T: the smaller is $b(0)$, the higher is T.

[23] The smaller concentration of the points at the beginning of the binding regime is due to the time function chosen to implement the algorithm.

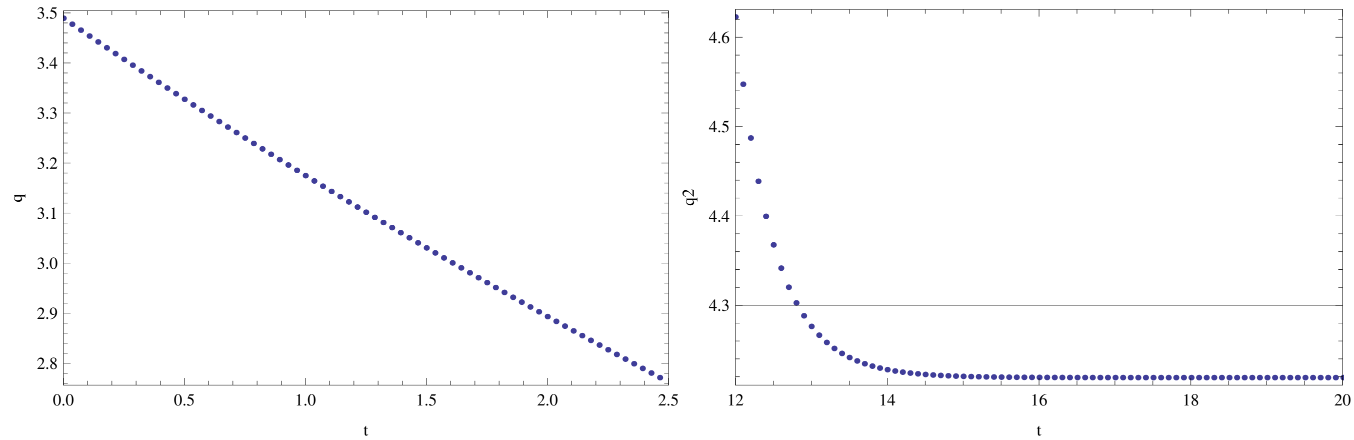

Figure 6.3: The transition inter regime: the Tobin's q

6.2 Comparative statics

In this section we investigate the effects of some relevant parameters on the transition between the non binding and the binding regime.

Numerical finding 6. *The optimal switching date T is an increasing function of the degree of market imperfection ϕ.*

An increase of ϕ relaxes the collaterals, hence the initial value of the public debt-capital ratio, $b(0)$, will be farther away from the constraint so that the economy will spend more time in the non binding regime.

Numerical finding 7. *The optimal switching date T is an increasing function of the interest rate $\hat{r}$. If the interest rate is too high, the economy will stay forever in the non binding regime and the public debt-capital ratio $b(t)$ will tend to zero.*

If the economy actually has to switch to the binding regime, the convergence conditions of Proposition 2 need to be satisfied, that is: $\hat{r} - \rho < (q_1 - 1)/h$. This condition identifies a treshold value of A below which the economy will not converge to the binding regime.[24] When the interest rate is lower than this treshold the economy does not benefit from financial openess. The numerical simulations show that the BGP value of (G/Y) is smaller than or equal to α if $\hat{r}$ is greater than or equal to the treshold, it is greater than α for $\hat{r}$ smaller than the treshold. The Kuhn-Tucker slackness condition (C.8) of Appendix C shows that the BGP value of (G/Y) needs to be grater than α in the binding regime (see Proposition 4). This means that the binding regime is not optimal. The stock of debt will become zero at a certain point in time, and the economy will stay in autarky.

The economic interpretation of this result is related to the amount of interests to be

[24]The value of the treshold is 0.13

payed. When this amount is too high, international finance is unattractive and the economy will remain in the non-binding regime. On the other hand, for low values of $\hat{r}$ the transition to the binding regime is shorter.[25]

Similar treshold effects exist for the TFP parameter and the investment adjustment costs parameter.

