

# Indeterminacy and Sunspot Fluctuations in Two-Sector RBC models: Theory and Calibration

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# Indeterminacy and sunspot fluctuations in two-sector RBC models: theory and calibration\*

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**Abstract:** *We analyze sunspot-driven fluctuations in the standard 2-sector RBC model with moderate increasing returns to scale. We provide a detailed theoretical analysis enabling us to derive relevant bifurcation loci and to characterize the steady-state local stability properties as a function of various structural parameters. With GHH preferences, we show that local indeterminacy occurs through flip and Hopf bifurcations for a large set of values of the elasticity of intertemporal substitution in consumption if the labor supply is sufficiently inelastic. With additively-separable preferences, we prove that local indeterminacy occurs through flip and Hopf bifurcations for any value of the elasticity of the labor supply, and can even be compatible with an arbitrarily low elasticity of intertemporal substitution in consumption. Finally, we provide a detailed quantitative analysis of the model. Computing, on a quarterly basis, a new set of empirical moments related to two broadly defined consumption and investment sectors, we are able to identify, among the set of admissible calibrations consistent with sunspot equilibria, the ones that provide the best fit of the data. The model properly calibrated solves several empirical puzzles traditionally associated with 2-sector RBC models.*

**Keywords:** *Indeterminacy, sunspots, two-sector model, sector-specific externalities, real business cycles*

*Journal of Economic Literature* Classification Numbers: C62, E32, O41.

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# 1 Introduction

The aim of this paper is to provide a detailed theoretical and empirical assessment of the sunspot-driven two-sector Real Business Cycle model with productive externalities and increasing returns to scale, considering alternative specifications for individual preferences. The recent literature suggests that by comparison to their one-sector equivalents, 2-sector RBC models are able to generate local indeterminacy with much lower degrees of increasing returns to scale.<sup>1</sup> Yet, this result has often been obtained under relatively narrow specifications for technology and/or preferences, without much attention to robustness and domain of validity issues. Starting from the Benhabib and Farmer [4]’s formulation with increasing social returns, we consider a wider class of preferences enabling us to thoroughly analyze the role of income effects, intertemporal substitution and labor supply elasticity in the emergence of local indeterminacy.

In a first step, we consider the popular specification of individual preferences proposed by Greenwood *et al.* [17] (GHH) characterized by the absence of income effects. In this case, it is known that with *constant* social returns, local indeterminacy occurs for sufficiently inelastic labor supply (Nishimura and Venditti [30]). Yet, for *increasing* social returns, this result has been extended only for the specific case of a logarithmic specification, and in fact essentially through numerical simulations (Guo and Harrison [19]).<sup>2</sup>

We prove here that this result holds quite generally, in particular *for a large set of values of the elasticity of intertemporal substitution (EIS) in consumption*. We also prove the existence of an upper bound on the labor supply elasticity under which local indeterminacy arises, and we show that this upper bound is decreasing with the degree of increasing social returns. We finally exhibit the existence of flip and Hopf bifurcations in the parameter space and we provide the analytical expressions for these bifurcations.<sup>3</sup> This allows us to show how a change in the EIS in consumption

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<sup>1</sup>While indeterminacy requires about 50% of increasing returns to scale in the one-sector RBC model of Benhabib and Farmer [3] and Farmer and Guo [11], this degree decreases to only 7% in its two-sector equivalent (see Benhabib and Farmer [4]). Indeterminacy also occurs with constant social returns to scale and decreasing private returns (Benhabib and Nishimura [5], Garnier *et al.* [12], Nishimura and Venditti [29]).

<sup>2</sup>In one-sector models with GHH preferences, the results are drastically different: Meng and Yip [26] and Nishimura *et al.* [28] have shown that local indeterminacy cannot arise. Jaimovich [23], using a specification that nests the GHH formulation as a special case, has proved that a minimum amount of income effect is necessary for local indeterminacy.

<sup>3</sup>See Grandmont [14, 15] for a simple presentation of bifurcation theory.

drastically affects the range of values for the other structural parameters for which the steady-state is locally indeterminate.

In a second step, we consider the standard case of additively separable preferences. Within this class of utility functions, it is known that under constant returns to scale, local indeterminacy occurs for sufficiently inelastic labor supply (Benhabib and Nishimura [5], Garnier *et al.* [12]). Yet, for increasing social returns, the literature has only focused on the case of an infinitely elastic labor supply (Benhabib and Farmer [4], Harrison [21]).

We prove here that local indeterminacy again occurs quite generally, in particular *no matter how elastic or inelastic the labor supply is*. This is true provided the EIS in consumption and the amount of externalities are in an intermediary range. We again provide the analytical expressions for the flip and Hopf bifurcation values. We finally prove that with an infinitely elastic labor supply, local indeterminacy and Hopf bifurcation occur for *arbitrarily low EIS in consumption*, provided that the degree of increasing social returns is larger than some (empirically plausible) lower bound. This conclusion is drastically different from what is known from the previous literature in which a large enough EIS is always assumed (Garnier *et al.* [12, 13], Harrison [21]).

The second contribution of the paper is *empirical*. While 2-sector RBC models are able to generate local indeterminacy and endogenous sunspot fluctuations with much lower degrees of increasing returns to scale than one-sector models, they also tend to make several inaccurate empirical predictions. For example, in the case of additively-separable preferences, the literature has identified several empirical puzzles associated with such models:<sup>4</sup> the *consumption cyclicality puzzle*, the *labor comovement puzzle*, and the *hours volatility puzzle*.

We first start by computing on a quarterly basis a new set of empirical moments related to two broadly defined consumption and investment sectors, adapting a methodology initially proposed in Baxter [2] but on an annual basis. Then, we show that, by considering a sufficiently general specification for individual preferences and appropriate calibrations, all three empirical puzzles mentioned above can be resolved. Improving the model's predictions requires to find better compromises between the various economic mechanisms — intertemporal elasticity of substitution, income effects, wage elasticity of labor supply — identified as crucial for the local stability properties of the model. We show that the best performing calibrations are typically close to the boundary of the set of admissible calibrations, near the Hopf

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<sup>4</sup>See e.g. Benhabib and Farmer [4], Harrison [21].

bifurcation locus identified in the theoretical analysis. This implies that appropriate calibrations must depart from the traditional logarithmic specification for the utility function extensively considered in the literature. Moreover, we are able to conclude that GHH preferences lead to better empirical results compared to the additively-separable specification.

The rest of this paper is organized as follows. We present the model and we characterize the intertemporal equilibrium and the steady state in the next section. In Section 3, the complete set of conditions for indeterminacy are derived respectively for GHH and additively separable preferences. In Section 4, we provide detailed simulations in order to discuss the ability of the two versions of the model to account for the main features of observed business cycles. Some concluding remarks are provided in Section 5, whereas all the technical details are provided in an Appendix.

## 2 The model

We consider a standard infinite-horizon two-sector real business-cycle model with productive externalities.

### 2.1 Production

The economy produces a consumption good,  $c$ , and an investment good,  $I$ , with Cobb-Douglas technologies which contain some positive sector-specific externalities. We consider the Benhabib and Farmer [4] model with identical technologies in both sectors at the private level and output externalities. We denote by  $Y_c$  and  $Y_I$  the outputs of sectors  $c$  and  $I$ , and by  $A$  the external effects. The *private production functions* are thus:

$$Y_{ct} = z_t K_{ct}^\alpha L_{ct}^{1-\alpha}, \quad Y_{It} = z_t A_t K_{It}^\alpha L_{It}^{1-\alpha}$$

where  $z_t$  is the level of total factor productivity which is assumed for now to be constant.

The externalities  $A$  depend on  $\bar{K}_I$  and  $\bar{L}_I$ , which denote the average use of capital and labor in sector  $I$ , and are equal to

$$A_t = \bar{K}_{It}^{\alpha\Theta} \bar{L}_{It}^{(1-\alpha)\Theta} \quad (1)$$

with  $\Theta \geq 0$ .<sup>5</sup> We assume that these economy-wide averages are taken as given by individual firms. Assuming that factor markets are perfectly competitive and that

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<sup>5</sup>We do not consider externalities in the consumption good sector as they do not play any crucial role in the existence of multiple equilibria.

capital and labor inputs are perfectly mobile across the two sectors, the first order conditions for profit maximization of the representative firm in each sector are

$$r_t = \frac{\alpha Y_{ct}}{K_{ct}} = p_t \frac{\alpha Y_{It}}{K_{It}}, \quad \omega_t = \frac{(1-\alpha)Y_{ct}}{L_{ct}} = p_t \frac{(1-\alpha)Y_{It}}{L_{It}} \quad (2)$$

where  $r_t$ ,  $p_t$  and  $\omega_t$  are respectively the rental rate of capital, the price the investment good and the real wage rate at time  $t$  all in terms of the price of the consumption good.

## 2.2 Preferences

We consider an economy populated by a large number of identical infinitely-lived agents. At each period a representative agent supplies elastically an amount of labor  $l \in [0, \ell]$ , with  $\ell > 1$  (possibly infinite) his endowment of labor. The agent derives utility from consumption  $c$  and labor  $l$  according to a function  $U(c, l)$  which satisfies:

**Assumption 1.**  $U(c, l)$  is  $\mathbf{C}^r$  over  $\mathbb{R}_+ \times [0, \ell]$  for  $r$  large enough, increasing with respect to consumption, decreasing with respect to labor and concave.

Actually, within these general properties for the utility function, we will consider two different specifications which are widely used in the literature.<sup>6</sup>

i) A Greenwood-Hercovitz-Huffman [17] (GHH) formulation such that

$$U(c, l) = \frac{\left(c - \frac{Bl^{1+\chi}}{1+\chi}\right)^{1-\sigma}}{1-\sigma} \quad (3)$$

with  $B > 0$  a normalization constant,  $\sigma \geq 0$  and  $\chi \geq 0$ . The essential feature of this specification is that the marginal rate of substitution between consumption and leisure depends on the latter only as

$$\frac{U_2(c, l)}{U_1(c, l)} = -Bl^\chi \quad (4)$$

This property illustrates the lack of income effect associated with the agent's labor supply. We observe also that the Frisch wage elasticity of the labor supply is given by  $1/\chi$  while the EIS in consumption is affected by both  $\sigma$  and  $\chi$  as we will show later on.

ii) An additively separable formulation such that

$$U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{Bl^{1+\chi}}{1+\chi} \quad (5)$$

with  $B > 0$  a normalization constant,  $\sigma \geq 0$  and  $\chi \geq 0$  which are respectively the inverse of the elasticity of intertemporal substitution in consumption and the inverse of the Frisch wage elasticity of the labor supply.

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<sup>6</sup>Others specifications are also considered in the literature but they usually yield local determinacy (Bennett and Farmer [6], Hintermaier [21], Nishimura *et al.* [28], Pintus [31]).

Note that in both formulations, consumption and leisure are normal goods. We finally assume without loss of generality that the total population is constant and normalized to one. Denoting  $I_t$  the gross investment,  $Y_t$  the GDP and  $k_t$  the household's capital stock, the budget constraint faced by the representative household is

$$c_t + p_t I_t = Y_t = r_t k_t + \omega_t l_t$$

Assuming that capital depreciates at rate  $\delta \in [0, 1]$  in each period, it follows that the law of motion of the capital stock is:

$$k_{t+1} = I_t + (1 - \delta)k_t \quad (6)$$

with  $k_0$  given. The household then maximizes its expected present discounted lifetime utility

$$\begin{aligned} \max_{k_t, c_t, l_t} \quad & E_0 \sum_{t=0}^{+\infty} \beta^t U(c_t, l_t) \\ \text{s.t.} \quad & c_t + p_t I_t = Y_t = r_t k_t + \omega_t l_t \\ & k_{t+1} = I_t + (1 - \delta)k_t \\ & k_0 \text{ given} \end{aligned} \quad (7)$$

with  $\beta \in (0, 1)$  the discount factor. The first-order conditions for this optimization problem are

$$U_1(c_t, l_t) = \beta E_t U_1(c_{t+1}, l_{t+1}) \left[ \frac{r_{t+1} + (1 - \delta)p_{t+1}}{p_t} \right] \quad (8)$$

$$U_1(c_t, l_t) \omega_t = -U_2(c_t, l_t) \quad (9)$$

Equation (8) is the standard stochastic Euler equation, and (9) corresponds to the trade-off between consumption and leisure. With GHH preferences, as suggested by (4), the income elasticity of intertemporal substitution in labor is zero.

### 2.3 Intertemporal equilibrium and steady state

We consider symmetric perfect-foresight equilibria which consist of prices  $\{r_t, p_t, \omega_t\}_{t \geq 0}$  and quantities  $\{c_t, l_t, k_{t+1}\}_{t \geq 0}$  that satisfy the household's and the firms' first-order conditions as given by (2) and (8)-(9). All firms of sector  $I$  being identical, we have  $\bar{K}_I = K_I$  and  $\bar{L}_I = L_I$ . The *social production function* in the investment good sector is defined as

$$Y_{It} = z_t K_{It}^{\alpha(1+\Theta)} L_{It}^{(1-\alpha)(1+\Theta)} \quad (10)$$

We have thus increasing social returns which size is measured by  $\Theta$ .

