# Should a Country Invest more in Human or Physical Capital?
# A Two-Sector Endogenous Growth Approach

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### Abstract

Should a country invest more in human or physical capital? The present paper addresses this issue, considering the impact of different factor intensities between sectors on both optimal human and physical capital accumulation. Using a two-sector overlapping generations setting with endogenous growth driven by human capital accumulation, we prove that relative factor intensity between sectors drastically shapes the welfare analysis: two *laissez-faire* economies with the same global capital share may generate physical capital excess or scarcity, with respect to the optimum. The model for the Japanese economy, that experienced a factor intensity reversal after the oil shock, is then calibrated. It is shown that Japan invested relatively too much in human capital before 1975, but has not invested enough since 1990. .

**Keywords:**   Endogenous growth, social optimum, two-sector model, factor intensity differential

*JEL Classification Number*: E20; H52; O41

## 1   Introduction

Over the past 25 years, access to higher education has increased whereas household saving has fallen sharply. Chand (2008) draws attention to this question explaining the decline in the saving rate as a result of a trade-off between physical and human capital accumulation. The model we propose allows the relative significance of physical capital accumulation *vis-à-vis* human capital accumulation to be considered from a different perspective. We adopt an optimal view point,

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seeking to establish whether the relative significance of physical capital accumulation *vis-à-vis* human capital accumulation is efficient.

The optimal amount a country should invest in human or physical capital is usually analyzed through a basic aggregated one-sector framework, even though economists agree that this approach is too restrictive to describe the production process. Representing the whole economy through a one-sector structure does not allow sectoral differences and relative price adjustments between sectors to be considered. Empirical evidence suggests that sectoral relative prices vary (Hsieh and Klenow (2007)), especially between rich and poor economies. Valentinyi and Herrendorf (2008) also show that factor intensity is sector-dependent in the US economy. Zuelta and Young (2012) emphasize that the US labor income share within the agricultural and manufacturing sector fell between 1958 and 1996 whereas it increased in the service sector. This means that the apparent stability of the US global labor share hides contrasted evolutions of sectoral labor shares.

In a recent study, Takahashi *et al.* (2012) measure the capital intensity difference between consumption and investment goods sectors in the post-war Japanese economy and in other main OECD countries. Before the 1973 oil shock, the Japanese investment sector was capital intensive. They observe a capital intensity reversal after the oil shock, with the consumption sector becoming capital intensive compared to the investment sector. They suggest that a capital intensive investment sector in Japan before 1973 may explain the high speed growth observed over this period, as suggested by the Rybczynski theorem[1]. A rise in physical capital endowment increases production in the investment goods sector more than in the consumption goods sector. As the investment sector produces physical capital, this leads to a magnification effect.[2] According to Takahashi *et al.* (2012), a one-sector framework fails to account for this phenomenon. Consequently, considering the one-sector aggregated model to describe the economy as a whole may bias the analysis and leads to some properties that a two-sector approach may capture being ignored.

We depart from the Glomm and Ravikumar (1992) overlapping generations (OLG) model introducing a two-sector two-factor production structure *à la* Galor (1992). In this model, growth is driven by human capital accumulation. We distinguish between a consumption and an "investment" sector which use both human and physical capital. In this specific setting, the good produced in the investment sector is used for education spending and investment. Due to the two-sector structure, the relative price between the two goods plays a crucial role in the factor allocation between sectors. As a result, it matters both for human capital accumulation and for

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[1]The theorem states that a rise in the endowment of one factor will lead to a more than proportional expansion of the output in the sector which uses that factor intensively, at constant relative goods price.

[2]Unlike Japan, in other OECD countries, the consumption sector is capital intensive. There is neither a magnification effect nor capital intensity reversal.

economic growth. We analyze optimal physical and human capital accumulations and explain how these allocations depend on sectoral characteristics.

Our paper considers the literature on optimal balanced path in OLG models with human capital (Boldrin and Montes (2005) and Docquier *et al.* (2007)). It also addresses a second strand of literature dealing with optimal path and efficiency in a two-sector framework (e.g., Uzawa (1964)or Cremers (2006)). Most papers using a two-sector model of endogenous growth with human and physical capital accumulation consider only a final good sector - producing a good which can either be consumed or invested in physical capital - and an education sector [Bond *et al.* (1996), Mino *et al.* (2008)][3].

A contribution of this paper is to fully characterize the socially optimal balanced growth path in a two-sector framework with paternalistic altruism. The first-best optimum is defined by a social planner who maximizes the discounted sum of utility of all future generations. We prove that the sectoral differences in terms of capital intensities become crucial to characterize the social optimum. We develop a setting where the average capital share is constant, whereas sectoral capital intensities in the two sectors may change. Consider two *laissez-faire* (LF) economies with the same characteristics, except for relative factor intensities. These economies may generate physical capital excess or scarcity, with respect to the optimum, even if the global factor share is identical between these two countries. In a one-sector model, the sectoral capital intensities differential would be ignored: optimal factor accumulation would be the same for these two economies.

We define a Relative Factor Accumulation (RFA) reversal as a situation where a change in sectoral capital intensities makes the optimal global capital intensity higher (lower) than the LF when it was initially lower (higher) than the LF. For example, in a country where the optimal global physical to human capital ratio is lower than the LF, an optimal policy would favor human capital investment. If sectoral changes lead to an RFA reversal - which means that the optimal global physical to human capital ratio becomes higher than the competitive one - the optimal policy would be to favor physical capital accumulation. We emphasize that such RFA reversal may arise depending on the level of individuals' impatience. Then, to achieve the first-best, the government should consider a relationship between these sectoral capital intensities and the time preference, which does not exist in a one-sector setting. To sum up, relative capital intensity between sectors is crucial to determine the scheme of optimal policy.

The model for Japan is calibrated and we find that it experiences an RFA reversal between 1970 and 2000, due to sectoral changes. In our model, such a reversal affects the design of optimal policy. Based on the analysis of sectoral capital intensities, the model predicts that Japan did not invest enough in education when the consumption sector was capital intensive compared to

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[3]Bond *et al.* (1996) use a continuous-time model and Mino *et al.* (2008) use a discrete-time model with infinitely-lived agents. To our knowledge, in the literature, the two-sector two-factor formalization with education sector and final good sector is not used in the OLG model with education spending.

the investment sector.

The remainder of this paper is organized as follows. In Section 2, we set up the theoretical model. Section 3 is devoted to the planner's solution. In Section 4, we compare optimal and *laissez-faire* solutions. In Section 5, we propose a calibration of the model for Japan. Finally, Section 6 concludes.

## 2   The Model

### 2.1   The production structure

We consider an economy producing a consumption good $Y_0$ and a capital good $Y_1$. Each good is produced using physical capital $K_i$ and human capital $H_i$, with $i = \{0, 1\}$, through a Cobb-Douglas production function. The representative firm in each industry faces the following technology:

$$Y_0 = A_0 K_0^{\alpha_0} H_0^{1-\alpha_0} \tag{1}$$

$$Y_1 = A_1 K_1^{\alpha_1} H_1^{1-\alpha_1} \tag{2}$$

$$\alpha_1, \alpha_0 \in (0, 1) \qquad A_1 > 0, A_0 > 0$$

Full employment of factors holds so that, $K_{0t} + K_{1t} = K_t$, and $H_{0t} + H_{1t} = H_t$, where $K_t$ and $H_t$ are respectively the total stock of physical capital and the aggregate human capital at time $t$. We denote the physical to human capital ratio in sector $i$ by $k_i = K_i/H_i$, and the share of human capital to sector $i$, $h_i = H_i/H$ and obtain:

$$y_0 = A_0 k_0^{\alpha_0} \; ; \quad y_1 = A_1 k_1^{\alpha_1} \tag{3}$$

$$h_0 k_0 + h_1 k_1 = k, h_0 + h_1 = 1 \tag{4}$$

First order conditions give

$$w_t = (1 - \alpha_1)A_1 k_{1t}^{\alpha_1} = P_{0t}(1 - \alpha_0)A_0 k_{0t}^{\alpha_0} \tag{5}$$

$$R_t = \alpha_1 A_1 k_{1t}^{\alpha_1 - 1} = P_{0t}\alpha_0 A_0 k_{0t}^{\alpha_0 - 1} \tag{6}$$

where $w_t$ represents the wage, $R_t$ the rental rate of capital, and $P_0$ the relative price of the consumption good in terms of the investment good. From (5) and (6), we derive the physical to human capital ratios as functions of the price of the consumption good:

$$k_{0t} = B\left(\frac{\alpha_0(1-\alpha_1)}{\alpha_1(1-\alpha_0)}\right)(P_{0t})^{\frac{1}{\alpha_1-\alpha_0}}$$

$$k_{1t} = B(P_{0t})^{\frac{1}{\alpha_1-\alpha_0}} \tag{7}$$

$$with \; B = \left(\frac{\alpha_0}{\alpha_1}\right)^{\frac{\alpha_0}{\alpha_1-\alpha_0}} \left(\frac{A_0}{A_1}\right)^{\frac{1}{\alpha_1-\alpha_0}} \left(\frac{1-\alpha_1}{1-\alpha_0}\right)^{\frac{\alpha_0-1}{\alpha_1-\alpha_0}}$$

In this model, there are as many mobile factors as sectors, so that factor returns depends only on the relative price and do not depend on the global capital intensity $k$.