Numerical finding 8. *The optimal switching date T is a decreasing function of the productivity parameter A. If the productivity parameter is too low, the economy will stay forever in the non binding regime and the public debt-capital ratio $b(t)$ will tend to zero.*

When the TFP parameter is smaller than the treshold,[26] the economy does not benefit from financial openess. As it has been shown for the parameter $\hat{r}$, for too low values of the technology parameter, the binding regime is not optimal and the economy will end up in autarky.

On the contrary, when A becomes very large, the optimal switching date approaches zero.[27] Hence a highly developed economy will immediately benefit from financial openess. These observations allow us to say that there is a treshold value of A that an economy must overcome in order to enjoy the benefits of financial globalization: less developed countries do not benefit from financial openess because the level of debt always remains smaller than its upper bound given by the constraint,[28] while more developed countries fully exploit the effect of financial liberalization.
In our model the parameter A plays a double role. A greater A rises the economy's productivity making the country more rich. This has in turns the effect of increasing the collaterals, allowing to the economy to borrow more from the rest of the world. Hence the economy will spend less time in the non binding regime reaching faster the advantages of financial globalization.

Numerical finding 9. *The optimal switching date T is an increasing function of the adjustment costs parameter h. If it is too high, the economy will stay forever in the non binding regime and the public debt-capital ratio $b(t)$ will tend to zero.*

An increase of the adjustment costs makes the transition slower. If h is greater than the treshold,[29] the economy remains in the non binding regime taking not benefit from financial openess. For low values of h the transition is very fast. This is consistent with the result of Section 3.2.2: if adjustment costs are absent, the economy switches to the binding regime

[25] We recall that the discount rate ρ is a lower bound for $\hat{r}$. For smaller interest rate, the non binding regime could not exist.

[26] Which is equal to: 0.386975.

[27] The simulations show that this appens for A bigger than 8

[28] In other terms $\frac{B}{K} < \phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$ always.

[29] For the chosen value for the technology parameter the treshold is very big.

at $t = 0$.

The above results are consistent with recent empirical literature on financial integration. There is no consensus about the effects of financial globalization on economic growth: the benefits depend on the countries' situation. In particular, Kose et al. (2009) show that there exists a certain treshold of financial and institutional development and economic features that an economy needs to attain in order to benefit from financial liberalization. Similary, Aizenman et al. (2011) show that economy's GDP growth is lower in countries with high levels of past debt and weak institutions.

This is consistent with our model's prediction: a country with a low level of productivity, with high adjustment costs of capital or facing a high interest rate will not benefit from financial liberalization. In order to enjoy the advantages of international borrowing, the economy must reach a certain level of financial and institutional development coupled with other favorable economic features. Moreover a high interest rate can be interpreted by the investitors as an index of high risk of a country and low institutions development, so that the economy will be excluded from the main international capital flows.

7 Conclusions

Recent government finance statistics of European countries show that countries which experience the greater public debt issues are those with the weaker tax revenue-GDP ratio. This suggests that public debt should be bounded by tax revenue. Hence we depart from the existing growth literature with collateral constraint since we assume that collaterals are flow-based instead than stock-based. We extended Barro (1990)-Turnovsky's (2009) small open economy model by assuming that government can finance public expenditures by borrowing on imperfect international financial markets: the stock of debt is bounded by tax revenue and creditors can just seize a fraction of collaterals in the case of default. Three are the possible states of the economy: autarky, non binding regime and binding regime. We give the conditions that the economy needs to meet in order to switch from the non-binding to the binding regime, which also define the corner solutions of the problem. First, in the absence of investment adjustment costs, the economy switches to the binding regime at the initial date. Second, if installation costs are positive, for the economy to switch to the binding regime the TFP should be sufficiently high and/or the adjustment costs and the world interest rate should be low enough. This is consistent with recent empirical literature on the benefits of financial globalization: the economy can fully benefit from financial openess only if it has attaind a certain treshold of institutional and financial development and economic characteristics.