The market clearing conditions for the consumption and investment goods give  $c_t = Y_{ct}$  and  $I_t = Y_{It}$ , while the market clearing conditions for capital and labor yield  $K_{ct} + K_{yt} = k_t$  and  $L_{ct} + L_{yt} = l_t$ . Any solution that also satisfies the transversality condition

$$\lim_{t \rightarrow +\infty} \beta^t U_1(c_t, l_t) p_t k_{t+1} = 0$$

is called an equilibrium path.

A steady state is defined by  $k_t = \bar{k}$ ,  $I_t = \delta\bar{k}$ ,  $l_t = \bar{l}$ . Using constant returns at the private level, note that the steady state can be equivalently defined in terms of the capital/labor ratio  $\bar{k} = \bar{k}/\bar{l}$  and the labor supply  $\bar{l}$ .

**Proposition 1.** *If the utility function  $U(c, l)$  is given by (3) or (5), there exists a unique steady state  $(\bar{k}, \bar{l})$ .*

We will consider in the following a family of economies parameterized by the EIS in consumption which depends on  $\sigma$  and the wage elasticity of labor which depends on  $\chi$ .

### 3 Theory

We provide a detailed local stability analysis of the steady-state successively for GHH and additively separable preferences.

#### 3.1 GHH preferences

In the case of GHH preferences as given by (3), the first-order conditions (8) and (9) become

$$\begin{aligned} \left(c_t - \frac{Bl_t^{1+\chi}}{(1+\chi)}\right)^{-\sigma} &= \beta E_t \left(c_{t+1} - \frac{Bl_{t+1}^{1+\chi}}{(1+\chi)}\right)^{-\sigma} \left[\frac{r_{t+1} + (1-\delta)p_{t+1}}{p_t}\right] \\ \omega_t &= Bl_t^\chi \end{aligned} \quad (11)$$

Linearizing this system in a neighborhood of the steady state yields to a characteristic polynomial which is given in Appendix 6.3. Since for a given value of  $\chi$ , the Trace and Determinant are linear functions of  $\sigma$ , we can use the geometrical methodology described in Grandmont *et al.* [16] to study the local stability properties of the steady state. Indeed, for a given  $\Theta$ , as  $\sigma$  is varied over  $(0, +\infty)$ , the Trace and Determinant move along a line denoted  $\Delta_\chi$  whose location depends on the value of  $\chi$ .

Let us denote

$$\underline{\chi} = \frac{\alpha\Theta}{1-\alpha(1+\Theta)} \equiv \underline{\chi}(\Theta) \quad (12)$$

We also introduce the following assumption:

**Assumption 2.**  $\alpha < 1/2$ ,  $\Theta \in (\underline{\Theta}, \bar{\Theta})$ , with  $\underline{\Theta} = \delta/(1-\delta)$ ,  $\bar{\Theta} = (1-\alpha)/\alpha$ , and  $\delta \in (0, \bar{\delta})$  with  $\bar{\delta}$  as given by equation (34) in Appendix 6.3.



Under a standard parameterization compatible with quarterly data with  $(\alpha, \delta, \beta) = (0.3, 0.025, 0.99)$ , Assumption 2 provides a wide range for the size of externalities as  $\underline{\Theta} \approx 0.0256$  and  $\bar{\Theta} \approx 2.33$ . Note that the lower bound  $\underline{\Theta}$  is perfectly in line with the estimates of Basu and Fernald [4].

When  $\Theta \in (\underline{\Theta}, \bar{\Theta})$ , we have the following geometrical configurations that provide a full picture of the local stability properties of the steady state:

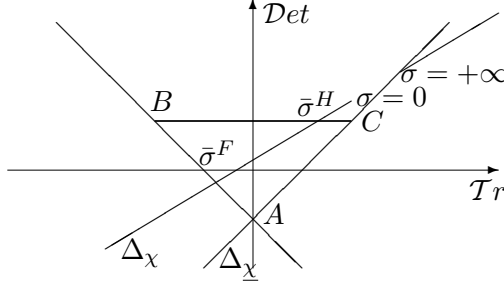


Figure 1: Local indeterminacy with GHH preferences when  $\chi > \underline{\chi}$ .

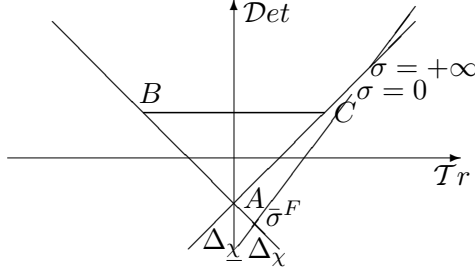


Figure 2: Local determinacy with GHH preferences when  $\chi \in [0, \underline{\chi})$ .

Let us denote  $\theta = \beta(1 - \delta)$ . It can be shown (see Appendix 6.3) that at the steady state, for given parameters  $(\beta, \alpha, \delta)$ , the EIS in consumption  $\epsilon_{cc}$  is a function of  $(\sigma, \chi)$ , namely

$$\epsilon_{cc}(\sigma, \chi) = \frac{1}{\sigma} \left( 1 - \frac{1-\alpha}{(1+\chi) \left( 1 - \frac{\beta\delta\alpha}{1-\theta} \right)} \right) \quad (13)$$

$\epsilon_{cc}(\sigma, \chi)$  is decreasing with respect to  $\sigma$  but increasing with respect to  $\chi$ .

The Hopf and flip bifurcation values are respectively defined as:

$$\bar{\sigma}^H = \frac{\Theta(1-\delta)(1-\beta)(1-\theta) \left[ \left( 1 - \frac{\beta\delta\alpha}{1-\theta} \right) - \frac{1-\alpha}{1+\chi} \right]}{\delta\alpha[1-\beta+(1-\theta)\Theta]} \quad (14)$$

and

$$\bar{\sigma}^F = \frac{(1-\theta)[(\chi+\alpha)\Theta[2(1-\delta)(1+\beta)+(1-\theta)\delta] - (1-\theta)\delta\chi(1-\alpha)(1+\Theta)] \left[ \left( 1 - \frac{\beta\delta\alpha}{1-\theta} \right) - \frac{1-\alpha}{1+\chi} \right]}{2(\chi+\alpha)\delta\alpha[1+\beta+(1-\theta)\Theta]} \quad (15)$$

We conclude therefore that:

**Proposition 2.** *Under Assumption 2, let  $\underline{\chi} = \alpha\Theta/[1 - \alpha(1 + \Theta)]$  as in (12). Then the following results hold:*

*i) If  $\chi > \underline{\chi}$  and  $\delta \in (0, \bar{\delta})$ , the steady state is saddle-point stable when  $\sigma > \bar{\sigma}^F$ , undergoes a flip bifurcation at  $\sigma = \bar{\sigma}^F$ , becomes locally indeterminate when  $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$ , undergoes a Hopf bifurcation when  $\sigma = \bar{\sigma}^H$  and becomes locally unstable when  $\sigma \in [0, \bar{\sigma}^H)$ .*

*ii) If  $\chi \in [0, \underline{\chi})$ , the steady state is locally unstable when  $\sigma > \bar{\sigma}^F$ , undergoes a flip bifurcation at  $\sigma = \bar{\sigma}^F$  and becomes saddle-point stable when  $\sigma \in [0, \bar{\sigma}^F)$ .*

For a large set of intermediary values for the EIS in consumption  $\epsilon_{cc}$  with  $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$ , local indeterminacy then occurs if the labor supply is sufficiently inelastic, i.e.  $\chi$  large enough, and is ruled out when  $\chi$  is close to 0, i.e. if the labor supply is infinitely elastic.<sup>7</sup> While a similar conclusion is well-known in models with constant social returns (see for instance Benhabib and Nishimura [5]), this is a new conclusion in models with increasing social returns. Moreover, as  $\partial \underline{\chi} / \partial \Theta > 0$ , the larger the externalities, the less elastic labor must be for the existence of indeterminacy. Note finally that since  $\alpha < 1/2$ , we have  $\underline{\Theta} < \alpha / (1 - \alpha) < \bar{\Theta}$ . This implies that all our results are compatible with standard negative slopes for the capital and labor equilibrium demand functions.

Proposition 2 allows us to get a better understanding of the conclusions obtained on a numerical basis by Guo and Harrison [19] in the particular case  $\sigma = 1$ . First, we provide an analytical expression for the threshold  $1/\underline{\chi}$  on the labor supply elasticity  $1/\chi$  above which local indeterminacy is ruled out. Second, we show how a change in the EIS in consumption drastically affects the range of values for the other parameters for which the steady-state is locally indeterminate. Third, on this basis, we exhibit the existence of a Hopf bifurcation, in addition to the flip bifurcation identified by Guo and Harrison [19]. This last feature will play an important role in our data confrontation analysis as it allows to get persistent and non-monotonous convergence to the steady-state. Indeed, on the basis of a better compromise between the various economic mechanisms — intertemporal elasticity of substitution, income effects, wage elasticity of labor supply — identified as crucial for the local stability properties of the model, we will show that the best performing calibrations are typically close to the Hopf bifurcation locus.

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<sup>7</sup>In two-sector models with constant social returns *à la* Benhabib and Nishimura [5] and GHH preferences, local indeterminacy remains compatible with an infinitely elastic labor supply provided  $\sigma$  is close enough to 0 (see Nishimura and Venditti [30]).

### 3.2 Additively separable preferences

In the case of additively separable preferences as given by (5), the first-order conditions (8) and (9) become

$$\begin{aligned} c_t^{-\sigma} &= \beta E_t \left[ \frac{r_{t+1} + (1-\delta)p_{t+1}}{p_t c_{t+1}^\sigma} \right] \\ \omega_t c_t^{-\sigma} &= B l_t^\chi \end{aligned} \quad (16)$$

Linearizing this system in a neighborhood of the steady state yields to a characteristic polynomial which is given in Appendix 6.4. As in the GHH case, for a given value of  $\chi$ , the Trace and Determinant are linear functions of  $\sigma$  and we can also use the geometrical methodology described in Grandmont *et al.* [16]. When  $\sigma$  is varied over  $(0, +\infty)$ , the Trace and Determinant move along a line denoted  $\Delta_\chi$  whose location depends on the value of  $\chi$ .

From now on, in order to simplify the analysis, we introduce the following restrictions:

**Assumption 3.**  $\alpha \in (1/4, 3/8)$ ,  $\beta \in (\hat{\beta}, 1)$ ,  $\delta < \hat{\delta}$  and  $\Theta \in (\underline{\Theta}, \bar{\Theta})$  with  $\hat{\beta} \equiv \max\{(1-2\alpha)/[(1-\delta)(1-\alpha)^2], (1-\alpha-\delta)/(1-\delta)(1-\alpha)\}$ ,  $\hat{\delta} \equiv [\beta(1-\alpha) - 2(1-\beta)]/\beta(2-\alpha)$ ,  $\underline{\Theta} = \delta/(1-\delta)$  and  $\bar{\Theta} = \alpha/(1-\alpha)$ .

Different dynamical configurations can be obtained depending on the size of the externalities. Let us introduce the following additional bounds:

$$\hat{\Theta} = \frac{\delta\alpha(1-\theta(1-\alpha))}{(1-\delta)(1-\theta)(1-\alpha)} \quad \text{and} \quad \tilde{\Theta} = \frac{\frac{2\delta\alpha(1+\beta)(1-\theta(1-\alpha))}{1-\theta} + \delta(1-\alpha)(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta})}{2[(1-\alpha-\delta)(1+\beta) + \delta\beta\alpha]} \quad (17)$$

Under Assumption 3 we have  $\underline{\Theta} < \hat{\Theta} < \tilde{\Theta} < \bar{\Theta}$ .

Note that using a standard parameterization compatible with quarterly data such that  $(\alpha, \delta, \beta) = (0.3, 0.025, 0.99)$ , Assumption 3 holds and is compatible with mild external effects since  $\hat{\beta} \approx 0.837$ ,  $\hat{\delta} \approx 0.3998$ ,  $\underline{\Theta} \approx 0.0256$ ,  $\hat{\Theta} \approx 0.1025$ ,  $\tilde{\Theta} \approx 0.1033$  and  $\bar{\Theta} \approx 0.4286$ . This implies as in the GHH case that all our results are compatible with standard negative slopes for the capital and labor equilibrium demand functions.

We start by considering the case of small externalities with  $\Theta \in (\underline{\Theta}, \hat{\Theta})$ . Let us introduce the same critical value  $\underline{\chi}$  as given by (12) in section 3.1. When  $\chi > \underline{\chi}$  we have the following geometrical configuration:

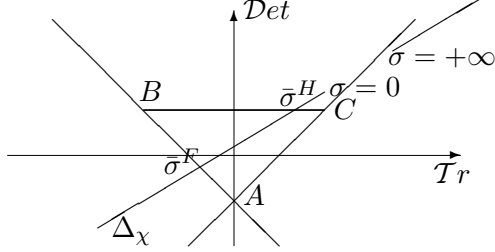


Figure 3: Local indeterminacy with  $\Theta \in (\underline{\Theta}, \hat{\Theta})$  and  $\chi \in (\underline{\chi}, +\infty)$ .

When  $\chi \in [0, \underline{\chi})$  we get a very similar picture

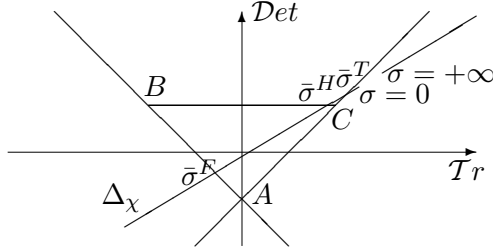


Figure 4: Local indeterminacy with  $\Theta \in (\underline{\Theta}, \hat{\Theta})$  and  $\chi \in [0, \underline{\chi})$ .