For simplicity, we assume a complete depreciation of the capital stock within one period, then: $K_{t+1} = I_t$.

## 2.2 Household's behavior

The economy is populated by finitely-lived agents. In each period $t$, $N$ persons are born, and they live for three periods. Following Glomm and Ravikumar (1992), we consider a paternalistic altruism whereby parents value the quality of education received by their children. In their first period of life, agents benefit from education. In their second period of life, when adult, they are endowed with $h_{t+1}$ efficiency units of labor that they supply inelastically to firms. Their income is allocated between current consumption, saving and investment in children's education. As we assume no population growth, we normalize the size of a generation to $N = 1$. In their third period of life, when old, agents retire. They consume the proceeds of their savings.

The preferences of a representative agent born at time $t - 1$ are represented by a log-linear utility function:

$$U(c_t, d_{t+1}, h_{t+1}) = (1 - \beta) \ln c_t + \beta \ln d_{t+1} + \gamma \ln h_{t+1} \tag{8}$$

$$0 < \beta < 1 \; ; \; 0 < \gamma < 1$$

where $\beta/(1 - \beta)$ is the psychological discount factor, $\gamma$ the degree of altruism, $c_t$ and $d_{t+1}$ correspond respectively to adult and old aggregate consumption, and $h_{t+1}$ child's human capital. Parents devote $e_t$ to their children's education. Human capital in $t + 1$ is given by:

$$h_{t+1} = be_t \tag{9}$$

Notice that even if there is no explicit externality in our simple human capital production function, grandparents' expenditure in education generate a positive intergenerational external effect. Indeed, parents decide for their child's education but do not consider the impact of this decision on their grandchild's human capital. The more educated children are, the more they earn when adults and invest in their own children's education.

An agent born at date $t - 1$ maximizes his utility function over his life cycle, with respect to budget constraints and human capital accumulation function :

$$\max_{s_t, e_t} U(c_t, d_{t+1}, h_{t+1}) = (1 - \beta) \ln c_t + \beta \ln d_{t+1} + \gamma \ln h_{t+1}$$

$$s.t \quad w_t h_t = P_{0t} c_t + s_t + e_t \tag{a}$$

$$s_t R_{t+1} = P_{0t+1} d_{t+1} \tag{b}$$

$$h_{t+1} = be_t \tag{c}$$

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First order conditions give the optimal education $e_t$, and the optimal saving $s_t$:

$$e_t = \frac{\gamma}{1+\gamma} w_t h_t \tag{10}$$

$$s_t = \frac{\beta}{1+\gamma} w_t h_t \tag{11}$$

## 2.3 Equilibrium

Since the size of the working age population is equal to one, the effective labor supply at time $t$ is $H_t = h_t$. The clearing condition in the capital market is $K_{t+1} = s_t$, and as by defintion $K_{t+1} = k_{t+1} H_{t+1}$, we have:

$$k_{t+1} = \frac{s_t}{h_{t+1}} \tag{12}$$

Using (9), (10), (11) and (12), we obtain the equilibrium physical to human capital ratio which is constant over time.

$$k_{t+1} = \frac{\beta}{b\gamma} = k \tag{13}$$

The consumption market clearing condition in period $t$ is:

$$c_t + d_t = Y_{0t}$$

$$c_t + d_t = A_0 k_{0t}{}^{\alpha_0} h_{0t} h_t \tag{14}$$

We define the growth rate as the growth rate of human capital:

$$1 + g_t = \frac{h_{t+1}}{h_t} \tag{15}$$

To highlight the impact of sectoral differences, we formulate this assumption:

**Assumption 1** *Let $\varepsilon$ be the factor intensity differential between consumption and investment sectors: $\varepsilon = \alpha_0 - \alpha_1$.*

Therefore, when $\varepsilon$ tends to zero, we have two identical sectors. And the larger $\varepsilon$ is, the larger will be differences between sectors. Hereafter, we focus on the Balanced Growth Path (BGP).

**Definition 1** *A Balanced Growth Path is an equilibrium path along which all variables grow at the same rate, that is the per-unit-of-effective-labor variables are constant.*

We can now characterize the economy's growth rate:

**Lemma 1** *On the balanced growth path, the growth rate is*

$$1 + g = \frac{\gamma}{(1+\gamma)} b A_1 w \tag{16}$$

*with*

$$w = (1 - \alpha_1)k_1^{\alpha_1}$$

*and*

$$k_1 = \frac{\beta}{b\gamma(1 - \alpha_1)} \frac{\alpha_1(1 + \gamma)(1 - \alpha_1 - \varepsilon)}{\varepsilon(1 - \beta) + \alpha_1(1 + \gamma)} \tag{17}$$

**Proof.** See Appendix 7.1.

The equilibrium growth rate depends on the degree of altruism $\gamma$ and on the psychological factor $\beta$, in the same way as in Michel and Vidal (2000).[4] In a two-sector framework, it is also shaped by the spread between sectoral factor intensities $\varepsilon$, through the wage.

**Proposition 1** *For $\alpha_1$ given, when $\varepsilon$ increases the competitive growth rate goes down.*

Indeed, at $\alpha_1$ given, the return on human capital is a decreasing function of $\varepsilon$. A rise in $\varepsilon$ (or in $\alpha_0$, when $\alpha_1$ is given) leads to a decrease in the marginal productivity of human capital of the consumption sector. Combining (5) and (6), we have

$$\frac{1 - \alpha_1}{\alpha_1}k_1 = \frac{1 - \alpha_0}{\alpha_0}k_0 \tag{18}$$

A rise in $\varepsilon$ shifts human capital from the consumption to the investment sector: capital intensity decreases in the investment sector and so does the wage. As a result, the wage is lower in an economy where the consumption sector is capital intensive ($\varepsilon > 0$), and so is the growth rate. When the investment sector is capital intensive ($\varepsilon < 0$), the higher $\varepsilon$ (in absolute value), the higher the growth rate. Conversely, when the consumption sector is capital intensive ($\varepsilon > 0$), the higher $\varepsilon$, the lower the growth rate. Notice that this feature of our endogenous growth model is consistent with Takahashi *et al.* (2012), who show that the investment good sector was capital intensive with respect to the consumption good sector ($\varepsilon > 0$) during the high speed growth period in Japan.

# 3   The social planner's problem

The social planner adopts a utilitarian viewpoint[5], and maximizes the discounted sum of all future generations' utilities while allocating output between the different activities. The maximization is subject to the clearing conditions on both good markets, the human capital accumulation equation, and the full employment of resources. In this two-sector setting, the planner has to allocate both capital stocks between the two sectors at the initial period ($t = 0$). We thus have two additional constraints with respect to the one sector case, corresponding to the full

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[4]Growth rate is an increasing function of $\beta$ and an increasing and then decreasing function of $\gamma$.