We compared our results with the existing endogenous growth literature. The optimal public spending-GDP ratio is greater than that of Barro-Turnovsky's model and the stock-based collateral constraint literature: the government is incentivated to finance public expenditures by debt because this relaxes the collateral constraint. Finally, we found that if financial markets imperfections are not too weak, countries with a high TFP experience higher growth rates under the flow-based collateral assumption than under stock-based collaterals. Otherwise the international interest rate must be sufficiently high. When instead the TFP is low the flow-based growth rate is always smaller than the stock-based growth rate.

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Appendix

A The role of investment adjustment costs in a small open economy model of endogenous growth

In the absence of investment adjustment costs the economy's current account writes as $\dot{B} = G + C + I + \hat{r}B - AG^{\alpha}K^{1-\alpha}$. The planner maximizes the intertemporal utility subject to the economy's current account and the law of accumulation of capital over an infinite horizon. The hamiltonian is

$$H = \ln Ce^{-\rho t} + \lambda_1 \left[G + C + I + \hat{r}B - AG^{\alpha}K^{1-\alpha} \right] + \lambda_2 I$$

The Prontryagin's first order conditions are

$$\frac{\partial H}{\partial C} = \frac{1}{C}e^{-\rho t} + \lambda_1 = 0 \tag{A.1}$$

$$\frac{\partial H}{\partial G} = \lambda_1 - \lambda_1 A\alpha G^{\alpha-1}K^{1-\alpha} = 0 \implies \frac{G}{K} = A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}; \frac{G}{Y} = \alpha \tag{A.2}$$

$$\frac{\partial H}{\partial I} = \lambda_1 + \lambda_2 = 0 \tag{A.3}$$

$$\frac{\partial H}{\partial B} = \lambda_1 \hat{r} = -\dot{\lambda}_1 \tag{A.4}$$

$$\frac{\partial H}{\partial K} = -\lambda_1 AG^{\alpha}(1-\alpha)K^{-\alpha} = -\dot{\lambda}_2 \tag{A.5}$$

$$\lim_{t\to\infty} \lambda_1 B = 0 \qquad \lim_{t\to\infty} \lambda_2 K = 0$$

The absence of installation costs implies, from equation (A.3), that the sum of the shadow value of debt and the shadow value of capital is zero. This means also that $\frac{\dot{\lambda}_1}{\lambda_1} = \frac{\dot{\lambda}_2}{\lambda_2}$. Combining this together with (A.3), (A.4) and (A.5) gives

$$\hat{r} = A(1-\alpha)\left(\frac{G}{K}\right)^{\alpha} = (1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}$$

where $\left(\frac{G}{K}\right)$ is given by (A.2). Hence the absence of investment adjustment costs constraints the exogenous interest rate to be equal to the internal marginal productivity of capital.

B Proof of Theorem 1

In this section we prove Theorem 1. We make use of the calculus of variations techniques applied to the sequence of the two control sub-problems shown in the main text.

Second Regime. The problem is

$$\max_{\{C,I\}} \qquad V_2 = \int_T^{+\infty} \ln C e^{-\rho t} dt$$

$$\text{s. t.} \qquad \dot{K} = I$$

$$\dot{B} = C + I + \left(\tfrac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \tfrac{h}{2}\tfrac{I^2}{K} - \tfrac{1}{\phi}B$$

$$\text{for a given } T, K(T), B(T)$$

The associated hamiltonian is

$$H^{(2)} = \ln C e^{-\rho t} + \lambda_1^{(2)} \left[C + I + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \frac{h}{2}\frac{I^2}{K} - \frac{1}{\phi}B \right] + \lambda_2^{(2)} I$$

We denote by the upper-script "*" the paths which satisfy the first order conditions. The corresponding value function depends on T, $B(T)$ and $K(T)$: $V_2^*(T, B(T), K(T))$. Moreover, since the problem is concave, the following envelope conditions apply

$$\frac{\partial V_2^*}{\partial T} = -H^{(2)*}(T) \tag{B.1}$$

$$\frac{\partial V_2^*}{\partial B(T)} = \lambda_1^{(2)}(T) \tag{B.2}$$

$$\frac{\partial V_2^*}{\partial K(T)} = \lambda_2^{(2)}(T) \tag{B.3}$$

First regime. The corresponding control sub-problem is

$$\max_{\{C,G,I\}} \qquad V_1 = \int_0^T \ln C e^{-\rho t} dt + V_2^*$$

$$\text{s. t.} \qquad \dot{K} = I$$

$$\dot{B} = C + I + G + \tfrac{h}{2}\tfrac{I^2}{K} + \hat{r}B - AG^{\alpha}K^{1-\alpha}$$