The only difference concerns the stability properties of the steady state when  $\sigma = 0$ . The Hopf, flip and transcritical bifurcation values are respectively defined as:

$$\bar{\sigma}^H = \frac{(1 - \frac{\beta\delta\alpha}{1-\theta}) \frac{\Theta(\chi+\alpha)(1-\delta)(1-\beta)}{\delta}}{-\Theta \frac{[(1-\alpha)(1-\theta)-\delta]}{\delta} + \frac{\alpha(1-\beta)[1-\theta(1-\alpha)]}{1-\theta} + \chi\alpha \frac{[1-\beta+\Theta(1-\theta)]}{1-\theta}} \quad (18)$$

$$\bar{\sigma}^F = \frac{(1 - \frac{\beta\delta\alpha}{1-\theta}) \frac{\Theta[\alpha(1+\theta)(2-\delta)+\chi[2(1-\delta)(1+\beta)+\delta\alpha(1-\theta)]] - \delta\chi(1-\alpha)(1-\theta)}{\delta(1-\alpha)}}{2 \left[ -\frac{\Theta[(1-\delta)(1-\alpha)(1+\beta)-\delta\alpha]}{\delta(1-\alpha)} + \frac{\alpha(1+\beta)[1-\theta(1-\alpha)]}{(1-\alpha)(1-\theta)} + \frac{\chi\alpha[1+\beta+\Theta(1-\theta)]}{(1-\alpha)(1-\theta)} \right] + (1-\theta) \left( 1 - \frac{\beta\delta\alpha}{1-\theta} \right)} \quad (19)$$

and

$$\bar{\sigma}^T = \frac{\Theta\alpha - \chi(1-\alpha)(1+\Theta)}{(1-\alpha)} \quad (20)$$

We conclude therefore that:

**Proposition 3.** *Under Assumption 3, let  $\Theta \in (\underline{\Theta}, \hat{\Theta})$  and  $\underline{\chi} = \alpha\Theta/[1 - \alpha(1 + \Theta)]$  as in (12). Then there exist  $\underline{\delta}, \bar{\delta}$ , with  $0 < \underline{\delta} < \bar{\delta} \leq \hat{\delta}$ , and  $\underline{\beta} \in [\hat{\beta}, 1)$  such that when  $\beta \in (\underline{\beta}, 1)$  and  $\delta \in (\underline{\delta}, \bar{\delta})$ , the following results hold:*

*i) If  $\chi \in (\underline{\chi}, +\infty)$ , the steady state is saddle-point stable when  $\sigma \in (\bar{\sigma}^F, +\infty)$ , undergoes a flip bifurcation at  $\sigma = \bar{\sigma}^F$ , becomes locally indeterminate when  $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$ , undergoes a Hopf bifurcation when  $\sigma = \bar{\sigma}^H$ , and becomes locally unstable when  $\sigma \in (0, \bar{\sigma}^H)$ .*

*ii) If  $\chi \in [0, \underline{\chi})$ , the steady state is saddle-point stable when  $\sigma \in (\bar{\sigma}^F, +\infty)$ , undergoes a flip bifurcation at  $\sigma = \bar{\sigma}^F$ , becomes locally indeterminate when  $\sigma \in$*

$(\bar{\sigma}^H, \bar{\sigma}^F)$ , undergoes a Hopf bifurcation when  $\sigma = \bar{\sigma}^H$ , becomes locally unstable when  $\sigma \in (\bar{\sigma}^T, \bar{\sigma}^H)$ , and becomes again saddle-point stable when  $\sigma \in [0, \bar{\sigma}^T)$ .

Proposition 3 provides new clear-cut conditions for the existence of local indeterminacy through the occurrence of flip and Hopf bifurcations.<sup>8</sup> We show indeed that local indeterminacy occurs for any value of  $\chi$ , i.e. no matter how elastic or inelastic is the labor supply. This is a new result in the literature as all the previous contributions dealing with two-sector models fundamentally rely either on a strongly elastic ( $\chi$  small) or a weakly elastic ( $\chi$  large) labor supply.<sup>9</sup>

*Remark:* Figure 4 clearly identifies a transcritical bifurcation value  $\bar{\sigma}^T$  as defined by (20) which is usually associated with the existence of multiple steady states. However, as proved by Proposition 1, the steady state is unique for both specifications of the utility function. It follows that this transcritical bifurcation is here, and in the rest of the paper, degenerate and only associated with a loss of stability of the unique steady state.

Let us consider the case of slightly larger externalities with  $\Theta \in (\hat{\Theta}, \bar{\Theta})$ . From now on we simplify the formulation by assuming that the labor supply is infinitely elastic, i.e.  $\chi = 0$ .<sup>10</sup> We have the following two geometrical configurations depending on whether  $\Theta \in (\hat{\Theta}, \tilde{\Theta})$  or  $\Theta \in (\tilde{\Theta}, \bar{\Theta})$  with  $\tilde{\Theta}$  as defined by (17):

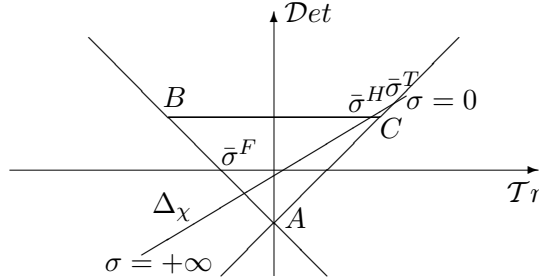


Figure 5: Local indeterminacy with  $\chi = 0$  and  $\Theta \in (\hat{\Theta}, \tilde{\Theta})$ .

<sup>8</sup>In Benhabib and Farmer [4] the existence of the Hopf bifurcation is mentioned but not proved, while in Harrison [21] the analysis focuses exclusively on the flip bifurcation through a numerical analysis.

<sup>9</sup>See Benhabib and Farmer [4], Benhabib and Nishimura [5], Garnier *et al.* [12, 13], Harrison [21]

<sup>10</sup>Of course all the following results still hold by continuity when  $\chi$  is positive but remains close enough to 0.

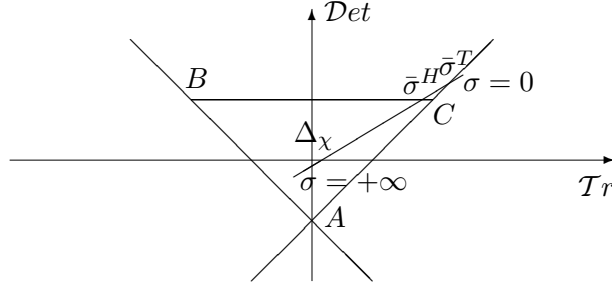


Figure 6: Local indeterminacy with  $\chi = 0$  and  $\Theta \in (\tilde{\Theta}, \bar{\Theta})$ .

With respect to Figures 4 and 5, Figure 6 shows that when  $\Theta$  is large enough, local indeterminacy arises for arbitrarily large values of  $\sigma$ . The Hopf, flip and transcritical bifurcation values here are obviously obtained by setting  $\chi = 0$  in equations (18), (19) and (20).

We conclude that:

**Proposition 4.** *Under Assumption 3, let  $\chi = 0$ . Then, there exist  $\underline{\delta}$ ,  $\bar{\delta}$ , with  $0 < \underline{\delta} < \bar{\delta} \leq \hat{\delta}$ , and  $\underline{\beta} \in [\hat{\beta}, 1)$  such that if  $\delta \in (\underline{\delta}, \bar{\delta})$  and  $\beta \in (\underline{\beta}, 1)$ , the following results hold:*

*i) For  $\Theta \in (\hat{\Theta}, \tilde{\Theta})$ , the steady state is saddle-point stable when  $\sigma \in (\bar{\sigma}^F, +\infty)$ , undergoes a flip bifurcation at  $\sigma = \bar{\sigma}^F$ , becomes locally indeterminate when  $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$ , undergoes a Hopf bifurcation when  $\sigma = \bar{\sigma}^H$ , becomes locally unstable when  $\sigma \in (\bar{\sigma}^T, \bar{\sigma}^H)$ , and becomes again saddle-point stable when  $\sigma \in [0, \bar{\sigma}^T)$ .*

*ii) For  $\Theta \in (\tilde{\Theta}, \bar{\Theta})$ , the steady state is locally indeterminate when  $\sigma \in (\bar{\sigma}^H, +\infty)$ , undergoes a Hopf bifurcation when  $\sigma = \bar{\sigma}^H$ , becomes locally unstable when  $\sigma \in (\bar{\sigma}^T, \bar{\sigma}^H)$ , and becomes saddle-point stable when  $\sigma \in [0, \bar{\sigma}^T)$ .*

Case ii) of Proposition 4 is a new result in the literature which proves that when the externalities are not too weak but still in line with the estimates of Basu and Fernald [4], local indeterminacy arises even though the EIS in consumption is arbitrarily small. More generally, Proposition 4 shows that, when the elasticity of labor is large enough, increasing slightly the size of externalities allows to decrease the value of the EIS in consumption compatible with local indeterminacy.

As in the GHH specification, the identification of the Hopf bifurcation in Propositions 3 and 4 will play an important role in our data confrontation analysis. Indeed, building again on a better compromise between intertemporal elasticity of substitution, income effects and wage elasticity of labor supply, we will show that the best performing calibrations are typically close to the Hopf bifurcation locus.

## 4 Quantitative analysis

In this section, we turn to the quantitative analysis of the 2-sector real business cycle model. Previous papers in the literature have already performed this data confrontation step and have identified three empirical regularities that are hardly accounted for by such models:<sup>11</sup> the consumption/investment cyclical puzzle (the inability to generate procyclical comovements of consumption and investment with output), the labor comovement puzzle (the inability to generate procyclical movement in sectoral hours worked), and the hours worked volatility puzzle (the inability to generate sufficiently volatile hours worked relatively to output).

The main contribution of this section is to show that many of these difficulties can actually be overcome by adopting a sufficiently general specification for preferences and an appropriate calibration for structural parameters. The theoretical characterization of the local stability properties of the model is important at this stage, as it helps to identify calibrations that, at the same time, (i) make the model sufficiently close to a Hopf bifurcation, thus generating persistent and non-monotonous dynamics of convergence to the steady-state;<sup>12</sup> (ii) exploit more adequately the various underlying economic mechanisms (intertemporal substitution in consumption, labor supply elasticity, etc.) important in the dynamic properties of the business cycle.

In the following we consider two standard assumptions about the role of the sunspot process: (i) sunspot shocks are purely exogenous and are the only source of disturbances, and (ii) sunspot shocks are perfectly correlated with fundamental disturbances — here, technological shocks affecting the total factor productivity level  $z_t$ . The latter is assumed to follow the exogenous stochastic process  $\log(z_{t+1}) = \rho \log(z_t) + \varepsilon_{t+1}$ , with  $0 < \rho < 1$ ,  $z_0$  given, where  $\varepsilon_t$  a normally distributed technological shock.<sup>13</sup> In this case, sunspots act as an *amplification mechanism* of

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<sup>11</sup>See e.g. Benhabib and Farmer [4], Harrison [21], Guo and Harisson [19].

<sup>12</sup>See Dufourt et al. (2011) for a more precise discussion of this point.

<sup>13</sup>In the case of exogenous shocks on fundamentals, one must also ensure that the model remains in the basin of attraction of the stable steady state. This is particularly true for calibrations near a subcritical Hopf bifurcation, since in this case the basin of attraction is delimited by an invariant

real shocks.

## 4.1 Data

In order to test the ability of 2-sector dynamic models to fit empirical facts, we constructed our own data set, adapting an approach introduced by Baxter [2] to identify two broadly defined consumption and investment sectors. Contrarily to Baxter, who worked with annual data, our data set is quarterly and thus more suitable for business cycle analysis.<sup>14</sup>

I. Volatility						
(absolute and relative standard deviations)						
$(x)$	$Y$	$C$	$pI$	$L$	$L_C$	$L_I$
$\sigma_x$	2.32	0.94	4.59	2.38	1.52	4.64
$\sigma_x/\sigma_Y$	1	0.41	1.98	1.03	0.66	2.00
II. Persistence						
(first-order autocorrelation)						
$(x)$	$Y$	$C$	$pI$	$L$	$L_C$	$L_I$
$\rho_x$	0.90	0.87	0.89	0.92	0.91	0.92
III. Covariations						
(contemporaneous correlations with output)						
$(x)$	$Y$	$C$	$pI$	$L$	$L_C$	$L_I$
$\text{corr}(x, Y)$	1	0.92	0.98	0.84	0.83	0.84
Sectoral labor comovements:						
$\text{corr}(L_C, L_I) = 0.97$						

Table 1: US Data - Cyclical properties

Table 1 provides the main summary statistics (second-order moments), after all series were detrended using the HP filter. The results display all the well-known stylized facts concerning aggregate consumption, investment, output and hours worked,

closed curve surrounding the steady-state. Thus, strictly speaking, the normal distribution must be truncated to avoid that an extreme realization of  $\epsilon_t$  makes endogenous variables leave the stable neighborhood of the steady state.