[5]See, Davin, Gente and Nourry (2012).

employment of resources at time 0. The planner's program is then given by:

$$\max_{c_t,d_t,K_{0t},K_{1t},H_{0t},H_{1t},K_{0t+1},K_{1t+1},H_{0t+1},H_{1t+1}} \sum_{t=-1}^{\infty} \delta^t \left((1-\beta)\ln c_t + \beta\ln d_{t+1} + \gamma\ln h_{t+1}\right) \quad (19)$$

$$\text{subject to: } \forall t \geqslant 0 \quad A_0 K_{0t}^{\alpha_0} H_{0t}^{1-\alpha_0} = c_t + d_t$$

$$A_1 K_{1t}^{\alpha_1} H_{1t}^{1-\alpha_1} = e_t + K_{t+1}$$

$$h_{t+1} = be_t$$

$$K_t = K_{0t} + K_{1t}$$

$$h_t = H_{0t} + H_{1t}$$

$$K_0 = K_{00} + K_{10}$$

$$h_0 = H_{00} + H_{10}$$

$$K_0, h_0, \text{ and } c_{-1} \quad \text{given}$$

We make use of the method of the infinite Lagrangian to characterize the optimal solution. The Lagrangian expression can be written as:

$$\mathcal{L} = \delta^{-1}((1-\beta)\ln c_{-1} + (1-\beta)\ln d_0 + \gamma\ln h_0) + \sum_{t=0}^{\infty}\delta^t\left((1-\beta)\ln c_t + \beta\ln d_{t+1} + \gamma\ln h_{t+1}\right) +$$

$$\sum_{t=0}^{\infty}\delta^t\left(q_{0t}\left(A_0 K_{0t}^{\alpha_0} H_{0t}^{1-\alpha_0} - c_t - d_t\right) + q_{1t}\left(A_1 K_{1t}^{\alpha_1} H_{1t}^{1-\alpha_1} - \frac{H_{0t+1} + H_{1t+1}}{b} - K_{0t+1} - K_{1t+1}\right)\right) \quad (20)$$

where $0 < \delta < 1$ is the discount factor, reflecting the social planner's time preference, $q_0$ and $q_1$ are multipliers associated with the resource constraints in both sectors.

We denote by a superscript asterisk (*) the optimal solution. The maximum of $\mathcal{L}$ with respect to $c_t$, $d_t$, $K_{0t}$, $K_{1t}$, $H_{0t}$, $H_{1t}$, $K_{0t+1}$, $K_{1t+1}$, $H_{0t+1}$, and $H_{1t+1}$, is reached when the following conditions are fulfilled for $t \geqslant 0$:

$$c_t^* = \frac{(1-\beta)}{q_{0t}^*} \quad ; \quad d_t^* = \frac{\beta}{\delta q_{0t}^*} \quad (21)$$

$$\delta q_{0t+1}^* \alpha_0 A_0 k_{0t+1}^{*\,\alpha_0-1} = q_{1t}^* \quad (22)$$

$$\delta q_{1t+1}^* \alpha_1 A_1 k_{1t+1}^{*\,\alpha_1-1} = q_{1t}^* \quad (23)$$

$$\delta q_{0t+1}^*(1-\alpha_0)A_0 k_{0t+1}^{*\,\alpha_0} = \frac{q_{1t}^*}{b} - \frac{\gamma}{h_{t+1}^*} \quad (24)$$

$$\delta q_{1t+1}^*(1-\alpha_1)A_1 k_{1t+1}^{*\,\alpha_1} = \frac{q_{1t}^*}{b} - \frac{\gamma}{h_{t+1}^*} \quad (25)$$

$$q_{00}^* \alpha_0 A_0 k_{00}^{*\,\alpha_0-1} = q_{10}^* \alpha_1 A_1 k_{10}^{*\,\alpha_1-1} \quad (26)$$

$$q_{00}^*(1-\alpha_0)A_0 k_{00}^{*\,\alpha_0} = q_{10}^*(1-\alpha_1)A_1 k_{10}^{*\,\alpha_1} \quad (27)$$

$$A_0 k_{0t}^{\alpha_0 *} h_{0t}^* h_t^* = c_t^* + d_t^* \tag{28}$$

$$A_1 k_{1t}^{* \alpha_1} h_{1t}^* h_t^* = h_{t+1}^* \left( \frac{1}{b} + k_{0t+1}^* h_{0t+1}^* + k_{1t+1}^* h_{1t+1}^* \right) \tag{29}$$

Transversality conditions are:

$$\lim_{t \to \infty} \delta^t q_{1t}^* K_{t+1}^* = 0, \quad \lim_{t \to \infty} \delta^t z_t^* h_{t+1}^* = 0 \tag{30}$$

where $z^*$ is the shadow price of human capital.

For initial conditions $c_{-1}$, $K_0$, $h_0$ and for all $t \geqslant 0$ an optimal solution is defined as satisfying equations (21) to (30).[6]

Eliminating shadow prices from the first order conditions (FOC) and rearranging the terms, we obtain conditions that characterize optimal solutions. From the FOC (21), sectoral implicit prices must equal the marginal utility of consumption. The intertemporal allocation of consumption between the two goods is thus obtained:

$$\beta c_t^* = \delta(1 - \beta) d_t^* \tag{31}$$

Optimal allocation is such that, worker's marginal utility for consumption and retired person's marginal utility for consumption - discounted by the factor $\delta$ - are the same.

From (21), (22) and (26), we obtain the optimal growth rate of consumption in $t + 1$:[7]

$$\frac{c_{t+1}^*}{c_t^*} - 1 = \frac{k_{0t+1}^{* \alpha_0 - 1}}{k_{0t}^{* \alpha_0 - 1}} \delta \alpha_1 A_1 k_{1t}^{* \alpha_1 - 1} - 1 \tag{32}$$

We analyze the optimal solution along the BGP and derive the following Lemma:

**Lemma 2** *Along the BGP, welfare is maximized according to the modified golden rule, $\delta f'(k_1^*) = 1 + g^*$. There exists a unique unstable steady state equilibrium $k_1^*$ :*

$$k_1^* = \frac{\alpha_1}{b(1 - \alpha_1)} \frac{\mathcal{S} + \delta \gamma (1 - \alpha_1)}{\mathcal{S} + \gamma (1 - \delta \alpha_1)} \tag{33}$$

*with $\mathcal{S} = ((1 - \beta)\delta + \beta)(\varepsilon(\delta - 1) + 1 - \alpha_1)) > 0$. Moreover, $\frac{\partial g^*}{\partial \gamma} > 0$, $\frac{\partial g^*}{\partial \varepsilon} > 0$ and $\frac{\partial g^*}{\partial \beta} < 0$.*

**Proof.** See Appendix 7.2.

The optimal growth rate corresponds to the modified golden rule in the investment good sector. Due to the decreasing returns in physical capital accumulation, there is a negative relationship

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[6]Since the problem is concave, these first order conditions (FOC) are sufficient.

[7]Integrating equation (21), we have $\frac{c_{t+1}^*}{c_t^*} = \frac{q_{0t}^*}{q_{0t+1}^*}$. With (22) and (23) we obtain, $\frac{c_{t+1}^*}{c_t^*} = \frac{q_{1t}^*}{q_{0t+1}^*} \frac{\alpha_1 A_1 k_{1t}^{* \alpha_1 - 1}}{\alpha_0 A_0 k_{0t}^{* \alpha_0 - 0}}$. Then, with (22) we deduce (32).

between the optimal growth rate $g^*$ and sectoral factor intensity $k_1$ and $k_0$.[8] According to the expression of $\mathcal{S}$, the optimal value $k_1^*$ decreases with $\gamma$ and rises with $\beta$ since $\delta < 1$. The more altruistic individuals are, the higher the optimal level of human capital and the lower the ratio $k_1^*$. Thus, when preferences for human capital go up the optimal growth rate increases. When $\beta$ goes up the agent becomes more patient. To improve his future consumption he has to invest in physical capital, which is the only way to smooth consumption. As a result, the physical to human capital ratio increases in both sectors and the optimal growth rate goes down. We recover the competitive equilibrium: a negative relationship between $\varepsilon$ and $k_1^*$, (for a given $\alpha_1$). Thus, the central planner implements the modified golden rule and the growth rate decreases with $k_1^*$. To sum up, for $\alpha_1$ given, when $\varepsilon$ increases the optimal growth rate is higher.