$$K(0), B(0) \text{ given, } T \text{ free}$$

and the corresponding hamiltonian is

$$H^{(1)} = \ln C e^{-\rho t} + \lambda_1^{(1)}\left[C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - AG^{\alpha}K^{1-\alpha}\right] + \lambda_2^{(1)} I$$

The value function can be rewritten in terms of the hamiltonian

$$V_1 = \int_0^T \left[H^{(1)} - \lambda_1^{(1)}\dot{B} - \lambda_2^{(1)}\dot{K}\right] dt + V_2^*\left(T, B(T), K(T)\right)$$

Standard integrations by parts yield

$$\int_0^T \lambda_1^{(1)}\dot{B} = \lambda_1^{(1)}(T)\, B(T) - \lambda_1^{(1)}(0)\, B(0) - \int_0^T \dot{\lambda}_1^{(1)} B dt$$

$$\int_0^T \lambda_2^{(1)}\dot{K} = \lambda_2^{(1)}(T)\, K(T) - \lambda_2^{(1)}(0)\, K(0) - \int_0^T \dot{\lambda}_2^{(1)} K dt$$

which allows to rewrite the value function as

$$V_1 = \begin{array}{c} \int_0^T \left[H^{(1)} + \dot{\lambda}_1^{(1)} B + \dot{\lambda}_2^{(1)} K\right] dt + V_2^*\left(T, B(T), K(T)\right) - \lambda_1^{(1)}(T)\, B(T) \\ + \lambda_1^{(1)}(0)\, B(0) - \lambda_2^{(1)}(T)\, K(T) + \lambda_2^{(1)}(0)\, K(0) \end{array}$$

First order variation of V_1 with respect to the state and the control variables' paths yields

$$\partial V_1 = \begin{array}{c} \int_0^T \left[\frac{\partial H^{(1)}}{\partial B}\partial B + \frac{\partial H^{(1)}}{\partial K}\partial K + \frac{\partial H^{(1)}}{\partial C}\partial C + \frac{\partial H^{(1)}}{\partial G}\partial G + \frac{\partial H^{(1)}}{\partial I}\partial I + \dot{\lambda}_1^{(1)}\partial B + \dot{\lambda}_2^{(1)}\partial K\right] dt \\ + \left(H^{(1)}(T) + \dot{\lambda}_1^{(1)}(T)\, B(T) + \dot{\lambda}_2^{(1)}(T)\, K(T)\right)\partial T \\ + \frac{\partial V_2^*(T,B(T),K(T))}{\partial T}\partial T + \frac{\partial V_2^*(T,B(T),K(T))}{\partial B(T)}\partial B(T) + \frac{\partial V_2^*(T,B(T),K(T))}{\partial K(T)}\partial K(T) \\ - \dot{\lambda}_1^{(1)}(T)\, B(T)\, \partial T - \lambda_1^{(1)}(T)\, \partial B(T) - \dot{\lambda}_2^{(1)}(T)\, K(T)\, \partial T - \lambda_2^{(1)}(T)\, \partial K(T) \end{array}$$

Reorganizing the terms

$$\partial V_1 = \begin{array}{c} \int_0^T \left[\left(\frac{\partial H^{(1)}}{\partial B} + \dot{\lambda}_1^{(1)} \right) \partial B + \left(\frac{\partial H^{(1)}}{\partial K} + \dot{\lambda}_2^{(1)} \right) \partial K + \frac{\partial H^{(1)}}{\partial C} \partial C + \frac{\partial H^{(1)}}{\partial G} \partial G + \frac{\partial H^{(1)}}{\partial I} \partial I \right] dt \\ + \left(H^{(1)}(T) + \frac{\partial V_2^*(T,B(T),K(T))}{\partial T} \right) \partial T + \left(\frac{\partial V_2^*(T,B(T),K(T))}{\partial B(T)} - \lambda_1^{(1)}(T) \right) \partial B(T) \\ + \left(\frac{\partial V_2^*(T,B(T),K(T))}{\partial K(T)} - \lambda_2^{(1)}(T) \right) \partial K(T) \end{array}$$