<sup>14</sup>A detailed description of the data used for the construction of aggregate and sectoral variables is provided in Appendix 6.5. All the series can be downloaded at: <http://greqam.univ-mrs.fr/spip.php?rubrique1438&a=921>



so that we do not comment them further. The main new features concern the co-movements of labor and hours worked in the consumption and investment sectors. As Table 1 indicates, hours worked in the consumption sector appear to be significantly less volatile than output and total hours worked. By contrast, hours worked in the investment sector are twice as volatile as output. Fluctuations in each variable are very persistent (the first-order autocorrelation coefficients are above 0.9) and are strongly positively correlated with output. Finally, hours worked in the consumption and in the investment sectors are very strongly positively correlated (0.97). Note that most of these features are consistent with those obtained by Baxter [2], with the main difference that we obtain substantially larger autocorrelation and contemporaneous correlation coefficients.<sup>15</sup> This is what we expected, taking account of the fact that our dataset is quarterly.

## 4.2 Data confrontation strategy: admissible calibrations

We focus in the following on a set of parameters 1) that are empirically admissible, and 2) that give rise to local indeterminacy. In practice, calibration of the most common parameters  $(\delta, \beta, \rho, \alpha)$  is not controversial, and we follow Guo and Harrison [19] by setting the quarterly depreciation rate of capital to  $\delta = 0.025$ , the quarterly subjective discount factor to  $\beta = 0.99$ , the autoregressive coefficient of technological process to  $\rho = 0.95$  and the capital-output elasticity coefficient (which is identical to the capital share of national income) to  $\alpha = 0.3$ . By contrast, calibration for  $\Theta$ ,  $\chi$  and  $\sigma$  is much more controversial, as a wide range of empirical estimates exists for these parameters. Based on our reading of the empirical literature on the degree of increasing returns to scale, the wage elasticity of the labor supply and the EIS in consumption,<sup>16</sup> we define a set of *empirically admissible calibrations* for  $\Theta$ ,  $\chi$  and

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<sup>15</sup>The Baxter analysis, based on HP-filtered annual data with smoothing coefficient  $\lambda = 400$ , reports relative volatility coefficient of hours worked in the consumption and investment sector of 0.87 and 3.37, respectively. The correlation between these two series is 0.87, and the first-order autoregressive coefficients are 0.32 and 0.46, respectively.

<sup>16</sup>See Burnside *et al.* [8], and Basu and Fernald [1] for a discussion of the degree of increasing returns to scale at the aggregate level or in the investment sector. For the wage elasticity of labor supply, see Blundell and MaCurdy [7] for low estimates at the micro level, Rogerson and Wallenius [33] for large estimates at the macro level, and Hansen [20] and Rogerson [32] for theoretical considerations leading to an infinite elasticity. For the EIS in consumption, see Campbell [9], Kocherlakota [25] for estimates lower than unity, and Gruber [18], Mulligan [27], Vissing-Jorgensen and Attanasio [34] for estimates larger than unity.

$\epsilon_{cc}(\chi, \sigma)$  as:

$$\Gamma_1 \equiv \{(\Theta, \chi, \sigma) \in \mathbb{R}_+^3 \mid \Theta \in (0, .44), \chi \in (0, +\infty), \epsilon_{cc}(\chi, \sigma) \in (0, 2)\}$$

### 4.3 Evaluation: GHH preferences

In the case of GHH preferences, Proposition 2 straightforwardly enables us to derive the set  $\Gamma_2$  of calibrations consistent with indeterminacy:

$$\Gamma_2 \equiv \{(\Theta, \chi, \sigma) \in \mathbb{R}_+^3 \mid \Theta \in (\underline{\Theta}, \bar{\Theta}), \chi > \underline{\chi}(\Theta), \sigma \in (\sigma_H(\Theta, \chi), \sigma_F(\Theta, \chi))\},$$

where  $\underline{\chi}(\Theta)$ ,  $\sigma_H(\Theta, \chi)$  and  $\sigma_F(\Theta, \chi)$  are defined by (12), (14) and (15), once a fixed calibration for  $(\beta, \delta, \alpha)$  is chosen. The set  $\Gamma = \Gamma_1 \cap \Gamma_2$  of *admissible calibrations* corresponds, in Figure 7, to the interior subset delimited by the flip and the Hopf bifurcation loci in the  $(\Theta, \chi, \sigma)$  space.

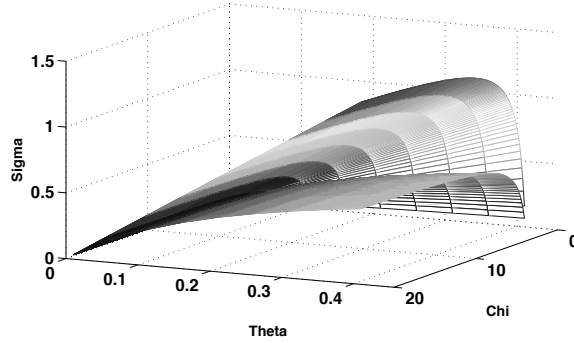


Figure 7: *Admissible calibrations* set in the  $(\Theta, \chi, \sigma)$  space – GHH

Note: The indeterminacy zone is the interior area delimited by the flip (upper curve) and the Hopf (lower curve) bifurcation loci.

A simple inspection of this figure leads to two immediate conclusions. First, the parameter governing the EIS in consumption,  $\sigma$ , has a strong influence on the dynamic properties of the model: for large  $\sigma$ 's, changes in the local stability properties of the steady-state only occur through a flip bifurcation in the  $(\Theta, \chi)$  plane, while for low  $\sigma$ 's both a Hopf and a flip bifurcation appear. Second, the figure shows that for any degree of increasing returns to scale  $\Theta$ , considering smaller values for  $\sigma$  enables to obtain indeterminacy with much larger labor supply elasticity (i.e. smaller  $\chi$ 's). Both features will turn out to be important for improving the model's predictions, as we show in the next two subsections.

**Logarithmic consumption case:**  $\sigma = 1$ . Figure 8 displays the indeterminacy area when the specification of individual preferences is restricted to the class of

logarithmic consumption utility functions ( $\sigma = 1$ ). As can be seen, the benchmark calibration considered by Guo and Harrison [19],  $\{GH1: \sigma = 1, \Theta = 0.3, \chi = 15\}$ , implying a value  $\epsilon_{cc} = 0.94$ , locates the model very close to the flip bifurcation locus.

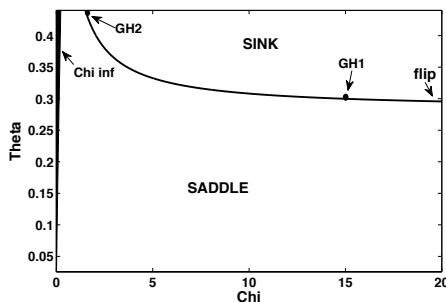


Figure 8: Cross section at  $\sigma = 1$  – GHH preferences

Table 2 reports the simulation results, with purely exogenous sunspot shocks in column  $S$ , and sunspot shocks that are perfectly correlated with technological shocks in column  $TS$ .<sup>17</sup> The statistics reported in Table 2 are the average of the second-order moments obtained from simulating 200 series of 190 observations (the length of our data set), and after detrending all series using the HP filter. HP-filtering explains why the results of Table 2 differ from those reported by Guo and Harrison [19]

As can be seen, and as shown by Guo and Harrison, the model’s performance under the GH1 calibration is very poor. The model exhibits a dramatic over-volatility of consumption and investment relatively to output, a dramatic under-volatility of total hours worked (the hours worked volatility puzzle), and a substantial over-volatility of hours worked in each of the consumption and the investment good sector. The investment sector is countercyclical (the consumption/investment cyclicity puzzle), as are the sectoral comovements in hours worked in the consumption and the investment sectors (the sectoral labor cyclicity puzzle).

Since the model under the GH1 calibration exhibits a large under-volatility of total hours worked, a better calibration requires to increase the elasticity of the labor supply (i.e., to decrease  $\chi$ ). However, as Figure 8 reveals, in the case  $\sigma = 1$ ,

<sup>17</sup>In the latter case, an assumption must be made about the initial response of the free endogenous control variable (which we assume to be the relative price of investment) following the exogenous technological shock. We assume that for moderate degrees of IRS in the investment sector (up to 30%), the relative price of investment decreases by 1% in response to a 1% positive technological shock. For larger degrees of increasing returns, we assume the decrease is 1.4%.

	Data	GH1		GH2	
		$S$	$TS$	$S$	$TS$
<u>Rel. std. dev.:</u>					
Cons. ( $C$ )	0.41	13.2	3.82	9.62	2.13
Inv. ( $pI$ )	1.98	47.3	16.3	34.1	10.4
Hours: ( $L_C + L_I$ )	1.03	0.06	0.06	0.38	0.38
cons. hours ( $L_C$ )	0.66	12.9	4.24	9.43	2.41
inv. hours. ( $L_I$ )	2.00	47.5	15.8	34.3	10.0
<u>Corr. with output</u>					
Cons. ( $C$ )	0.92	0.29	-0.35	0.34	-0.35
Inv. ( $pI$ )	0.98	-0.20	0.59	-0.21	0.70
hours: ( $L_C + L_I$ )	0.84	0.99	0.99	0.99	0.99
cons hours. ( $L_C$ )	0.83	0.22	-0.54	.28	-0.56
inv. hours. ( $L_I$ )	0.84	-0.22	0.55	-0.23	0.68
Corr. ( $L_C, L_I$ )	0.97	-0.99	-0.99	-0.99	-0.99

Table 2: Model properties, GHH preferences with  $\sigma = 1$

the possibilities of doing so while remaining in the set of admissible calibrations are severely limited. Even considering the largest bound for the degree of increasing returns,  $\Theta = 0.44$ ,<sup>18</sup> the minimal value for the inverse of labor supply elasticity consistent with indeterminacy is  $\chi = 1.6$ . As shown in Table 2, under this alternative calibration  $\{GH2: \sigma = 1, \Theta = 0.44, \chi = 1.6\}$ , the model still does not deliver satisfying results.

**Generalized GHH preferences.** As shown by Figure 7, decreasing  $\sigma$  enables to get indeterminacy with much larger labor supply elasticities. Assume for example that the degree of increasing returns in the investment sector is set to  $\Theta = 0.42$ , a value close to the upper range of empirically credible estimates but still implying a downward sloping aggregate labor demand curve. In this case, the lower bound imposed on the preference parameter  $\chi$  is  $\underline{\chi}(0.42) \approx 0.22$ , and the Hopf bifurcation value for  $\sigma$  is  $\sigma^H \approx 0.17$ . The calibration  $\{DNV1: \sigma = 0.2, \chi = 0.23, \Theta = 0.42\}$

<sup>18</sup>Note that for this value of  $\Theta$ , which is the upper bound (point estimate + one standard deviation) of the degree of IRS in the durable manufacturing industry obtained by Basu and Fernald (1997), the *aggregate* degree of IRS in the model is only 9.4% (since the share of investment in GDP is 21%).

locates the model close to this Hopf bifurcation value while generating a value  $\epsilon_{cc} = 1.38$  well within the range of admissible values for the EIS in consumption.

	Data	DNV1		DNV2	
		$S$	$TS$	$S$	$TS$
<u>Rel. std. dev.:</u>					
Cons. ( $C$ )	0.41	3.24	0.78	3.31	0.87
Inv. ( $pI$ )	1.98	10.7	2.15	11.0	2.13
Hours: ( $L_C + L_I$ )	1.03	0.81	0.81	0.86	0.87
cons. hours. ( $L_C$ )	0.66	3.16	0.61	3.26	0.75
inv. hours. ( $L_I$ )	2.00	10.7	1.99	11.0	2.13
<u>Corr. with output</u>					
Cons. ( $C$ )	0.92	0.43	0.95	0.43	0.92
Inv. ( $pI$ )	0.98	-0.05	0.90	-0.05	0.82
hours: ( $L_C + L_I$ )	0.84	0.99	0.99	0.99	0.99
cons hours. ( $L_C$ )	0.83	0.38	0.91	0.39	0.90
inv. hours. ( $L_I$ )	0.84	-0.06	0.89	-0.06	0.80
Corr. ( $L_C, L_I$ )	0.97	-0.95	0.62	-0.94	0.45

Table 3: Model properties, generalized GHH

Table 3 presents the simulation results associated with this calibration. As can be seen, the model still does not deliver good results when sunspot shocks are assumed to be the only source of disturbances. Yet, the model performs substantially better when sunspots are perfectly correlated with technological shocks. All the relative volatilities are now close to their empirical counterparts, and the contemporaneous cross-correlations with output have the correct sign. In particular, the model solves the “consumption/investment cyclical puzzle” and the “labor comovement puzzle”. The “labor-volatility puzzle” is also substantially alleviated, as the relative volatility of hours worked in the model (0.81) is now much closer to its empirical counterpart.

Finally, for robustness check, we present the results obtained with a more conservative degree of increasing returns to scale in the investment sector  $\Theta = 0.3$ . In this case, the preference parameters  $\chi$  and  $\sigma$  are also reduced close to their new lower bounds,  $\underline{\chi}(0.3) = 0.1475$  and  $\sigma_H(0.3, 0.15) \approx 0.12$ . This calibration  $\{DNV2 \text{ calibration: } \sigma = 0.12, \chi = 0.15, \Theta = 0.3\}$  implies a value  $\epsilon_{cc} = 1.88$  for the EIS in consumption. Results are displayed in the corresponding entry of Table 3.