Using the equilibrium in the investment sector (29), along the BGP we have:

$$h_1^* = \frac{(1+g^*)(\frac{1}{b}+k^*)}{A_1 k_1^{*\alpha_1}} \tag{34}$$

From equations (4), (22) to (25)[9], and (34), we can write $k^*$ as a function of $k_1^*$ and obtain the following global physical to human capital ratio:

$$k^* = \frac{b(\varepsilon + \alpha_1)(1-\alpha_1)k_1^* - \delta\alpha_1\varepsilon}{\alpha_1 b((1-\alpha_1) + \varepsilon(\delta-1))} \tag{35}$$

Through equations (2), ( 34) and (35) it can be seen that optimal capital intensity $(k^*)$, and optimal factor allocation between sectors $(k_1^*$ and $h_1^*)$ depend on $\varepsilon$. The optimal ratio $k^*$ is an increasing function of $\varepsilon$. As a result, a change in the spread between sectoral factor intensities affects the optimal allocation of factors between sectors and the optimal factor accumulation.

## 4  *Laissez-faire* and the social optimum

We are interested in the trade-off between investment in education and investment in physical capital. More precisely, we consider the role of the two-sector feature on this trade-off. The physical to human capital ratio, $k$, is an indicator of the relative physical to human capital investment in the economy. Indeed, the higher $k^*$ the more the government has to invest in physical capital relative to human capital. We know from the previous section that the optimal ratio $k^*$ is affected by the relative factor intensity differential $\varepsilon$. As mentioned in the Introduction, empirical studies show that an apparent constant average factor share may hide sectoral factor share $(\alpha_0$ and $\alpha_1)$ changes. Let us consider the average capital share $\bar{\alpha} = (\alpha_0 + \alpha_1)/2$, and the deviation from this average, $\Upsilon$, with $\Upsilon \in [\max\{-\bar{\alpha}, \bar{\alpha}-1\} \; ; \; \min\{\bar{\alpha}, 1-\bar{\alpha}\}]$. As a result, the

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[8]Along the BGP, growth rate is given by $1 + g^* = \delta A_1 \alpha_1 k_1^{*\alpha_1-1}$, hence it is decreasing in $k_1^*$. Using equations (22) to (25), we can express the growth rate as $1 + g^* = \delta A_1 \alpha_1 \left(k_0^* \frac{\alpha_1(1-\alpha_0)}{\alpha_0(1-\alpha_1)}\right)^{\alpha_1-1}$, hence it is decreasing in $k_0^*$.

[9]Equations (22) to (25) gives the standard relationship $\frac{k_{1t+1}^*}{k_{0t+1}^*} = \left(\frac{1-\alpha_0}{1-\alpha_1}\right)\frac{\alpha_1}{\alpha_0}$.

consumption and investment sector physical capital share are respectively given by $\alpha_0 = \bar{\alpha} + \Upsilon$ and $\alpha_1 = \bar{\alpha} - \Upsilon$, and the factor intensity differential is given by $\varepsilon = 2\Upsilon$. This section aims to examine how changes in sectoral factor shares affect the ratio $k^*$, whereas the average capital share remains the same. We obtain the optimal physical to human capital ratio $k^*$ as a function of $\Upsilon$:

**Lemma 3** *The physical to human capital ratio at the first-best as a function of $\Upsilon$ is given by:*

$$k^* = \frac{(\bar{\alpha} + \Upsilon(1 - 2\delta)) \left[\Upsilon(\psi(2\delta - 1) + \delta\gamma) + (1 - \bar{\alpha})\psi\right] + \delta\gamma\left[(1 - \bar{\alpha})\bar{\alpha} - \Upsilon(1 + \bar{\alpha}(1 - 2\delta))\right]}{b\left[\Upsilon(\psi(2\delta - 1) + \delta\gamma) + (1 - \bar{\alpha})\psi + \gamma(1 - \delta\bar{\alpha})\right](1 - \bar{\alpha} - \Upsilon(1 - 2\delta))} \quad (36)$$

*where $\psi = ((1 - \beta)\delta + \beta)$.*
*Moreover, $\partial k^*/\partial\Upsilon > 0$ (resp. $< 0$) when $\beta < \tilde{\beta}$ (resp. $\beta > \tilde{\beta}$), with : $\tilde{\beta} = \delta((2\delta - 1) + \gamma)/(2\delta - 1)(\delta - 1)$.*

**Proof.** see Appendix 7.3.

This Lemma underlines the non-trivial effects of sectoral factor share movements on the aggregate physical to human capital ratio. In an economy where agents are impatient ($\beta$ low), an increase in capital intensity in the consumption sector ($\Upsilon > 0$) raises the optimal physical capital accumulation even when the capital intensity in the investment sector decreases. Conversely, when the consumption sector becomes more intensive in human capital ($\Upsilon < 0$), it is optimal to increase human capital accumulation relatively to physical capital investment. This means that when the weight that agent gives to future consumption is sufficiently low, the changes in the consumption sector have a more substantial impact on the optimal factor accumulation than the changes in the investment sector. We observe the opposite result when agents value their future consumption more ($\beta$ high), *i.e*, the effect of changes in the investment sector prevails over those in the consumption sector.

Due to externalities, the relative physical to human capital ratio in the decentralized economy, given by (13), differs from the first-best. The positive externality in education entails an under-accumulation of human capital in *laissez-faire*, whereas we can observe under- or over-accumulation of physical capital. As a result, depending on $\beta$, the competitive ratio $k$ may be higher or lower than the first-best $k^*$. A higher (lower) optimal capital intensity than the competitive one means that there is an under-accumulation (over-accumulation) of physical capital. In this case, optimal policy should favor investment in infrastructure (education) rather than in education (infrastructure). Regarding the optimal and *laissez-faire* physical to human capital ratio, the relative importance of factor accumulation may switch, and we can formulate the following definition:

**Definition 2** *There is a relative factor accumulation (RFA) reversal, when the sign of the term $\mathcal{K} = k^* - k_{LF}$ changes.*

From equations (13) and (36), we know that $\mathcal{K} \equiv \mathcal{K}(\Upsilon, \beta)$.

For simplicity's sake, we formulate this assumption, as relevant for a developed economy:

**Assumption 2** $\bar{\alpha} < 1/2$, *thus* $\Upsilon \in [-\bar{\alpha}\,;\,\bar{\alpha}]$.

Using Lemma 3 and Definition 2, we compare the RFA at the optimum and the *laissez-faire*, according to sectoral factor intensity changes:

**Proposition 2** *Under Assumption 2, there exist two critical bounds $\bar{\beta}_1$ and $\bar{\beta}_2$ such that:*
*i) For $\bar{\beta}_1 < \bar{\beta}_2 < \tilde{\beta}$: When $\beta < \bar{\beta}_1$, optimal physical to human capital ratio is always higher than the* laissez-faire. *When $\bar{\beta}_1 < \beta < \bar{\beta}_2$, $\exists\ \overline{\Upsilon}$ characterizing RFA reversal. When $\beta > \bar{\beta}_2$, optimal physical to human capital ratio is always lower than the laissez-faire.*
*ii) For $\tilde{\beta} < \bar{\beta}_2 < \bar{\beta}_1$: When $\beta > \bar{\beta}_1$, competitive physical to human capital ratio is always higher than the first-best. When $\bar{\beta}_2 < \beta < \bar{\beta}_1$, $\exists\ \overline{\Upsilon}$ characterizing RFA reversal. When $\beta < \bar{\beta}_2$, competitive physical to human capital ratio is always lower than the first best.*