A trajectory is (locally) optimal if any (local) departure from it decreases the value function: $\partial V_1 \leq 0$ for any $\partial B(t)$ and $\partial K(t)$, $t \in (0, T)$, for any $\partial C(t)$ and $\partial G(t)$, $t \in [0, T]$, and for any ∂T, $\partial B(T)$ and $\partial K(T)$. This gives the following necessary conditions for an interior maximizer:

$$\begin{cases} \frac{\partial H^{(1)}}{\partial C} = 0 \\ \frac{\partial H^{(1)}}{\partial G} = 0 \\ \frac{\partial H^{(1)}}{\partial I} = 0 \\ \frac{\partial H^{(1)}}{\partial B} = -\dot{\lambda}_1^{(1)} \\ \frac{\partial H^{(1)}}{\partial K} = -\dot{\lambda}_2^{(1)} \\ H^{(1)}(T) + \frac{\partial V_2^*(T,B(T),K(T))}{\partial T} = 0 \\ \frac{\partial V_2^*(T,B(T),K(T))}{\partial B(T)} - \lambda_1^{(1)}(T) = 0 \\ \frac{\partial V_2^*(T,B(T),K(T))}{\partial K(T)} - \lambda_2^{(1)}(T) = 0 \end{cases}$$

The first five equations are the standard Pontryagin conditions, the last three may be interpreted as the optimality conditions with respect to the switching time T and the free state values $B(T)$ and $K(T)$. Combining these last three equations with equations (B.1), (B.2) and (B.3) obtained in the first sub-problem we obain the matching conditions stated in Theorem 1.

C The method of Lagrange

In this section we give the solution of the binding regime derived by applying the method of Lagrange. The method is perfectly equivalent to determine the intertemporal equilibrium of the model, but it is not able to determine in our case the endogenous switching time T. As it is shown in Section 3 and in Appendix B, the multi-stage optimal control allows to determine T, $B(T)$ and $K(T)$ by using the continuity conditions.

$$\mathcal{L} = \ln C e^{-\rho t} + \lambda_1 \left[G + C + I + \frac{h}{2} \frac{I^2}{K} + \hat{r} B - A G^{\alpha} K^{1-\alpha} \right] + \lambda_2 I + \eta \left[\phi A G^{\alpha} K^{1-\alpha} - B \right]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial C} = \frac{1}{C}e^{-\rho t} + \lambda_1 = 0 \tag{C.1}$$

$$\frac{\partial \mathcal{L}}{\partial G} = \lambda_1 - \lambda_1 A\alpha G^{\alpha-1}K^{1-\alpha} + \eta\phi A\alpha G^{\alpha-1}K^{1-\alpha} = 0 \tag{C.2}$$

$$\frac{\partial \mathcal{L}}{\partial I} = \lambda_1\left[1 + h\frac{I}{K}\right] + \lambda_2 = 0 \tag{C.3}$$

$$\frac{\partial \mathcal{L}}{\partial B} = \lambda_1\hat{r} - \eta = -\dot{\lambda}_1 \tag{C.4}$$

$$\frac{\partial \mathcal{L}}{\partial K} = -\lambda_1\left[\frac{h}{2}\frac{I^2}{K^2} + AG^{\alpha}\left(1-\alpha\right)K^{-\alpha}\right] + \eta\phi AG^{\alpha}\left(1-\alpha\right)K^{-\alpha} = -\dot{\lambda}_2 \tag{C.5}$$

$$\eta \geq 0 \quad \eta\left(\phi AG^{\alpha}K^{1-\alpha} - B\right) = 0 \tag{C.6}$$

$$\lim_{t\to\infty}\lambda_1 B = 0 \qquad \lim_{t\to\infty}\lambda_2 K = 0$$

By using the shadow value of public debt λ_1 as the numeraire and defining $q = -\frac{\lambda_2}{\lambda_1}$, we find $\frac{I}{K} = \frac{q-1}{h}$ from equation (C.3). Hence the growth rate of capital is $\frac{\dot{K}}{K} = \frac{q-1}{h}$.