As can be verified, the model’s predictions are only marginally deteriorated, and the overall performance remains, we believe, quite satisfactory.

#### 4.4 Separable preferences

We now briefly turn to the case of separable preferences. We restrict our analysis to the configuration with  $\chi = 0$  since, even in the best performing case of correlated sunspots, the relative volatility of hours worked remains too low compared to its empirical counterpart. Considering larger values for  $\chi$  would further worsen this result.

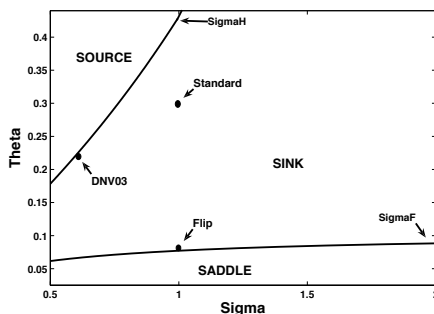


Figure 9: Cross section at  $\chi = 0$  – Separable preferences

Figure 9 displays the indeterminacy-determinacy areas obtained from Propositions 3 and 4 in the case  $\chi = 0$ . In the figure, the set of *admissible calibrations* is the *sink* area located to the right of the vertical line  $\sigma = 0.5$  (remember that  $\epsilon_{cc} = 1/\sigma$  in the separable preferences case). As can be seen, when  $\sigma = 1$ , indeterminacy prevails for a wide range of calibrations for  $\Theta$ . Setting  $\Theta$  close to its minimum value consistent with indeterminacy locates the model close to the flip bifurcation locus. Simulation results obtained under this calibration are reported in Table 4 *{Flip calibration:  $\sigma = 1, \chi = 0, \Theta = 0.08$ }*. The model displays all the well-known difficulties associated with the 2-sector model, independently of whether sunspots are autonomous or correlated disturbances.

Increasing the degree of increasing returns to scale helps to improve the model’s performance, but only in the case of correlated sunspots. For example, setting  $\Theta$  to its largest value in the set of admissible calibration *{Standard calibration:  $\sigma = 1, \chi = 0, \Theta = 0.44$ }*, Table 4 reveals that consumption and labor in the consumption sector are now procyclical, and that sectoral hours worked in the consumption and the investment sectors are also positively correlated. Yet, the sectoral labor volatility puzzle remains as the relative volatility of hours worked in the consumption sector is zero, compared to 0.66 in the data.

	Data	Flip		Standard		DNV3	
		$S$	$TS$	$S$	$TS$	$S$	$TS$
<u>Rel. std. dev.:</u>							
Cons. ( $C$ )	0.41	0.47	0.42	0.51	0.48	0.79	0.48
Inv. ( $pI$ )	1.98	6.34	6.00	6.13	3.09	6.87	3.24
Hours: ( $L_C + L_I$ )	1.03	1.45	1.36	1.40	0.58	1.35	0.76
cons. hours ( $L_C$ )	0.66	0.00	0.00	0.01	0.00	0.32	0.21
inv. hours. ( $L_I$ )	2.00	6.79	6.37	6.56	2.69	7.27	3.03
<u>Corr. with output</u>							
Cons. ( $C$ )	0.92	-0.94	-0.81	-0.73	0.94	-0.65	0.89
Inv. ( $pI$ )	0.98	0.99	0.99	0.98	0.98	0.95	0.97
hours: ( $L_C + L_I$ )	0.84	0.99	0.98	0.97	0.96	0.96	0.98
cons hours. ( $L_C$ )	0.83	-0.94	-0.81	-0.73	0.94	-0.65	0.89
inv. hours. ( $L_I$ )	0.84	0.99	0.98	0.97	0.98	0.94	0.95
Corr. ( $L_C, L_I$ )	0.97	-0.97	-0.90	-0.87	0.80	-0.88	0.70

Table 4: Model properties, separable preferences

To understand this last result, it is useful to analyze the functioning of the labor market and the way the allocation of hours worked between sectors is achieved in the model. Combining the labor supply equation  $\omega_t c_t^{-\sigma} = B l_t^X$ , see (16), with the sectoral labor demand equation in the consumption sector,  $\omega_t = (1 - \alpha) c_t / L_{ct}$ , see (2), we obtain, setting  $\chi = 0$ :

$$(1 - \alpha) c_t^{1-\sigma} = B L_{ct}$$

Thus, when  $\sigma = 1$ , the relationship simplifies to  $L_{ct} = (1 - \alpha) / B$  and hours worked in the consumption sector are *constant* in the business cycle. This shows that inability to generate sufficiently volatile hours worked in the consumption sector is *endemic* to the case  $\sigma = 1$ . In particular, it does not depend on the degree of increasing returns to scale  $\Theta$ .

In the case  $\sigma > 1$ , hours worked in the consumption sector become volatile, but they are also negatively correlated with consumption (and with other macroeconomic variables). Thus, solving the sectoral labor *volatility* puzzle implies to reintroduce the sectoral labor *cyclical* puzzle, even if large degrees of increasing returns to scale are considered.

In the end, the only way to generate sufficiently volatile and procyclical movements in hours worked in the consumption sector is to assume  $\sigma < 1$ . But as Figure

9 emphasized, decreasing  $\sigma$  also shrinks the range of values for  $\Theta$  for which the model is indeterminate. Since sufficiently large increasing returns are also required to solve the “sectoral labor *cyclical* puzzle”, there is clearly a tradeoff involved in the joint calibration of  $\sigma$  and  $\Theta$ . In any case, successful calibrations require to associate with any value of  $\sigma$  the *largest* value for  $\Theta$  in the set of admissible calibrations. In other word, improving the model’s performance requires to consider calibrations that locate the model close to the Hopf bifurcation locus in Figure 9.

The calibration  $\{DNV3 \text{ calibration: } \sigma = 0.6, \chi = 0, \Theta = 0.22\}$  analyzed in Table 4, shows that the model still does not deliver satisfying results when sunspot shocks are assumed to be an autonomous source of disturbances. For example, it does not solve the consumption and the sectoral hours worked cyclical puzzles.<sup>19</sup> But in the case of perfectly correlated sunspots, investment and hours worked are now strongly procyclical, and sectoral hours worked are strongly positively correlated. The main difference with the *standard calibration* is that the relative volatility of hours worked in the consumption sectors has increased from 0 to 0.21. Although this remains too low compared to the empirical value of 0.66, the model clearly goes in the right direction in this dimension.

In the end, comparing the results of the last two subsections, the GHH specification enables to obtain slightly better empirical results compared to the additively-separable specification. In both cases, improving the model’s predictions required to increase the value of the EIS in consumption compared to the traditional logarithmic consumption utility function.

## 5 Concluding comments

Although multi-sector real business cycle models are based on a pervasive feature of the data and require smaller degrees of increasing returns to scale for indeterminacy than aggregate models, they are usually criticized on a few shortcomings : the labor comovement and consumption-investment cyclical puzzles and the low volatility of aggregate hours. Many contributions have tried to solve these shortcomings and

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<sup>19</sup>Note that our results in this regard appear to be in contradiction with those obtained by Benhabib and Farmer [4] and Harrison [21], who report positive correlations between consumption and output for sufficiently large degrees of increasing returns. The contradiction is only apparent, however, for these papers considered *non-filtered* simulated data. If we followed this route and did not filter our simulated data, the contemporaneous correlations implied by our model would also have been positive. This shows that the positive correlation reported in these papers is due to low frequency movements in consumption and output that are removed by the HP filter.



the traditional strategy is to increase the degree of increasing returns. The main contribution of this paper is to show that a strong interaction between theoretical and empirical analysis allows to significantly improve the empirical predictions of the model.

We have provided a detailed theoretical analysis of the local stability properties of the steady state in order to get a full picture of the configurations giving rise to local indeterminacy and sunspot fluctuations. With GHH preferences, we have shown that local indeterminacy occurs through flip and Hopf bifurcations for a large set of values of the EIS in consumption if the labor supply is sufficiently inelastic. With additively-separable preferences, we have proved that local indeterminacy occurs through flip and Hopf bifurcations for any value of the elasticity of the labor supply, and can even be compatible with an arbitrarily low EIS in consumption. Moreover, in both cases the existence of expectation-driven fluctuations is consistent with a mild amount of increasing returns.

Finally, building on the detailed theoretical analysis with both specifications of preferences, by increasing the value of the EIS in consumption in order to be close to the Hopf bifurcation locus, we have been able to find, among the set of admissible parameters configurations consistent with sunspot equilibria, the ones that provide the best fit of the data relatively to the *consumption cyclical puzzle*, the *labor comovement puzzle*, and the *hours volatility puzzle*.

## 6 Appendix

### 6.1 Proof of Proposition 1

Equation (6) evaluated at the steady state gives  $I = Y_I = \delta k$ . Moreover, we derive from (2) that  $r/p = \alpha Y_I / K_I = \alpha \delta k / K_I$ . It follows from (8) that  $K_I = \beta \delta \alpha k / (1 - \theta)$  with  $\theta = \beta(1 - \delta)$ . Merging equations (2) gives  $L_I / K_I = L_c / K_c = l/k$  and thus we get from (10)

$$\bar{k} = l^{\frac{(1-\alpha)(1+\Theta)}{1-\alpha(1+\Theta)}} \delta^{\frac{\Theta}{1-\alpha(1+\Theta)}} \left( \frac{\beta\alpha}{1-\theta} \right)^{\frac{1+\Theta}{1-\alpha(1+\Theta)}} \equiv l^{\frac{(1-\alpha)(1+\Theta)}{1-\alpha(1+\Theta)}} \bar{\kappa} \quad (21)$$

Consider now  $c = Y_c = K_c^\alpha L_c^{1-\alpha} = (k - K_I) \left( \frac{l}{k} \right)^{1-\alpha}$ . Substituting (21) into this expression gives

$$\bar{c} = \left( 1 - \frac{\beta\delta\alpha}{1-\theta} \right) \bar{\kappa}^\alpha l^{\frac{1-\alpha}{1-\alpha(1+\Theta)}} \equiv \bar{\psi} l^{\frac{1-\alpha}{1-\alpha(1+\Theta)}} \quad (22)$$

From the expression of the prices in (2) we derive

$$\bar{\omega} = (1 - \alpha) \bar{\kappa}^\alpha l^{\frac{\alpha\Theta}{1-\alpha(1+\Theta)}} \equiv \bar{\phi} l^{\frac{\alpha\Theta}{1-\alpha(1+\Theta)}}, \quad \bar{r} = \alpha l^{\frac{-(1-\alpha)\Theta}{1-\alpha(1+\Theta)}} \bar{\kappa}^{\alpha-1}, \quad \bar{p} = \frac{\beta}{1-\theta} \bar{r} \quad (23)$$

Equation (9) now gives  $U_1/U_2 = w$ .

i) In the case of a GHH utility function, we get

$$\bar{\phi} = Bl^{\chi - \frac{\alpha\Theta}{1-\alpha(1+\Theta)}} \quad (24)$$

Therefore, if  $\chi - \alpha\Theta/(1 - \alpha(1 + \Theta)) \neq 0$  and  $\ell$  is large enough, there generically exists a unique steady state for labor

$$\bar{l} = (\bar{\phi}/B)^{\frac{1-\alpha(1+\Theta)}{\chi(1-\alpha(1+\Theta))-\alpha\Theta}} \quad (25)$$

ii) In the case of an additively separable utility function, we get

$$\bar{\phi}\bar{\psi}^\sigma = Bl^{1+\chi - \frac{(1+\sigma)(1-\alpha)}{1-\alpha(1+\Theta)}} \quad (26)$$

Therefore, if  $1 + \chi - (1 + \sigma)(1 - \alpha)/(1 - \alpha(1 + \Theta)) \neq 0$  and  $\ell$  is large enough, there generically exists a unique steady state for labor

$$\bar{l} = (\bar{\phi}\bar{\psi}^\sigma/B)^{\frac{1-\alpha(1+\Theta)}{(1+\chi)(1-\alpha(1+\Theta))-(1+\sigma)(1-\alpha)}} \quad (27)$$

□

## 6.2 Computation of the linearized dynamical system

Let us introduce the following elasticities:

$$\epsilon_{cc} = -\frac{U_1(c,l)}{U_{11}(c,l)c}, \quad \epsilon_{lc} = -\frac{U_2(c,l)}{U_{21}(c,l)c}, \quad \epsilon_{cl} = -\frac{U_1(c,l)}{U_{12}(c,l)l}, \quad \epsilon_{ll} = -\frac{U_2(c,l)}{U_{22}(c,l)l} \quad (28)$$

Since  $U(c, \ell)$  is decreasing and concave with respect to  $l$ , the elasticity  $\epsilon_{ll}$  is negative. We easily derive a useful relationship between  $\epsilon_{lc}$  and  $\epsilon_{cl}$ .