**Proof.** see Appendix 7.4

Proposition 2 provides conditions for the existence of an RFA reversal generated by sectoral changes. The results are depicted by Figure 1 which represents $\mathcal{K}$.
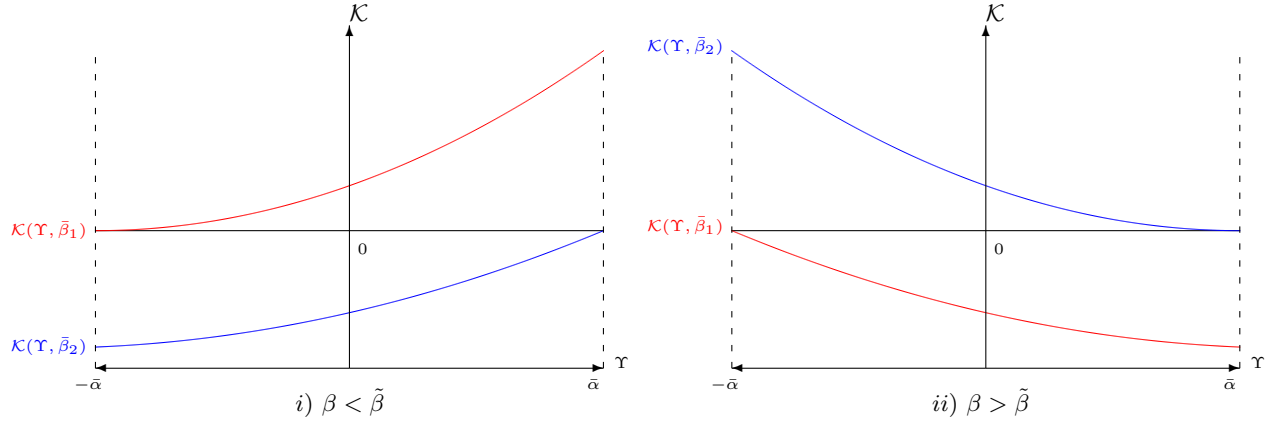


**Figure 1:** *Gap between the optimal and the laissez-faire physical to human capital ratio*

Figure 1 allows three regimes in each case to be distinguished. Consider the first case, $i$). When agents are impatient, such as $\beta < \bar{\beta}_1 < \bar{\beta}_2 < \tilde{\beta}$, the optimal physical to human capital ratio is higher than the *laissez-faire*, whatever the relative factor intensity between sectors ($\varepsilon$).[10] In this case, sectoral factor share movements do not lead to an RFA reversal. When $\bar{\beta}_1 < \beta < \bar{\beta}_2 < \tilde{\beta}$, the optimal ratio may be higher or lower than the *laissez-faire* depending on the factor intensities differential between sectors. In a country where the consumption sector is relatively more capital intensive, the optimal policy would favor physical capital relative to human capital investment. Indeed, the higher $\varepsilon$, the more likely is under-accumulation of physical capital. Conversely, in a country with a consumption sector that is human capital intensive, the optimal policy would

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[10]As $\varepsilon = 2\Upsilon$, we can interpret the results in terms of factor intensity differential between sectors.

favor human capital investment. Lastly, for a lower degree of time preference ($\beta$ high), such that $\bar{\beta}_1 < \bar{\beta}_2 < \beta < \tilde{\beta}$, the agent accumulates too much physical capital relative to human capital, and hence, there is no RFA reversal. We obtain opposite results in the case $ii$).

The results of Proposition 2 are due to the fact that optimal physical and human capital accumulation depend on the two-sector structure. Thus, considering a two-sector model with human and physical capital, we emphasize an important result: when countries experience a factor intensity reversal, as was the case in the postwar Japanese economy, the scheme of optimal capital accumulation may be affected.

We can formulate the following corollary, which gives indications about optimal policy allowing to achieve the first-best optimum:

**Corollary 1** *i) For $\bar{\beta}_1 < \beta < \bar{\beta}_2 < \tilde{\beta}$: $\exists \; \bar{\Upsilon}$, such that if $\Upsilon > \bar{\Upsilon}$ (resp. $\Upsilon < \bar{\Upsilon}$ ) it is efficient to invest relatively more (resp. less) in physical capital.*
*ii) For $\tilde{\beta} < \bar{\beta}_2 < \beta < \bar{\beta}_1$: $\exists \; \bar{\Upsilon}$, such that if $\Upsilon > \bar{\Upsilon}$ (resp. $\Upsilon < \bar{\Upsilon}$ ) it is efficient to invest relatively more (resp. less) in human capital.*

## 5  Discussion: the Japanese case

In a recent paper, Takahashi *et al.* (2012) measure the capital intensities in the investment and consumption sectors in Japan. They notice a capital intensity reversal after the 1973 oil shock. Indeed, before 1973, the Japanese economy characterized by a relatively capital intensive investment sector, whereas the consumption sector was capital intensive after the oil shock. Using their computations, we calibrate our model to emphasize the existence of an RFA reversal in Japan, generated by changes at sectoral level. We assume that each period has a length of 30 years. Following Evans and Sezer (2004), estimation for the yearly discount factor for Japan is 0.985 and corresponds to $\beta$ is 0.39.[11] The social discount factor $\delta$ is 0.95.[12] We calibrate the scale parameter in the human capital accumulation function, $b$, such that the sectoral capital intensities ($k_0$ and $k_1$) match the ones computed by Takahashi *et al.* (2012) (see, Table 1). The altruism factor drives human capital accumulation and so the growth rate of the economy. For more robustness, we consider two possible scenarios: $\gamma = 0.45$ and $\gamma = 0.65$.

Data in Table 1 come from Takahashi *et al.* (2012) (for $k_0$, $k_1$, $k$), and the present author's computations (for $\alpha_1$, $\alpha_0$, $\bar{\alpha}$, $\Upsilon$, $\varepsilon$). Using equations (3), (18) and sectoral data, we determine capital share in both sectors and average capital share in 1955. Then, we compute for each year the sectoral capital share considering equation (18) and $\alpha_0 = \bar{\alpha} + \Upsilon$ and $\alpha_1 = \bar{\alpha} - \Upsilon$, in order to compute $\Upsilon$.

Using Lemma 3, Proposotion 2, Table 1 and calibrations for $\delta$, $\beta$ and $\gamma$, we obtain the following critical bounds:

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[11]According to Evans and Sezer (2004), the pure time preference rate $r$ is 1.5%. Thus, the discount factor is given by $\left(\frac{1}{1+r}\right)^{30} = 0.639$, which corresponds in our model to $\beta/(1-\beta)$. Hence we get $\beta = 0.39$.

[12]Based on the Ramsey formula (1928), Evans and Sezer (2004) estimate the social time preferences rate at 5% for Japan. Thus, the social discount factor $\delta^t$ is given by $(1/1,05)^t$.

|        | 1955    | 1960    | 1965    | 1970    | 1975    | 1980   | 1985   | 1990    | 1995    | 2000    |
|--------|---------|---------|---------|---------|---------|--------|--------|---------|---------|---------|
| $k_0$  | 0,8378  | 1,1166  | 1,8020  | 2,9716  | 4,7794  | 6,5399 | 8,8666 | 12,5774 | 16,6877 | 18,1378 |
| $k_1$  | 1,0361  | 1,2171  | 2,0187  | 3,4760  | 4,8492  | 6,2347 | 8,5745 | 11,2214 | 14,8834 | 18,3598 |
| $\alpha_1$ | 0,4065 | 0,3917 | 0,3949 | 0,4000 | 0,3832 | 0,3759 | 0,3776 | 0,3681 | 0,3680 | 0,3829 |
| $\alpha_0$ | 0,3565 | 0,3714 | 0,3681 | 0,3630 | 0,3798 | 0,3872 | 0,3855 | 0,3950 | 0,3950 | 0,3801 |
| $\bar{\alpha}$ | 0,3815 | 0,3815 | 0,3815 | 0,3815 | 0,3815 | 0,3815 | 0,3815 | 0,3815 | 0,3815 | 0,3815 |
| $\Upsilon$ | 0,0000 | -0,0102 | -0,0134 | -0,0185 | -0,0017 | 0,0056 | 0,0040 | 0,0135 | 0,0135 | -0,0014 |
| $\varepsilon$ | -0,0501 | -0,0203 | -0,0268 | -0,0370 | -0,0034 | 0,0113 | 0,0079 | 0,0269 | 0,0270 | -0,0029 |

Source: Takahashi *et al.* (2012) and the present author's computations

**Table 1:** *Data for Japanese Economy*

| Parameters | Values |
|------------|--------|
| $\tilde{\beta}$ | -32.72 |
| $\bar{\beta}_1$ | 1.7123 |
| $\bar{\beta}_2$ | 0.0073 |
| $\bar{\Upsilon}$ | -0.0013 |