From equation (C.4) $\frac{\eta}{\lambda_1} = \hat{r} + \frac{\dot{\lambda}_1}{\lambda_1}$. Using the definition of q this can be rewritten as $\frac{\eta}{\lambda_1} = \hat{r} + \frac{\dot{\lambda}_2}{\lambda_2} - \frac{\dot{q}}{q}$. We can use it in equation (C.5) that we rewrite in terms of $\frac{G}{Y}$ by using the prodiction function $\frac{G}{K} = A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}}$:

$$-\frac{\dot{\lambda}_2}{\lambda_2} = \frac{\frac{(q-1)^2}{2h} + \left(1 - \hat{r}\phi + \frac{\dot{q}}{q}\phi\right)A^{\frac{1}{1-\alpha}}\left(1-\alpha\right)\left(\frac{G}{Y}\right)^{\frac{\alpha}{1-\alpha}}}{q - \phi A^{\frac{1}{1-\alpha}}\left(1-\alpha\right)\left(\frac{G}{Y}\right)^{\frac{\alpha}{1-\alpha}}} \tag{C.7}$$

From equation (C.4) $\hat{r} - \frac{\eta}{\lambda_1} = -\frac{\dot{\lambda}_1}{\lambda_1}$. We can rewrite $\frac{\eta}{\lambda_1}$ by using equation (C.2) that has been rewritten in terms of $\left(\frac{G}{Y}\right)$: $\frac{\eta}{\lambda_1} = \frac{1}{\phi}\left[1 - \frac{1}{\alpha}\left(\frac{G}{Y}\right)\right]$, which states that, in order to be in the binding regime, the Lagrange multiplier is positive if and only if

$$\left(\frac{G}{Y}\right) > \alpha \tag{C.8}$$

Hence

$$\hat{r} - \frac{1}{\phi}\left[1 - \frac{1}{\alpha}\left(\frac{G}{Y}\right)\right] = -\frac{\dot{\lambda}_1}{\lambda_1}$$

and this one determines the growth rate of consumption

$$\frac{\dot{C}}{C} = -\frac{\dot{\lambda}_1}{\lambda_1} - \rho = \hat{r} - \frac{1}{\phi}\left[1 - \frac{1}{\alpha}\left(\frac{G}{Y}\right)\right] - \rho \tag{C.9}$$

Since $-\frac{\dot{\lambda}_1}{\lambda_1} = -\frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{q}}{q}$

$$\hat{r} - \frac{1}{\phi}\left[1 - \frac{1}{\alpha}\left(\frac{G}{Y}\right)\right] - \frac{\dot{q}}{q} = -\frac{\dot{\lambda}_2}{\lambda_2} \tag{C.10}$$

Combining (C.7) and (C.10) we can find the dynamics of $\dot{q}$:

$$\frac{\dot{q}}{q} = \frac{1}{\alpha}\left(\frac{1}{\phi\tau}\right)\left(\frac{G}{Y}\right) + \hat{r} - \frac{1}{\phi\tau} - \frac{1}{q}\frac{1-\alpha}{\alpha}A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} - \frac{(q-1)^2}{2hq} \tag{C.11}$$

We define $c \equiv \frac{C}{K}$ and $b \equiv \frac{B}{K}$ hence the dynamical system is equal to the one obtained in the paper

$$\begin{cases} \frac{\dot{q}}{q} = \frac{1}{\alpha}\left(\frac{1}{\phi}\right)\left(\frac{G}{Y}\right) + \hat{r} - \frac{1}{\phi} - \frac{1}{q}\frac{1-\alpha}{\alpha}A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} - \frac{(q-1)^2}{2hq} \\ \frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \frac{\dot{K}}{K} = \hat{r} - \frac{1}{\phi} + \frac{1}{\alpha}\frac{1}{\phi}\left(\frac{G}{Y}\right) - \rho - \frac{q-1}{h} \\ \frac{\dot{b}}{b} = \frac{\dot{B}}{B} - \frac{\dot{K}}{K} = \frac{1}{b}\left[c + \frac{q-1}{h} + A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} + \frac{(q-1)^2}{2h} + \hat{r}b - A^{\frac{1}{1-\alpha}}\left(\frac{G}{Y}\right)^{\frac{\alpha}{1-\alpha}}\right] - \frac{q-1}{h} \end{cases} \tag{C.12}$$