**Lemma 6.1.** *At the steady state*

$$\epsilon_{cl} = -\frac{(1-\frac{\beta\delta\alpha}{1-\theta})}{1-\alpha}\epsilon_{lc} \quad (29)$$

*Proof:* Using (28) and the second equation in (8), we get  $\epsilon_{cl} = -\epsilon_{lc}(c/wl)$ . At the steady state the result follows from (22) and (23). □

Considering the first-order conditions (8) and (9) together with the 2 constraints in the optimization program (7), we get the following 3 equations

$$\begin{aligned} U_1(c_t, l_t) &= \beta U_1(c_{t+1}, l_{t+1}) \left[ \frac{r_{t+1} + (1-\delta)p_{t+1}}{p_t} \right] \\ \omega_t &= -\frac{U_2(c_t, l_t)}{U_1(c_t, l_t)} \\ c_t + p_t[k_{t+1} - (1-\delta)k_t] &= r_t k_t + \omega_t l_t \end{aligned}$$

Using (23), total differentiation of these equations in a neighborhood of the steady state gives after simplifications:

$$\begin{aligned}
& \frac{dk_{t+2}}{k} n - \frac{dk_{t+1}}{k} \frac{1+\Theta(1-\theta)}{\beta} M_{11} + \frac{dk_t}{k} \frac{1+\Theta(1-\theta)}{\beta} N_{11} \\
& - \frac{dl_{t+1}}{l} \frac{(1-\alpha)(1-\theta)(1+\Theta)\bar{\kappa}}{\beta\alpha} M_{12} + \frac{dl_t}{l} \frac{(1-\alpha)(1-\theta)(1+\Theta)\bar{\kappa}}{\beta\alpha} N_{12} = 0 \\
& \frac{dk_{t+1}}{k} m - \frac{dk_t}{k} \frac{1+\Theta(1-\theta)}{\beta} N_{21} - \frac{dl_t}{l} \frac{(1-\alpha)(1-\theta)(1+\Theta)\bar{\kappa}}{\beta\alpha} N_{22} = 0
\end{aligned} \tag{30}$$

with

$$\begin{aligned}
m &= \frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}, \quad n = \frac{1}{\epsilon_{cc}} - \frac{(1-\delta)(1-\theta)\Theta\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\delta\alpha} \\
M_{11} &= \frac{1+\beta+\Theta(1-\theta)}{1+\Theta(1-\theta)} \left[ \frac{1}{\epsilon_{cc}} + \frac{\left[(1-\theta)(1-\alpha)(1+\Theta) - \frac{1+\theta(1-\delta)}{\delta}\Theta\right](1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\alpha[1+\beta+\Theta(1-\theta)]} \right] \\
M_{12} &= \frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}} - (1-\theta) \left(1 - \frac{\beta\alpha\delta}{1-\theta}\right), \quad N_{11} = \frac{1}{\epsilon_{cc}} - \frac{\frac{1-\delta}{\delta}\Theta(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\alpha[1+\Theta(1-\theta)]} \\
N_{12} &= \frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}, \quad N_{21} = \frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}} - \frac{(1+\Theta)(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{1+\Theta(1-\theta)} \\
N_{22} &= \frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}} - \left(\frac{1}{\epsilon_{ll}} - \frac{1}{\epsilon_{cl}} - \alpha\right) \frac{\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{1-\alpha}
\end{aligned} \tag{31}$$

### 6.3 GHH preferences

We need to compute the elasticities of the GHH utility function. We get from (28) and (29) that at the steady state

$$\epsilon_{cc} = \epsilon_{lc}, \quad \frac{1}{\epsilon_{ll}} - \frac{1}{\epsilon_{cl}} = -\chi < 0 \quad \text{with} \quad \epsilon_{cc} = \frac{1}{\sigma} \left(1 - \frac{1-\alpha}{(1+\chi)\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)}\right) \tag{32}$$

and

$$\epsilon_{cl} = -\epsilon_{cc} \frac{\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{1-\alpha} = \frac{1}{\sigma} \left(\frac{1}{1+\chi} - \frac{1-\frac{\beta\delta\alpha}{1-\theta}}{1-\alpha}\right) \tag{33}$$

From all this we get  $m = N_{12} = 0$  and

$$\begin{aligned}
n &= \frac{\delta\alpha - \epsilon_{cc}(1-\delta)\Theta\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\epsilon_{cc}\delta\alpha} \\
M_{11} &= \frac{\frac{\alpha[1+\beta+\Theta(1-\theta)]}{1-\theta} + \epsilon_{cc} \left[ (1-\theta)(1-\alpha)(1+\Theta) - \frac{1+\theta(1-\delta)}{\delta}\Theta \right] \left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\epsilon_{cc}\alpha\left(\frac{1+\Theta(1-\theta)}{1-\theta}\right)} \\
M_{12} &= -(1-\theta) \left(1 - \frac{\beta\alpha\delta}{1-\theta}\right), \quad N_{11} = \frac{\alpha\left(\frac{1+\Theta(1-\theta)}{1-\theta}\right) - \epsilon_{cc}\frac{1-\delta}{\delta}\Theta\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\epsilon_{cc}\alpha\left(\frac{1+\Theta(1-\theta)}{1-\theta}\right)} \\
N_{21} &= -\frac{(1+\Theta)(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{1+\Theta(1-\theta)}, \quad N_{22} = \frac{(\chi+\alpha)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{1-\alpha}
\end{aligned}$$

From (22), (23) and (32) we have

$$\epsilon_{cc} \left(1 - \frac{\beta\alpha\delta}{1-\theta}\right) = \frac{1}{\sigma} \left[ \left(1 - \frac{\beta\delta\alpha}{1-\theta}\right) - \frac{1-\alpha}{1+\chi} \right]$$

We then derive from (30) the characteristic polynomial  $P_{\epsilon_{cc}}(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D} = 0$  with

$$\begin{aligned}
\mathcal{D} = \mathcal{D}_\chi(\sigma) &= \frac{(1-\theta)\delta \left\{ \sigma\alpha\left(\frac{1+\Theta(1-\theta)}{1-\theta}\right) - \frac{1-\delta}{\delta}\Theta \left[ \left(1-\frac{\beta\delta\alpha}{1-\theta}\right) - \frac{1-\alpha}{1+\chi} \right] \right\}}{\beta \left\{ \sigma\delta\alpha - (1-\delta)(1-\theta)\Theta \left[ \left(1-\frac{\beta\delta\alpha}{1-\theta}\right) - \frac{1-\alpha}{1+\chi} \right] \right\}} \\
\mathcal{T} = \mathcal{T}_\chi(\sigma) &= 1 + \mathcal{D}_\chi(\sigma) + \frac{(1-\theta)^2\delta[\chi(1-\alpha(1+\Theta)) - \alpha\Theta] \left[ \left(1-\frac{\beta\delta\alpha}{1-\theta}\right) - \frac{1-\alpha}{1+\chi} \right]}{\beta(\chi+\alpha) \left\{ \sigma\delta\alpha - (1-\delta)(1-\theta)\Theta \left(1-\frac{\beta\alpha\delta}{1-\theta}\right) \left[ \left(1-\frac{\beta\delta\alpha}{1-\theta}\right) - \frac{1-\alpha}{1+\chi} \right] \right\}}
\end{aligned}$$

We derive from this that when  $\sigma$  is varied over the interval  $[0, +\infty)$ ,  $\mathcal{D}$  and  $\mathcal{T}$  are linked through a linear relationship  $\mathcal{D} = \Delta_\chi(\mathcal{T}) = \mathcal{T}\mathcal{S}_\chi + \mathcal{C}$  with a slope

$$\mathcal{S}_\chi = \frac{(\chi+\alpha)(1-\delta)\Theta^2}{\chi\delta(1-\alpha)(1+\Theta) - (\chi+\alpha)\Theta[1-(1-\delta)(1+\Theta)]}$$

In other words,  $\Delta_\chi(\mathcal{T})$  corresponds to a half-line in the  $(\mathcal{T}, \mathcal{D})$  plane with a starting point  $(\mathcal{T}_\chi(+\infty), \mathcal{D}_\chi(+\infty))$  obtained when  $\sigma = +\infty$  such that:

$$\begin{aligned}\mathcal{D}_\chi(+\infty) &= \frac{(1-\theta)(1+\Theta)+\theta}{\beta} = \mathcal{D}(+\infty) \\ \mathcal{T}_\chi(+\infty) &= \frac{(1-\theta)(1+\Theta)+(\theta+\beta)}{\beta} = \mathcal{T}(+\infty) = 1 + \mathcal{D}(+\infty)\end{aligned}$$

It follows that when  $\sigma = +\infty$ ,  $P_{+\infty}(1) = 0$  and  $P_{+\infty}(-1) = 2(1 + \mathcal{D}(+\infty))$ . Note also that  $\mathcal{D}(+\infty) > 1$ .

When  $\epsilon_{cc}$  increases the  $\Delta_\chi(\mathcal{T})$  half-line crosses the triangle  $ABC$  depending on the slope and on the location of the end-point  $(\mathcal{T}_\chi(0), \mathcal{D}_\chi(0))$  obtained when  $\sigma = 0$ . Assume from now on that  $\alpha < 1/2$  and  $\Theta \in (\underline{\Theta}, (1-\alpha)/\alpha)$  with  $\underline{\Theta} \equiv \delta/(1-\delta)$ . It follows that the slope satisfies  $\mathcal{S}_\chi > 0$  for any  $\chi$ . Moreover, we have  $\mathcal{S}_\chi \leq 1$  if and only if  $\chi \geq \alpha\Theta/[1-\alpha(1+\Theta)] \equiv \underline{\chi}$ . Notice also that  $\partial\mathcal{S}_\chi/\partial\chi < 0$  with  $\lim_{\chi \rightarrow +\infty} \mathcal{S}_\chi = (1-\delta)\Theta^2 / [(1-\delta)\Theta^2 + (1-\alpha(1+\Theta))\delta] < 1$  and  $\partial\mathcal{D}_\chi(\sigma)/\partial\sigma > 0$ . We have indeed:

$$\begin{aligned}\mathcal{D}_\chi(0) = \lim_{\sigma \rightarrow 0} \mathcal{D}_\chi(\sigma) &= \frac{1}{\beta} = \mathcal{D}(0) \\ \mathcal{T}_\chi(0) = \lim_{\sigma \rightarrow 0} \mathcal{T}_\chi(\sigma) &= 1 + \frac{1}{\beta} - \delta(1-\theta) \frac{\chi(1-\alpha(1+\Theta)) - \alpha\Theta}{(\chi+\alpha)\theta\Theta}\end{aligned}$$

We then derive that when  $\sigma = 0$

$$\begin{aligned}P_0(1) &= \frac{(1-\theta)\delta[\chi(1-\alpha(1+\Theta)) - \alpha\Theta]}{(\chi+\alpha)\theta\Theta} \\ P_0(-1) &= \frac{\chi\{\Theta[2(1-\theta(1-\delta)) + \delta(1-\theta)\alpha] - \delta(1-\theta)(1-\alpha)\} + \alpha(1+\theta)(2-\delta)\Theta}{(\chi+\alpha)\theta\Theta}\end{aligned}$$

It follows that  $P_0(1) > 0$  if and only if  $\chi > \underline{\chi}$ . Moreover, it can be easily shown that when  $\Theta > \underline{\Theta}$ ,  $P_0(-1) > 0$  for any  $\chi \geq 0$ .

The Hopf bifurcation value value  $\bar{\sigma}^H > 0$  such that  $\mathcal{D}_\chi(\sigma) = 1$  is given by

$$\bar{\sigma}^H = \frac{\Theta(1-\delta)(1-\beta)(1-\theta) \left[ \left(1 - \frac{\beta\delta\alpha}{1-\theta}\right) - \frac{1-\alpha}{1+\chi} \right]}{\delta\alpha[1-\beta+(1-\theta)\Theta]}$$

Similarly, the flip bifurcation value value  $\bar{\sigma}^F > 0$  such that  $P_\sigma(-1) = 1 + \mathcal{T}_\chi(\sigma) + \mathcal{D}_\chi(\sigma) = 0$  is given by

$$\bar{\sigma}^F = \frac{(1-\theta)[(\chi+\alpha)\Theta[2(1-\delta)(1+\beta)+(1-\theta)\delta] - (1-\theta)\delta\chi(1-\alpha)(1+\Theta)] \left[ \left(1 - \frac{\beta\delta\alpha}{1-\theta}\right) - \frac{1-\alpha}{1+\chi} \right]}{2(\chi+\alpha)\delta\alpha[1+\beta+(1-\theta)\Theta]}$$

We easily derive that

$$\lim_{\chi \rightarrow +\infty} \mathcal{T}_\chi(\bar{\sigma}^H) = 2 - \frac{\delta[1-\beta+(1-\theta)\Theta](1-\alpha(1+\Theta))}{\theta\Theta^2} \leq 2$$

It follows that  $\lim_{\chi \rightarrow +\infty} \mathcal{T}_\chi(\bar{\sigma}^H) > -2$  if and only if

$$h(\delta) \equiv \delta^2\beta\Theta(1-\alpha(1+\Theta)) + \delta [(1-\beta)(1+\Theta)(1-\alpha(1+\Theta)) + 4\beta\Theta^2] - 4\beta\Theta^2 < 0$$