**Table 2:** *Critical bounds for $\gamma = 0.65$*

In accordance with Lemma 3, Proposition 2, and Table 2, $k^*$ is decreasing with $\Upsilon$ in Japan and there is an RFA reversal when $\bar{\Upsilon} = -0.0013$. As a result, from 1955 to 1975, optimal physical to human capital ratio was too high in Japan, whereas the opposite was true after 1975.
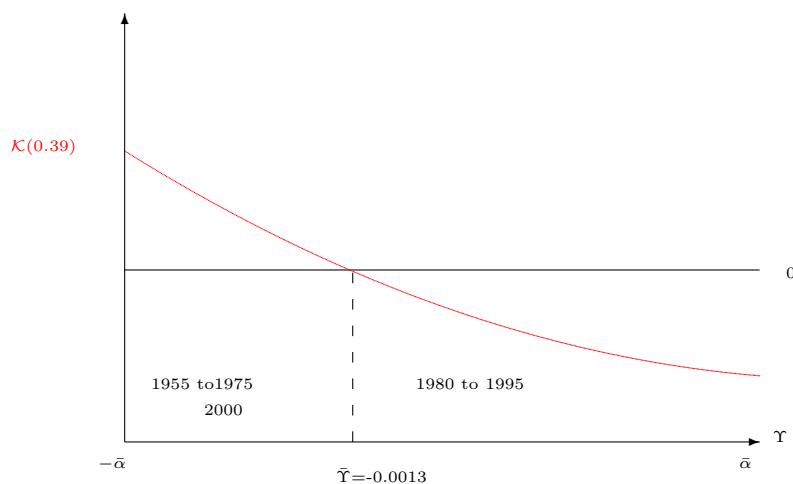


**Figure 2:** *Japanese case*

Two sources of inefficiency are likely to interact in an OLG model with both human and physical capital. A positive intergenerational externality in education and inefficiencies in physical capital accumulation. As a result, a physical to human capital ratio lower than the first best can be due to under-accumulation of human capital and/or over-accumulation of physical capital. Whatever the case, our calibrations stress that Japan should have invested more intensively in education from 1980 to 1995 (see figure 2). We examine whether Japanese economic policy was
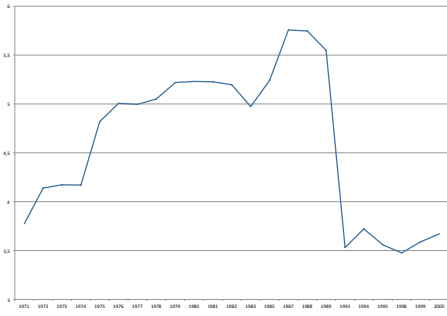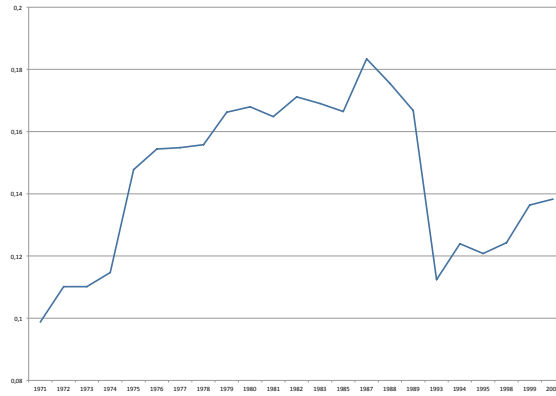
**Figure 3:** *Public expenditure on education (% of GDP)*

Source: UNdata
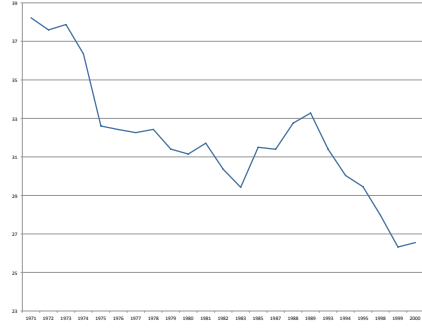


**Figure 4:** *Gross domestic savings (% of GDP)*

Source: The World Bank

in line with the optimal solution. We focus on the relative factor accumulation and thus analyze the government's intervention in education relative to domestic saving, which determines physical capital accumulation. The evolution of public education spending and gross domestic saving in Japan from 1970 to 2000 is considered.



**Figure 5:** *Ratio of public expenditure on education over domestic saving*

Figures 3, 4 and 5 show that from 1970 to 1985, public education spending went up and saving went down, reducing the physical to human capital ratio. This policy was not optimal between 1970 and 1975, because the ratio is already lower than the first-best. From 1975 to 1985, policy seems qualitatively appropriate as the optimal ratio is lower than the *laissez-faire*. We notice that public education expenditure fell sharply in the early 1990s. Moreover, it decreased more strongly than saving whereas our model highlights that it would have been optimal to encourage public education spending. Indeed, during this period, relative capital intensity between sectors continues to increase in favor of the consumption sector, hence the negative spread between the optimal and the *laissez-faire* physical to human capital ratio goes up and has to be offset by a more intensive education policy. As a result, considering sectoral differences, we assert that Japanese policy was not optimal.

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# 6 Conclusion

The unbalanced nature of industry factor shares highlights how crucial it is to consider at least two-sector models to design optimal policy. Indeed, whereas an aggregated model hides the difference in factor intensities across sectors, a multi-sector model allows such characteristics to be taken into account. In this paper, we develop a two-sector model and we underline the importance of factor intensities differences to design an optimal balanced growth path and optimal policy. We conclude that changes in sectoral factor shares may imply a relative factor intensity reversal and thus affect the optimal accumulation of human and physical capital. A factor intensity differential between sectors should then be considered to determine the scheme of optimal policy. We have shown that the two-sector model is tractable enough to conduct such an analysis. Our calibrations for the Japanese economy illustrate our results.

# 7 Appendix

## 7.1 Proof of Lemma 1

From the consumer budget constraints (a) and (b) and equations (10) and (11), we have:

$$P_{0t}c_t = w_t h_t \frac{1-\beta}{1+\gamma} \tag{37}$$

$$P_{0t+1}d_{t+1} = R_{t+1}\left(\frac{\beta}{1+\gamma}w_t h_t\right) \tag{38}$$

Substituting these last expressions in the consumption goods market equilibrium (14) gives:

$$\frac{1}{(1+\gamma)}\left(w_t h_t(1-\beta) + R_t\beta w_{t-1}h_{t-1}\right) = P_{0t}A_0 k_{0t}{}^{\alpha_0}h_t h_{0t} \tag{39}$$

We divide this expression by $h_t$ and substitute (10) for $h_{t-1}$:

$$\frac{1}{(1+\gamma)}\left(w_t(1-\beta) + R_t\frac{\beta}{b\gamma}(1+\gamma)\right) = P_{0t}A_0 k_{0t}{}^{\alpha_0}h_{0t} \tag{40}$$

As full employment of factors holds, we have $k_1 h_1 + k_0 h_0 = k$. From (4) this can be written:

$$k_1(1-h_0) + k_0 h_0 = k \quad \Rightarrow \quad h_0 = \frac{k-k_1}{k_0-k_1} \tag{41}$$

Including (41) in equation (40) we obtain:

$$\frac{1}{(1+\gamma)}\left(w_t(1-\beta) + R_t\frac{\beta}{b\gamma}(1+\gamma)\right) = P_{0t}A_0 k_{0t}^{\alpha_0}\frac{k_t-k_{1t}}{k_{0t}-k_{1t}}$$

According to (13) for $t > 0$, $k_{t+1} = k_t = k$. Using (7) and (13):

$$\frac{1}{(1+\gamma)}\left(w_t(1-\beta) + R_t k(1+\gamma)\right) =$$

$$P_{0t} A_0 \left(\frac{\alpha_0(1-\alpha_1)}{\alpha_1(1-\alpha_0)}\right)^{\alpha_0} B^{\alpha_0}(P_{0t})^{\frac{\alpha_0}{\alpha_1-\alpha_0}}\left(\frac{k-k_{1t}}{B(P_{0t})^{\frac{1}{\alpha_1-\alpha_0}}\left(\frac{\alpha_0-\alpha_1}{\alpha_1(1-\alpha_0)}\right)}\right)$$

fixing $D \equiv \frac{(\alpha_0(1-\alpha_1))^{\alpha_0}(\alpha_1(1-\alpha_0))^{1-\alpha_0}}{\alpha_0-\alpha_1}$ :

$$\frac{1}{(1+\gamma)}\left(w_t(1-\beta) + R_t k(1+\gamma a)\right) = P_{0t} A_0 D k_{1t}^{\alpha_0-1}(k-k_{1t})$$