D The role of adjustment costs in the determination of the switching date

In Appendix A we have already shown that the small open economy endogenous growth model without investment adjustment costs needs the equality between the exogenous interest rate and marginal return of domestic capital to exist. In this section we prove that if we assume that $(1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} > \hat{r}$, then the constraint binds for all $t \geq 0$. The first order conditions with respect to C, I, G, B and K are given by equations(A.1)-(A.5). By solving equation (A.4) we obtain $\lambda(t) = \lambda_1(0)e^{-\hat{r}t}$. By substituting into (A.5) and solving for $\lambda_2(t)$ we obtain

$$\lambda_2(t) = -\frac{(1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\lambda_1(0)e^{-\hat{r}t}}{\hat{r}} + d$$

where $d \equiv \lambda_2(0) + \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\lambda_1(0)}{\hat{r}}$. It follows that $\lambda_1(t) + \lambda_2(t) > 0$ for $t > 0$ if and only if $de^{\hat{r}t} > -\lambda_1(0)\left[1 - \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}}{r}\right]$. For this inequality to hold it is sufficient that

$d > -\lambda_1(0)\left[1 - \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}}{r}\right]$ which happens when $(1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} > \hat{r}$, provided that $\lambda_1(0) + \lambda_2(0) > 0$. Hence the the constraint is binding from $t = 0$.

E The stock-based collaterals optimal growth model

$$\begin{array}{lc}
\max & U(C) = \int_0^{+\infty} \ln C e^{-\rho t} dt \\
\text{s. t.} & \dot{K} = I \\
& \dot{B} = C + I + G + \hat{r}B - AG^{\alpha}K^{1-\alpha} \\
& B = \phi K \\
& \text{for a given } K(0), B(0)
\end{array}$$

We use the constraints $\dot{K} = I$ and $B = \phi K$ to eliminate K:

$$\dot{B} = -\frac{\phi}{1-\phi}\left[C + G + \hat{r}B - A\left(\frac{1}{\phi}\right)^{1-\alpha} G^{\alpha}B^{1-\alpha}\right]$$

Hence the Hamiltonian is

$$H = \ln C e^{-\rho t} - \mu\frac{\phi}{1-\phi}\left[C + G + \hat{r}B - A\left(\frac{1}{\phi}\right)^{1-\alpha} G^{\alpha}B^{1-\alpha}\right]$$

and the first order conditions write as

$$\frac{1}{C}e^{-\rho t} - \mu\frac{\phi}{1-\phi} = 0$$

$$-\mu\frac{\phi}{1-\phi}\left[1 - A\alpha\left(\frac{1}{\phi}\right)^{1-\alpha} G^{\alpha-1}B^{1-\alpha}\right] = 0$$

$$\Longrightarrow \left(\frac{G}{K}\right)^{(2)} = (\alpha A)^{\frac{1}{1-\alpha}} \iff \left(\frac{G}{Y}\right) = \alpha$$

$$-\mu\frac{\phi}{1-\phi}\left[\hat{r} - A(1-\alpha)\left(\frac{G}{B}\right)^{\alpha}\left(\frac{1}{\phi}\right)^{1-\alpha}\right] = -\mu$$

$$\text{TVC} \lim_{t\to+\infty} \mu(t) B(t) = 0$$

Hence the economy's growth rate turns out to be equal to

$$g_{\text{stock-based}} = \frac{\phi}{1-\phi}\left[A^{\frac{1}{1-\alpha}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\frac{1}{\phi} - \hat{r}\right] - \rho \tag{E.1}$$

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