Let  $\Omega = [(1-\beta)(1+\Theta)(1-\alpha(1+\Theta)) + 4\beta\Theta^2]^2 + 4\beta^2\Theta^3(1-\alpha(1+\Theta))$ . Therefore, there exists  $\bar{\delta} \in (0, 1)$  as given by

$$\bar{\delta} = \frac{\sqrt{\Omega} - [(1-\beta)(1+\Theta)(1-\alpha(1+\Theta)) + 4\beta\Theta^2]}{2\beta\Theta(1-\alpha(1+\Theta))} \quad (34)$$

such that when  $\delta \in (0, \bar{\delta})$ ,  $\lim_{\chi \rightarrow +\infty} \mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$ . Moreover, we have  $\partial \mathcal{T}_\chi(\bar{\sigma}^H)/\partial \chi < 0$  and  $\mathcal{T}_{\underline{\chi}}(\bar{\sigma}^H) = 2$ , so that  $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$  for any  $\chi > \underline{\chi}$ . Therefore, the  $\Delta_\chi$  line, when  $\chi > \underline{\chi}$ , is located as in Figure 1. On the contrary, when  $\chi \in [0, \underline{\chi})$ , the  $\Delta_\chi$  line is located as in Figure 2 and the steady state is either saddle-point stable or unstable.  $\square$

#### 6.4 Additively separable preferences

In the case of additively separable preferences as given by (5), we get

$$\frac{1}{\epsilon_{cc}} = \sigma, \quad \frac{1}{\epsilon_{ll}} = -\chi \text{ and } \frac{1}{\epsilon_{cl}} = \frac{1}{\epsilon_{lc}} = 0 \quad (35)$$

From this we get  $m = N_{12} = \sigma$  and

$$\begin{aligned} n &= \sigma - \frac{(1-\delta)(1-\theta)\Theta(1-\frac{\beta\alpha\delta}{1-\theta})}{\delta\alpha} \\ M_{11} &= \sigma \left( \frac{1+\beta+\Theta(1-\theta)}{1+\Theta(1-\theta)} \right) + \frac{[(1-\theta)(1-\alpha)(1+\Theta) - \frac{1+\theta(1-\delta)}{\delta}\Theta] \Theta (1-\frac{\beta\alpha\delta}{1-\theta})}{\alpha \left( \frac{1+\Theta(1-\theta)}{1-\theta} \right)} \\ M_{12} &= \sigma - (1-\theta) \left( 1 - \frac{\beta\alpha\delta}{1-\theta} \right), \quad N_{11} = \sigma - \frac{\frac{1-\delta}{\delta}\Theta(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta})}{\alpha[1+\Theta(1-\theta)]} \\ N_{21} &= \sigma - \frac{(1+\Theta)(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta})}{1+\Theta(1-\theta)}, \quad N_{22} = \sigma + \frac{(\chi+\alpha)(1-\frac{\beta\alpha\delta}{1-\theta})}{(1-\alpha)} \end{aligned}$$

We then derive from (30) the following characteristic polynomial  $P_\sigma(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D} = 0$  with

$$\begin{aligned} \mathcal{D} = \mathcal{D}_\chi(\sigma) &= \frac{\sigma \left[ -\frac{1-\delta}{\delta}\Theta + (1+\Theta)\alpha + \frac{(\chi+\alpha)\alpha}{(1-\alpha)} \left( \frac{1+\Theta(1-\theta)}{1-\theta} \right) \right] - (1-\frac{\beta\alpha\delta}{1-\theta}) \frac{(1-\delta)\Theta(\chi+\alpha)}{\delta(1-\alpha)}}{\sigma \left[ -\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{(\chi+\alpha)\beta\alpha}{(1-\theta)(1-\alpha)} \right] - (1-\frac{\beta\alpha\delta}{1-\theta}) \frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}} \\ \mathcal{T} = \mathcal{T}_\chi(\sigma) &= 1 + \mathcal{D}_\chi(\sigma) + \frac{(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta}) \left[ \sigma + \frac{\chi(1-\alpha(1+\Theta)) - \Theta\alpha}{1-\alpha} \right]}{\sigma \left[ -\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{(\chi+\alpha)\beta\alpha}{(1-\theta)(1-\alpha)} \right] - (1-\frac{\beta\alpha\delta}{1-\theta}) \frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}} \end{aligned}$$

We derive from this that when  $\sigma$  is varied over the interval  $[0, +\infty)$ ,  $\mathcal{D}$  and  $\mathcal{T}$  are linked through a linear relationship  $\mathcal{D} = \Delta_\chi(\mathcal{T}) = \mathcal{T}\mathcal{S}_\chi + \mathcal{C}$  with a slope  $\mathcal{S}_\chi = (\partial \mathcal{D}_\chi(\sigma)/\partial \sigma)/(\partial \mathcal{T}_\chi(\sigma)/\partial \sigma)$ . Let us introduce the following notation:

$$\mathcal{N} = \frac{(1-\frac{\beta\alpha\delta}{1-\theta})}{\left\{ \sigma \left[ -\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{(\chi+\alpha)\beta\alpha}{(1-\theta)(1-\alpha)} \right] - (1-\frac{\beta\alpha\delta}{1-\theta}) \frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)} \right\}^2}$$

We easily derive that

$$\begin{aligned}\frac{\partial \mathcal{D}_\chi(\sigma)}{\partial \sigma} &= -\mathcal{N} \frac{\theta \Theta^2 (\chi + \alpha) \alpha (1 + \chi)}{\delta (1 - \alpha)^2} < 0 \\ \frac{\partial \mathcal{T}_\chi(\sigma)}{\partial \sigma} &= -\mathcal{N} \beta \alpha \frac{\Theta (\chi + \alpha) (1 + \chi) \left\{ \frac{(1 - \alpha)(1 - \delta)(1 - \theta)(1 + \Theta)}{\chi + \alpha} + (1 - \delta)(1 + \Theta) - 1 \right\} + \chi \delta (1 - \alpha)(1 + \Theta)(1 + \chi - \theta(1 - \alpha))}{\delta (1 - \alpha)^2}\end{aligned}$$

and thus

$$\mathcal{S}_\chi = \frac{(1 - \delta) \Theta^2 (\chi + \alpha) (1 + \chi)}{\Theta (1 + \chi) \left\{ (1 - \alpha)(1 - \delta)(1 - \theta)(1 + \Theta) + (\chi + \alpha) [(1 - \delta)(1 + \Theta) - 1] \right\} + \chi \delta (1 - \alpha)(1 + \Theta)(1 + \chi - \theta(1 - \alpha))}$$

Under Assumption 3, we conclude that if  $\Theta > \underline{\Theta}$ , then  $\partial \mathcal{T}_\chi(\sigma) / \partial \sigma < 0$  and thus  $\mathcal{S}_\chi > 0$ . We also derive that  $\mathcal{S}_\chi < 1$  if and only if

$$\begin{aligned}& \chi^2 \delta (\Theta \alpha - (1 - \alpha)) - \chi \left\{ \Theta^2 (1 - \alpha)(1 - \delta)(1 - \theta) \right. \\ & \left. + \Theta \left[ (1 - \alpha)(1 - \theta) - \delta [1 + \alpha(1 - \theta(1 - \alpha))] \right] + \delta (1 - \alpha)(1 - \theta(1 - \alpha)) \right\} \\ & - \Theta \left\{ (1 - \alpha)(1 + \Theta)(1 - \theta)(1 - \delta) - \delta \alpha \right\} \equiv g(\chi) < 0\end{aligned}\quad (36)$$

Under Assumption 3, there exists  $\tilde{\delta} \leq \hat{\delta}$  such that  $g(\chi) < 0$  for any  $\chi \geq 0$  if  $\delta < \tilde{\delta}$ .

In other words,  $\Delta_\chi(\mathcal{T})$  corresponds to a half-line in the  $(\mathcal{T}, \mathcal{D})$  plane with the starting point  $(\mathcal{T}_\chi(+\infty), \mathcal{D}_\chi(+\infty))$  obtained when  $\sigma = +\infty$  such that

$$\begin{aligned}\mathcal{D}_\chi(+\infty) &= \frac{-\frac{(1 - \delta)\Theta}{\delta} + (1 + \Theta)\alpha + \frac{(\chi + \alpha)\alpha}{(1 - \alpha)} \left( \frac{1 + \Theta(1 - \theta)}{1 - \theta} \right)}{-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{\beta\alpha(\chi + \alpha)}{(1 - \theta)(1 - \alpha)}} \\ \mathcal{T}_\chi(+\infty) &= 1 + \mathcal{D}_\chi(+\infty) + \frac{(1 - \theta) \left( 1 - \frac{\beta\alpha\delta}{1 - \theta} \right)}{-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{\beta\alpha(\chi + \alpha)}{(1 - \theta)(1 - \alpha)}}\end{aligned}$$

and the end-point  $(\mathcal{T}_\chi(0), \mathcal{D}_\chi(0))$  obtained when  $\sigma = 0$  such that:

$$\mathcal{D}_\chi(0) = \frac{1}{\beta}, \quad \mathcal{T}_\chi(0) = 1 + \mathcal{D}_\chi(0) - \delta(1 - \theta) \frac{\chi(1 - \alpha(1 + \Theta)) - \alpha\Theta}{\theta\Theta(\chi + \alpha)}$$

For the starting point, when  $\sigma = +\infty$ , i.e.  $\epsilon_{cc} = 0$ , we get:

$$\begin{aligned}P_{+\infty}(1) &= -\frac{(1 - \theta) \left( 1 - \frac{\beta\alpha\delta}{1 - \theta} \right)}{-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{\beta\alpha(\chi + \alpha)}{(1 - \theta)(1 - \alpha)}} \\ P_{+\infty}(-1) &= \frac{2 \left[ -\frac{(1 - \delta)(1 + \beta)\Theta}{\delta} + (1 + \beta + \Theta)\alpha + \frac{(\chi + \alpha)\alpha}{(1 - \alpha)} \left( \frac{1 + \Theta(1 - \theta)}{1 - \theta} \right) \right] + (1 - \theta) \left( 1 - \frac{\beta\alpha\delta}{1 - \theta} \right)}{-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{\beta\alpha(\chi + \alpha)}{(1 - \theta)(1 - \alpha)}}\end{aligned}$$

For the end point, when  $\sigma = 0$ , i.e.  $\epsilon_{cc} = +\infty$ , we get:

$$\begin{aligned}P_0(1) &= \frac{(1 - \theta) \delta [\chi(1 - \alpha(1 + \Theta)) - \alpha\Theta]}{(\chi + \alpha) \theta \Theta} \\ P_0(-1) &= \frac{\chi \{ \Theta [2(1 - \theta(1 - \delta)) + \delta(1 - \theta)\alpha] - \delta(1 - \theta)(1 - \alpha) \} + \alpha(1 + \theta)(2 - \delta)\Theta}{(\chi + \alpha) \theta \Theta}\end{aligned}$$

It follows immediately that  $P_0(1) > 0$  if and only if  $\chi > \alpha\Theta / [1 - \alpha(1 + \Theta)] \equiv \underline{\chi}$ , while it can be easily shown that  $P_0(-1) > 0$  if  $\Theta > \underline{\Theta}$ .

Finally, solving  $\mathcal{D}_\chi(\sigma) = 1$  gives

$$\bar{\sigma}^H = \frac{(1-\frac{\beta\delta\alpha}{1-\theta})\Theta(\chi+\alpha)(1-\delta)(1-\beta)}{-\Theta[(1-\alpha)(1-\theta)-\delta]+\frac{\delta\alpha(1-\beta)[1-\theta(1-\alpha)]}{1-\theta}+\frac{\chi\delta\alpha[1-\beta+\Theta(1-\theta)]}{1-\theta}} \quad (37)$$

Under Assumption 3, there exists  $\delta_0 \in (0, 1)$  as given by

$$\delta_0 = (1-\beta) \frac{-[\Theta[1+\chi\alpha-2\beta(1-\alpha)]+\alpha[1+\chi-\beta(1-\alpha)]]+\sqrt{\Lambda}}{2\beta[\Theta[1+\chi\alpha-\beta(1-\alpha)]+\alpha(1-\alpha)(1-\beta)]} \quad (38)$$

with

$$\Lambda = [\Theta[1+\chi\alpha-2\beta(1-\alpha)]+\alpha[1+\chi-\beta(1-\alpha)]]^2 + 4\Theta\beta(1-\alpha)[\Theta[1+\chi\alpha-\beta(1-\alpha)]+\alpha(1-\alpha)(1-\beta)]$$

such that  $\bar{\sigma}^H > 0$  if and only if  $\delta \in (\delta_0, 1)$ . Moreover, there exists  $\tilde{\beta} \in [\hat{\beta}, 1)$  such that  $\delta_0 < \tilde{\delta} \leq \hat{\delta}$  when  $\beta \in (\tilde{\beta}, 1)$ . Assuming  $\beta \in (\tilde{\beta}, 1)$  and  $\delta \in (\delta_0, \tilde{\delta})$ , we then derive

$$\mathcal{T}_\chi(\bar{\sigma}^H) = 2 - \frac{(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta})[\bar{\sigma}^H + \frac{[1-\alpha(1+\Theta)](\chi-\underline{\chi})}{1-\alpha}]}{\bar{\sigma}^H \left[ \frac{\theta(\Theta-\hat{\Theta})}{\delta} - \frac{\chi\beta\alpha}{(1-\theta)(1-\alpha)} \right] + (1-\frac{\beta\alpha\delta}{1-\theta}) \frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}}$$

Under Assumption 3, the denominator of the ratio in  $\mathcal{T}_\chi(\bar{\sigma}^H)$  is positive for all  $\chi \geq 0$ .