We replace $P_{0t}$ using (7) and factor returns by (5) and (6):

$$\frac{1}{(1+\gamma)}\left((1-\alpha_1)A_1 k_{1t}^{\alpha_1}(1-\beta) + \alpha_1 A_1 k_{1t}^{\alpha_1-1}k(1+\gamma)\right) = \frac{k_{1t}^{\alpha_1-\alpha_0}}{B^{\alpha_1-\alpha_0}}A_0 D k_{1t}^{\alpha_0-1}(k-k_{1t})$$

$$with \quad \frac{D}{B^{\alpha_1-\alpha_0}} = \frac{(\alpha_0(1-\alpha_1))^{\alpha_0}(\alpha_1(1-\alpha_0))^{1-\alpha_0}}{(\alpha_0-\alpha_1)}\left(\frac{\alpha_1}{\alpha_0}\right)^{\alpha_0}\left(\frac{A_1}{A_0}\right)\left(\frac{1-\alpha_0}{1-\alpha_1}\right)^{\alpha_0-1} \equiv \frac{(1-\alpha_1)\alpha_1}{(\alpha_0-\alpha_1)}\frac{A_1}{A_0}$$

As a result, the physical to human capital ratio in the investment good sector, $k_1$, is constant and we finally obtain, for $t > 0$:

$$k_1 = k\frac{\alpha_1}{1-\alpha_1}\frac{(1-\alpha_1)-(\alpha_0-\alpha_1)}{\left(\frac{1}{1+\gamma}\right)(1-\beta)(\alpha_0-\alpha_1)+\alpha_1}$$

To express the equilibrium growth rate, we use (15) with (9) and (10):

$$1 + g_t = \frac{\gamma}{1+\gamma}bAw_t$$

As equilibrium physical to human capital ratio is constant at secotral level, from (5), the return of human capital is constant as well. We have $w_t = w_{t+1} = w$, hence we obtain a balanced growth path along which the variables chosen by agents ($s_t$, $e_t$, $c_t$ and $d_{t+1}$) grow at the same constant rate as individual human capital, $g$.

$\square$

## 7.2 Proof of Lemma 2

Using equations (28) and (29) at time $t+1$, and the relationship $h_i^* = H_i^*/H^*$ (with $i = \{0,1\}$), gives:

$$A_1 k_{1t+1}^{*\alpha_1} H_{1t+1}^* - \frac{h_{t+2}^*}{b} - k_{1t+2}^* H_{1t+2}^* - k_{0t+2}^* H_{0t+2}^* = 0 \tag{42}$$

and

$$A_0 k_{0t+1}^{*\alpha_0} H_{0t+1}^* - c_{t+1}^* - d_{t+1}^* = 0$$

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Integrating (31) in the last equation, we can write:

$$H_{0t+1}^* = \frac{c_{t+1}^*}{A_0 k_{0t+1}^{*\alpha_0}} \psi \tag{43}$$

with $\psi = \left(\frac{(1-\beta)\delta+\beta}{(1-\beta)\delta}\right)$.

Using FOC (23) and (25), we obtain the following relationship:

$$h_{t+1}^* = \frac{1}{q_{1t}^*}\left(\frac{\gamma b \alpha_1}{\alpha_1 - (1-\alpha_1)k_{1t+1}^* b}\right) \tag{44}$$

At each time, $h = H_1 + H_0$. Considering this relation, we can rewrite (42):

$$A_1 k_{1t+1}^{*\alpha_1}(h_{t+1}^* - H_{0t+1}^*) - \frac{h_{t+2}^*}{b} - k_{1t+2}^*(h_{t+2}^* - H_{0t+2}^*) - k_{0t+2}^* H_{0t+2}^* = 0$$

and substitute $h_{t+1}^*$ and $h_{t+2}^*$ from equation (44) and $H_{0t+1}^*$ and $H_{0t+2}^*$ from equation (43). We obtain :

$$A_1 k_{1t+1}^{*\alpha_1}\left[\frac{1}{q_{1t}^*}\left(\frac{\gamma b \alpha_1}{\alpha_1 - (1-\alpha_1)k_{1t+1}^* b}\right) - \frac{c_{t+1}^*}{A_0 k_{0t+1}^{*\alpha_0}}\psi\right] = \frac{1}{q_{1t+1}^*}\left(\frac{\gamma \alpha_1}{\alpha_1 - (1-\alpha_1)k_{1t+2}^* b}\right)$$

$$+ k_{1t+2}^*\left[\frac{1}{q_{1t+1}^*}\left(\frac{\gamma b \alpha_1}{\alpha_1 - (1-\alpha_1)k_{1t+2}^* b}\right) - \frac{c_{t+2}^*}{A_0 k_{0t+2}^{*\alpha_0}}\psi\right] + \frac{c_{t+2}^*}{A_0 k_{0t+2}^{*\alpha_0}}\psi$$

Simplify by $q_{1t+1}^*$ and using equations (21) to (23) we have:

$$\frac{k_{1t+1}^*}{\delta}\left(\frac{\gamma b}{\alpha_1 - (1-\alpha_1)k_{1t+1}^* b}\right) - \frac{k_{1t+1}^*}{k_{0t+1}^*}(1-\beta)\frac{\alpha_0}{\alpha_1}\psi =$$

$$\left(\frac{\gamma \alpha_1}{\alpha_1 - (1-\alpha_1)k_{1t+2}^* b}\right) + k_{1t+2}^*\left(\frac{\gamma b \alpha_1}{\alpha_1 - (1-\alpha_1)k_{1t+2}^* b}\right) - \frac{k_{1t+2}^*}{k_{0t+2}^*}\delta(1-\beta)\alpha_0\psi + \delta(1-\beta)\alpha_0\psi \tag{45}$$

From (22) to (25), optimal solution satisfies the equality of the marginal rate of transformation between the two sectors:

$$\frac{(1-\alpha_1)A_1 k_{1t+1}^{*\alpha_1}}{\alpha_1 A_1 k_{1t+1}^{*\alpha_1-1}} = \frac{(1-\alpha_0)A_0 k_{0t+1}^{*\alpha_0}}{\alpha_0 A_0 k_{0t+1}^{*\alpha_0-1}}$$

Thus, we obtain the following relationships between $k_{0t+1}^*$ and $k_{1t+1}^*$:

$$\frac{k_{1t+1}^*}{k_{0t+1}^*} = \left(\frac{1-\alpha_0}{1-\alpha_1}\right)\frac{\alpha_1}{\alpha_0}$$

Introduce it in (45) gives:

$$\frac{k^*_{1t+1}}{\delta}\left(\frac{\gamma b}{\alpha_1-(1-\alpha_1)k^*_{1t+1}b}\right)-\left(\frac{1-\alpha_0}{1-\alpha_1}\right)(1-\beta)\psi =$$

$$\left(\frac{\gamma\alpha_1}{\alpha_1-(1-\alpha_1)k^*_{1t+2}b}\right)(1+k^*_{1t+2}b)-\left(\frac{1-\alpha_0}{1-\alpha_1}\right)\delta(1-\beta)\alpha_1\psi+\delta(1-\beta)\alpha_0\psi \tag{46}$$

We have a dynamic equation in $k^*_1$. We differentiate it in order to study the stability of the equilibrium:

$$\frac{\gamma b\left(\alpha_1-(1-\alpha_1)k^*_1b+(1-\alpha_1)bk^*_1\right)}{\delta(\alpha_1-(1-\alpha_1)k^*_1b)^2}dk^*_{1t+1}=\frac{\gamma\alpha_1\left(b(\alpha_1-(1-\alpha_1)k^*_1b)+(1-\alpha_1)b(1+k^*_1b)\right)}{(\alpha_1-(1-\alpha_1)k^*_1b)^2}dk^*_{1t+2}$$

After simplifications:

$$\frac{b\alpha_1}{\delta}dk^*_{1t+1}=\alpha_1 b(\alpha_1+(1-\alpha_1))dk^*_{1t+2}$$

$$\frac{dk^*_{1t+2}}{dk^*_{1t+1}}=\frac{1}{\delta}>1 \tag{47}$$

According to (47), the stationary capital ratio given by the first best solution is unstable. Therefore, after one period, the economy jumps to the balanced growth path characterized by the optimal physical to human capital ratio $k^*_1$.