1 - Consider first the case of small externalities  $\Theta \in (\underline{\Theta}, \hat{\Theta})$  with

$$\hat{\Theta} = \frac{\delta\alpha(1-\theta(1-\alpha))}{(1-\delta)(1-\theta)(1-\alpha)}$$

It follows easily that  $\mathcal{D}_\chi(+\infty) > 1/\beta = \mathcal{D}_\chi(0)$ ,  $P_{+\infty}(1) < 0$  and  $P_{+\infty}(-1) > 0$  for any  $\chi \geq 0$ . Therefore, assuming  $\beta \in (\tilde{\beta}, 1)$  and  $\delta \in (\delta_0, \tilde{\delta})$ , provided  $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$ , we conclude that i) when  $\chi > \underline{\chi}$ , the  $\Delta_\chi$  line is located as in Figure 3, and ii) when  $\chi < \underline{\chi}$ , the  $\Delta_\chi$  line is located as in Figure 4.

Let us then provide conditions to get  $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$ .

i) When  $\chi > \underline{\chi}$  the numerator of the ratio in  $\mathcal{T}_\chi(\bar{\sigma}^H)$  is positive and thus  $\mathcal{T}_\chi(\bar{\sigma}^H) < 2$ . Let us then denote  $\bar{\delta} = \tilde{\delta}$  and  $\underline{\beta} = \tilde{\beta}$ . Note that, as  $\lim_{\delta \rightarrow 0} \hat{\Theta} = 0$ , we get  $\lim_{\delta \rightarrow 0} \mathcal{S} = 0$  and thus  $\lim_{\delta \rightarrow 0} \mathcal{T}_\chi(\bar{\sigma}^H) = -\infty$ . Therefore, there exists  $\underline{\delta} \in [\delta_0, \bar{\delta})$  such that when  $\beta \in (\underline{\beta}, 1)$  and  $\delta \in (\underline{\delta}, \bar{\delta})$ ,  $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$ .

ii) Let  $\underline{\beta} = \tilde{\beta}$  and  $\beta \in (\underline{\beta}, 1)$ . When  $\chi < \underline{\chi}$ , we need to study the numerator of the ratio in  $\mathcal{T}_\chi(\bar{\sigma}^H)$ . Note that

$$\bar{\sigma}^H + \frac{[1-\alpha(1+\Theta)](\chi-\underline{\chi})}{1-\alpha} > \bar{\sigma}^H - \frac{\alpha\Theta}{1-\alpha} = \frac{(1-\frac{\beta\delta\alpha}{1-\theta})\Theta(\chi+\alpha)(1-\delta)(1-\beta)(1-\alpha)-\alpha\Theta den\sigma^H}{(1-\alpha)den\sigma^H}$$

with  $den\sigma^H$  the denominator of  $\sigma^H$ . We have shown previously that  $den\sigma^H > 0$  if and only if  $\delta \in (\delta_0, 1)$ , with  $\lim_{\delta \rightarrow \delta_0} den\sigma^H = 0$ . Moreover, we derive from the expression of  $\sigma^H$  that  $\lim_{\delta \rightarrow 1} \sigma^H = 0$ . Therefore, we conclude that there exists  $\delta_1 \in (\delta_0, \tilde{\delta}]$  such that the numerator of the ratio in  $\mathcal{T}_\chi(\bar{\sigma}^H)$  is positive for all  $\chi \geq 0$  if  $\delta \in (\delta_0, \delta_1)$ . It follows that  $\mathcal{T}_\chi(\bar{\sigma}^H) < 2$  for all  $\chi \geq 0$  if  $\delta \in (\delta_0, \delta_1)$ . Let us then denote  $\bar{\delta} = \delta_1$ . As in case i), since  $\lim_{\delta \rightarrow 0} \hat{\Theta} = 0$ , we get  $\lim_{\delta \rightarrow 0} \mathcal{S} = 0$  and thus  $\lim_{\delta \rightarrow 0} \mathcal{T}_\chi(\bar{\sigma}^H) = -\infty$ . Therefore, there exists  $\underline{\delta} \in [\delta_0, \bar{\delta})$  such that when  $\delta \in (\underline{\delta}, \bar{\delta})$ ,  $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$ .

2 - Consider now the case of larger externalities  $\Theta > \hat{\Theta}$ . To simplify the analysis we assume that  $\chi = 0$ . We know that the end point satisfies  $\lim_{\chi \rightarrow 0} P_0(1) < 0$  and  $P_0(-1) > 0$ . The starting point satisfies:

$$\lim_{\chi \rightarrow 0} \mathcal{D}_\chi(+\infty) \equiv \mathcal{D}_0(+\infty) = \frac{(1-\alpha-\delta)(\Theta-\Theta_1)}{\beta(1-\alpha)(1-\delta)(\Theta-\hat{\Theta})}$$

and

$$\lim_{\chi \rightarrow 0} P_{+\infty}(1) = \frac{\delta(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta})}{\beta(1-\delta)(\Theta-\hat{\Theta})} > 0$$

$$\lim_{\chi \rightarrow 0} P_{+\infty}(-1) = \frac{2[(1-\alpha)(1+\beta)-\delta(1+\beta(1-\alpha))](\Theta-\hat{\Theta})}{\beta(1-\alpha)(1-\delta)(\Theta-\hat{\Theta})}$$

with

$$\tilde{\Theta} = \frac{\frac{2\alpha\delta(1+\beta)(1-\theta(1-\alpha))}{1-\theta} + \delta(1-\alpha)(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta})}{2[(1-\alpha-\delta)(1+\beta) + \delta\beta\alpha]} < \Theta_1 = \frac{\delta\alpha(1-\theta(1-\alpha))}{(1-\theta)(1-\alpha-\delta)} < \bar{\Theta}$$

We know that  $\mathcal{D}_\chi(0) > 1$ ,  $\lim_{\chi \rightarrow 0} P_0(1) < 0$  and  $P_0(-1) > 0$  for any  $\chi$ .

a) Assume first that  $\hat{\Theta} < \Theta < \Theta_1$ . We get  $\mathcal{D}_0(+\infty) < 0$  and  $\lim_{\chi \rightarrow 0} P_{+\infty}(-1) < 0$  if  $\Theta \in (\hat{\Theta}, \tilde{\Theta})$  while  $\lim_{\chi \rightarrow 0} P_{+\infty}(-1) > 0$  if  $\Theta \in (\tilde{\Theta}, \Theta_1)$ . We conclude that when  $\Theta \in (\hat{\Theta}, \tilde{\Theta})$ , provided  $\mathcal{T}_0(\bar{\sigma}^H) \in (-2, 2)$ , the  $\Delta_\chi$  line is located as in Figure 5. When  $\Theta \in (\tilde{\Theta}, \Theta_1)$ , and provided  $\mathcal{T}_0(\bar{\sigma}^H) \in (-2, 2)$ , the  $\Delta_\chi$  line is located as in Figure 6.

b) Assume now that  $\Theta \in (\Theta_1, \bar{\Theta})$ . We get under Assumption 3  $\mathcal{D}_0(+\infty) \in (0, 1)$  and  $\lim_{\chi \rightarrow 0} P_{+\infty}(-1) > 0$ . Therefore, provided  $\mathcal{T}_0(\bar{\sigma}^H) \in (-2, 2)$ , the  $\Delta_\chi$  line is located as in Figure 6.

The last step consists finally in showing that  $\mathcal{T}_0(\bar{\sigma}^H) \in (-2, 2)$ . Let us note first that  $\mathcal{T}_0(\bar{\sigma}^H)$  can be written as follows:

$$\mathcal{T}_0(\bar{\sigma}^H) = 2 - \frac{\delta(1-\alpha)(1-\theta)(1-\frac{\beta\alpha\delta}{1-\theta})(\bar{\sigma}^H - \frac{\alpha}{1-\alpha})}{\bar{\sigma}^H(1-\alpha)\theta(\Theta-\hat{\Theta}) + \Theta(1-\frac{\beta\alpha\delta}{1-\theta})\theta\alpha}$$

with

$$\bar{\sigma}^H = \frac{(1-\frac{\beta\delta\alpha}{1-\theta})\Theta\alpha(1-\delta)(1-\beta)}{-\Theta[(1-\alpha)(1-\theta)-\delta] + \frac{\delta\alpha(1-\beta)[1-\theta(1-\alpha)]}{1-\theta}}$$

As we have shown previously,  $\bar{\sigma}^H > 0$  if and only if  $\delta \in (\delta_0, 1)$  with  $\delta_0$  as given by (38). Moreover, there exists  $\tilde{\beta} \in [\hat{\beta}, 1)$  such that  $\delta_0 < \tilde{\delta} \leq \hat{\delta}$  when  $\beta \in (\tilde{\beta}, 1)$ . From now on, let  $\beta \in (\tilde{\beta}, 1)$  and  $\delta \in (\delta_0, \tilde{\delta})$ . As  $\Theta > \hat{\Theta}$ , the denominator of the ratio in  $\mathcal{T}_0(\bar{\sigma}^H)$  is positive. We then need to study the numerator of the ratio in  $\mathcal{T}_0(\bar{\sigma}^H)$ .

Note that

$$\bar{\sigma}^H - \frac{\alpha\Theta}{1-\alpha} = \frac{(1-\frac{\beta\delta\alpha}{1-\theta})\Theta\alpha(1-\delta)(1-\beta)(1-\alpha) - \alpha\Theta \text{den}\sigma^H}{(1-\alpha)\text{den}\sigma^H}$$

with  $\text{den}\sigma^H$  the denominator of  $\sigma^H$ . We have shown previously that  $\text{den}\sigma^H > 0$  if and only if  $\delta \in (\delta_0, 1)$ , with  $\lim_{\delta \rightarrow \delta_0} \text{den}\sigma^H = 0$ . Moreover, we derive from the expression of  $\sigma^H$  that  $\lim_{\delta \rightarrow 1} \sigma^H = 0$ . Therefore, we conclude that there exists  $\delta_1 \in (\delta_0, \tilde{\delta}]$  such that the numerator of the ratio in  $\mathcal{T}_0(\bar{\sigma}^H)$  is positive and  $\mathcal{T}_0(\bar{\sigma}^H) < 2$  if  $\delta \in (\delta_0, \delta_1)$ . Let us then denote  $\bar{\delta} = \delta_1$ . We conclude that there exists  $\underline{\delta} \in [\delta_0, \bar{\delta})$  such that when  $\delta \in (\underline{\delta}, \bar{\delta})$ ,  $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$ .



Finally, we may compute the bifurcation values of  $\sigma$ . The Hopf bifurcation value  $\bar{\sigma}^H$  is given by (18). The flip bifurcation value  $\bar{\sigma}^F$  is such that  $P_\sigma(-1) = 1 + \mathcal{T}_\chi(\sigma) + \mathcal{D}_\chi(\sigma) = 0$  and is given by (19). Finally, the transcritical bifurcation value  $\bar{\sigma}^T$  is such that  $P_\sigma(-1) = 1 - \mathcal{T}_\chi(\sigma) + \mathcal{D}_\chi(\sigma) = 0$  and is given by (20).  $\square$

## 6.5 Data set with quarterly series

Our definition for aggregate consumption, aggregate investment and aggregate output is standard. We use quarterly data from the Bureau of Economic Analysis and define consumption as the sum of personal consumption expenditure in non-durable goods and services, and investment as the sum of private fixed investment and personal consumption expenditures in durable goods.<sup>20</sup> To obtain per capita variables, both series are divided by the population aged 16 and over. Finally, output is the sum of consumption and investment thus defined.

In order to construct series for hours worked in the consumption and investment sectors, we used data collected by the Current Employment Statistics program and available on the FRED database at the Federal Reserve Bank of Saint-Louis. The CES program provides series on the number of production and non-supervisory employees (together with their average weekly hours of work) at the sectoral level for the period 1964-Q1 to 2011-Q2.<sup>21</sup> Following Baxter's approach, we allocate hours worked in a specific industry either to the consumption or the investment sector according to the predominant final use of the output of this industry as consumption or investment goods (see the NIPA input-output tables). We thus define labor in the investment sector as the total number of (production and non-supervisory) employees in the construction and in the manufacturing durable goods sectors, multiplied by the average weekly hours of work in each sector. Similarly, we define hours in the consumption sector as the total number of employees, multiplied by the corresponding series on average weekly hours worked, in the manufacturing nondurable goods sector plus the *Trade, transportation and utilities, Information, Leisure and hospitality* and *Other services* sectors. We discarded series on the *Mining and logging* and *Professional and business services* sectors because of the mixed final use of the

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<sup>20</sup>Those data can easily be found in NIPA tables provided by the BEA. See in particular NIPA table 1.2.3: Real Gross Domestic Product by major type of product, quantity indexes, 2005 = 100.

<sup>21</sup>Industries are defined according to the following classification, which slightly differs from the standard NAICS classification: (1) Mining and Logging, (2) Construction, (3) Manufacturing (durable and nondurable) Goods, (4) Trade, Transportation and Utilities, (5) Information, (6) Professional and Business Services, (7) Leisure and Hospitality, and (8) Other Services.

output of these sectors as consumption or investment goods. However, we verified that the results were not sensitive to this choice. Finally, total hours are defined as the sum of hours worked in each sector. To obtain per capita variables, all these series were divided by the population aged 16 and over.

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