We then compute the optimal physical to human capital ratio in the investment sector along the BGP. From Definition 1, it is characterized by $k^*_{1t+2}=k^*_{1t+1}=k_1{}^*$. We simplify equation (46) and obtain:

$$\frac{\gamma}{\alpha_1-(1-\alpha_1)k^*_1b}\left(\frac{k^*_1b-\alpha_1\delta-k^*_1b\delta\alpha_1}{\delta}\right)=\delta(1-\beta)\psi\left(\alpha_0-\alpha_1\frac{1-\alpha_0}{1-\alpha_1}\right)+(1-\beta)\psi\left(\frac{1-\alpha_0}{1-\alpha_1}\right)$$

From (44), $h^*$ exists only when $k^*_1 > \alpha_1/b(1-\alpha_1)$. Therefore, we can write:

$$\gamma k^*_1 b(1-\delta\alpha_1)-\gamma\delta\alpha_1=\frac{\delta(1-\beta)\psi}{1-\alpha_1}(\delta(\alpha_0-\alpha_1)+(1-\alpha_0))(\alpha_1-(1-\alpha_1)k^*_1b) \tag{48}$$

Equation (48) leads to:

$$\gamma(1-\delta\alpha_1)k^*_1b+\mathcal{S}k^*_1b=\gamma\delta\alpha_1+\mathcal{S}\left(\frac{\alpha_1}{1-\alpha_1}\right)$$

where $\mathcal{S}=\delta(1-\beta)\psi(\delta(\alpha_0-\alpha_1)+(1-\alpha_0))$

Therefore, we have an expression for $k^*_1$ given by (33).

From Definition 1 and (32) the optimal growth rate along the BGP is given by:

$$1+g^*=\delta\alpha_1 A_1 k^{*\alpha_1-1}_1$$

This corresponds to the two-sector modified golden rule as $1 + g^* = \delta f'(k_1)$.

Using equation (33) we compute the following derivatives:

$$\frac{\partial k_1^*}{\partial \gamma} = \frac{\delta - 1}{\delta b(1 - \alpha_1)^2} \frac{\mathcal{S}}{(\gamma(1 - \delta \alpha_1) + \mathcal{S})^2} < 0 \quad \Rightarrow \quad \frac{\partial g^*}{\partial \gamma} > 0$$

$$\frac{\partial k_1^*}{\partial \beta} = \frac{\alpha_1 \gamma (1 - \delta)^2}{b} \frac{(1 - \alpha_0) + \delta(\alpha_0 - \alpha_1)}{(\gamma(1 - \delta \alpha_1) + \mathcal{S})^2} > 0 \quad \Rightarrow \quad \frac{\partial g^*}{\partial \beta} < 0$$

$$\frac{\partial k_1^*}{\partial \varepsilon} = -\frac{\alpha_1}{b(1 - \alpha_1)} \frac{\psi \gamma (1 - \delta)^2}{(\mathcal{S} + \gamma(1 - \delta \alpha_1))^2} < 0 \quad \Rightarrow \quad \frac{\partial g^*}{\partial \varepsilon} > 0$$

Lemma 2 follows.

□

## 7.3   Proof of Lemma 3

We establish equation (36) from (35), substituting $\varepsilon = 2\Upsilon$ and $\alpha_1 = \bar{\alpha} - \Upsilon$. The derivative with respect to $\Upsilon$ is:

$$\frac{\partial k^*}{\partial \Upsilon} = \frac{-(\psi + \gamma)(\psi(2\delta - 1) + \delta \gamma)}{(\psi \Upsilon(2\delta - 1) + \psi(1 - \bar{\alpha}) + \gamma(1 - \delta \bar{\alpha}) + \delta \gamma \Upsilon)^2}$$

The sign of this derivative is given by the term $-(\psi(2\delta - 1) + \delta \gamma)$. Including the expression of $\psi$, $\frac{\partial k^*}{\partial \Upsilon} > 0$ when $\beta(2\delta - 1)(\delta - 1) - \delta((2\delta - 1) + \gamma) > 0$. We deduce the properties of $k^*(\Upsilon)$ given in Lemma 3.

□

## 7.4   Proof of Proposition 2

We compare the optimal and the *laissez-faire* physical to human capital ratio. The *laissez-faire* ratio is given by (12) and the optimal by (36).

According to Lemma 3, we have two possible cases:

- When $\beta < \tilde{\beta}$, $k^*$ is increasing in $\Upsilon$. We examine $k^*(\Upsilon)$ at the limits of its definition set: $k^*(-\bar{\alpha}) = \frac{2\delta \bar{\alpha}}{b(1 - 2\delta \bar{\alpha})}$ and $k^*(\bar{\alpha}) = \frac{2(\beta(1 - \delta) + \delta)(1 - \delta)\bar{\alpha}}{b(\gamma + (2\bar{\alpha}(\delta - 1) + 1)(\beta(1 - \delta + \delta))}$.

  Comparing to the *laissez-faire* ratio, we have $k^*(-\bar{\alpha}) < k_{LF}$ if $\beta > \frac{\gamma 2\delta \bar{\alpha}}{(1 - 2\delta \bar{\alpha})} \equiv \bar{\beta}_1$ and, respectively, $k(-\bar{\alpha}) > k_{LF}$ when $\beta < \bar{\beta}_1$. In the second case, we have $k_{LF} < k^* \; \forall \; \Upsilon$, as the function $k^*$ is increasing in $\Upsilon$. Consider the case $\beta > \bar{\beta}_1$, we have $k^*(\bar{\alpha}) > k_{LF}$ under the following condition:

$$-\beta^2(1 - \delta)(2\bar{\alpha}(\delta - 1) + 1) + \beta \left[ \gamma(2\bar{\alpha}(1 - \delta)^2 - 1) + \delta(2\bar{\alpha}(1 - \delta) - 1) \right] + 2\gamma \delta \bar{\alpha}(1 - \delta) > 0$$

This polynomial in $\beta$ admits a unique positive solution $\bar{\beta}_2$ such that when $0 < \beta < \bar{\beta}_2$, we have $k^*(\bar{\alpha}) > k_{LF}$. Respectively, when $\beta > \bar{\beta}_2$, we have $k^*(\bar{\alpha}) < k_{LF}$. The expression of $\bar{\beta}_2$ is given by:

$$\bar{\beta}_2 = \frac{(1-\delta)2\bar{\alpha}((1-\delta)\gamma + \delta) - \delta - \gamma + \sqrt{\Delta}}{2(1-\bar{\alpha})(1 + 2\bar{\alpha}(\delta - 1))}$$

With $\Delta = (1-\delta)2\bar{\alpha}((1-\delta)\gamma + \delta) - \delta - \gamma - 4(1-\delta)^2\delta\gamma\bar{\alpha}(1 - 2\bar{\alpha}(1-\delta))$.

When $\bar{\beta}_1 < \beta < \bar{\beta}_2$, there exists a critical level $\bar{\Upsilon}$ such that $k_{LF} = k^*$, with:

$$\bar{\Upsilon} = \frac{\psi\beta(\bar{\alpha} - 1) + \beta\gamma(\delta\bar{\alpha} - 1) + \bar{\alpha}\gamma(\psi + \delta\gamma)}{2\psi\delta(\beta + \gamma) - \psi(\beta + \gamma) + \delta\gamma(\beta + \gamma)}$$

- When $\beta > \tilde{\beta}$. Likewise, we obtain the opposite compared to the previous case. Ratio $k^*$ is decreasing in $\Upsilon$, thus, when $\beta > \bar{\beta}_1$, $k_{LF} > k^* \; \forall \; \Upsilon$. Conversely when $\beta < \bar{\beta}_2$, $k^* > k_{LF} \; \forall \; \Upsilon$. In this case, we have $\bar{\beta}_2 < \bar{\beta}_1$.

Proposition 2 follows.

$\square$

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